Shifted Systems of Linear Diff Eq's

$$\frac{d\vec{x}}{dt} = A(\vec{x} - \vec{a})$$

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$

Method of "Transient" Solution:

= "Real Solution" minus "An Equilibrium"

Shifted Systems

$$\boxed{\frac{d\vec{x}}{dt} = A(\vec{x} - \vec{a})}$$

$\vec{x} = \vec{x}(t)$, A : constant square matrix
$\vec{a}$: constant vector

Solution Method:

- Set $\vec{y} = \vec{x} - \vec{a}$ { Note $\vec{a}$ is an equilibrium. $\vec{y} = \vec{x} - \vec{a}$ is the so-called "transient" sol.
- $\frac{d\vec{y}}{dt} = A\vec{y}$. Solve this to get $\vec{y}(t)$.
- Solutions $\vec{x}(t) = \vec{a} + \vec{y}(t)$. [equilibrium + transient]

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}}$$

$\vec{x} = \vec{x}(t)$, A : constant square matrix
$\vec{b}$: constant vector

Solution Method:

- Find an equilibrium, by solving $A\vec{x} + \vec{b} = \vec{0}$.
  Denote this equilibrium by $\vec{a}$.
- Transient Sol. $\vec{y} = \vec{x} - \vec{a}$ satisfies $\frac{d\vec{y}}{dt} = A\vec{y}$.
- Solve for $\vec{y}(t)$.
- Solutions $\vec{x}(t) = \vec{a} + \vec{y}(t)$.

Remark: If equilibria do not exist, this method fails. Use other methods, e.g. "variation of parameters".

Example. Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \left(\vec{x} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$. Sketch the phase portrait.

Solution

Set $\vec{y} = \vec{x} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (transient sol)

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} -1 & 10 \\ -2 & 3 \end{bmatrix} \vec{y}$$

- **Eigenvalues**: $\det \begin{bmatrix} -1-\lambda & 10 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 17$.

  $\Leftrightarrow \lambda_{1,2} = 1 \pm 4i$

- **Eigenvectors** for $\lambda_1 = 1+4i$: $[A - (1+4i)I]\vec{y} = \vec{0}$

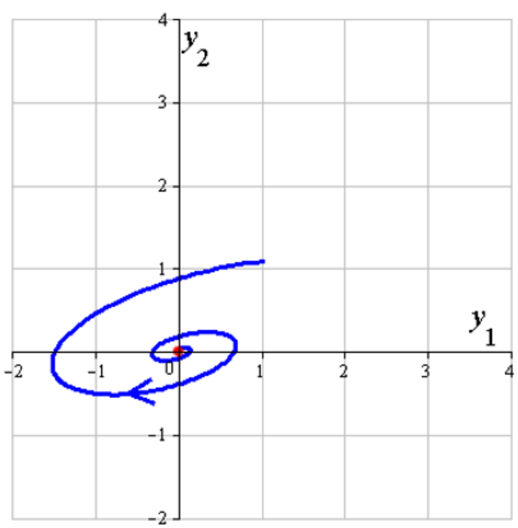
  $\begin{bmatrix} -2-4i & 10 \\ -2 & 2-4i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow -2y_1 + (2-4i)y_2 = 0 \Leftrightarrow y_1 = (1-2i)y_2 \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$

  An eigenvector for $\lambda_1 = 1+4i$: $\vec{u}_1 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

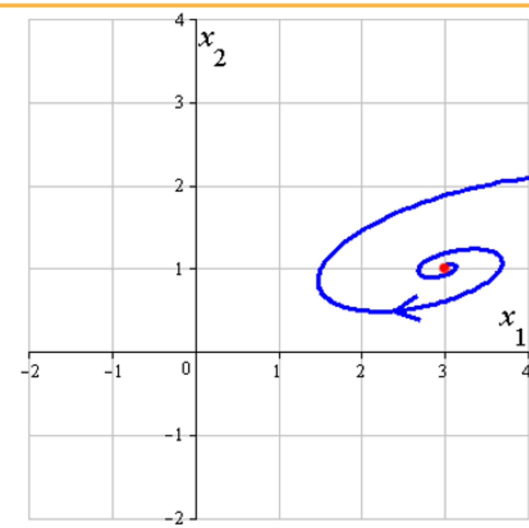
$$\vec{y}(t) = C_1 e^t \left\{ \cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} + C_2 e^t \left\{ \sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$$

Gen. Solutions

$$\vec{x}(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_1 e^t \left\{ \cos(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} + C_2 e^t \left\{ \sin(4t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(4t) \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$$



$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium.



$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an equilibrium.

Example. Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} -12 \\ 14 \end{bmatrix}$. Sketch the phase portrait.

Type : $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$ where $A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -12 \\ 14 \end{bmatrix}$.

Solution

• Find an equilibrium : $A\vec{x} + \vec{b} = \vec{0}$, $A\vec{x} = -\vec{b}$, $\begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{x} = \begin{bmatrix} 12 \\ -14 \end{bmatrix}$

Either row reduce, or $\vec{x} = -A^{-1}\vec{b}$: $\cdots\cdot \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (equilibrium)

• Set $\vec{y} = \vec{x} - \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (transient sol.)

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} -12 \\ 14 \end{bmatrix} \Leftrightarrow \frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \left(\vec{x} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right) \Leftrightarrow \frac{d\vec{y}}{dt} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \vec{y}$$

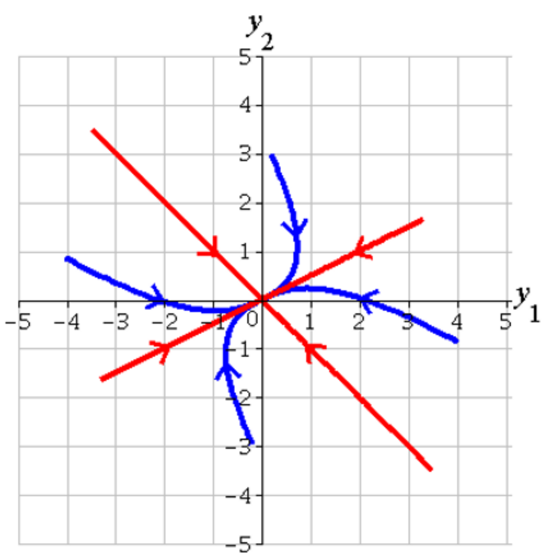
• Eigenvalues and Eigenvectors : $\lambda_1 = -2$, $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = -5$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Gen. Sol's of $\frac{d\vec{y}}{dt} = A\vec{y}$:

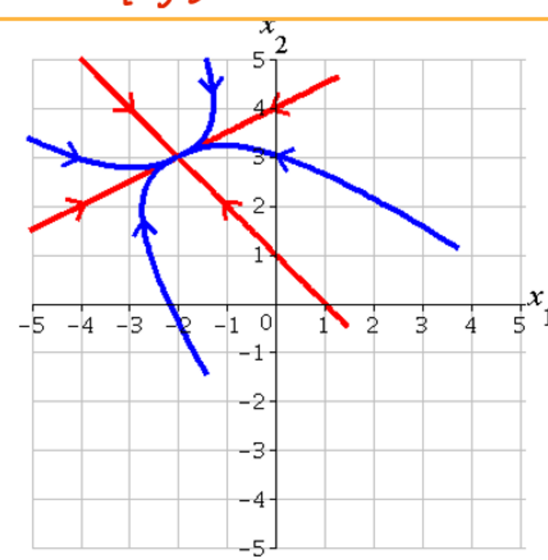
$\vec{y}(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Gen. Sol's of $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$:

$$\boxed{\vec{x}(t) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$



$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium.



$\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is an equilibrium.

Example. Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 12 \\ 4 \end{bmatrix}$. Sketch the phase portrait.

Solution

• Find equilibria: $A\vec{x} + \vec{b} = \vec{0}$, $A\vec{x} = -\vec{b}$, $\begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}\vec{x} = \begin{bmatrix} -12 \\ -4 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} -3x_1 - 6x_2 = -12 \\ -x_1 - 2x_2 = -4 \end{cases} \Leftrightarrow x_1 + 2x_2 = 4 \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 - 2x_2 \\ x_2 \end{bmatrix}$$ (infinitely many equilibria)

Choose one: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (a particular equilibrium)

• Set $\vec{y} = \vec{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (transient sol.)

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 12 \\ 4 \end{bmatrix} \Leftrightarrow \frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}\left(\vec{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) \Leftrightarrow \frac{d\vec{y}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}\vec{y}$$

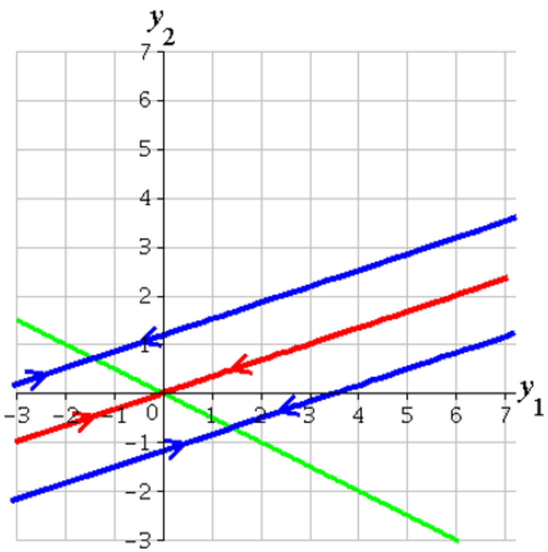
• Eigenvalues and Eigenvectors: $\lambda_1 = 0$, $\vec{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\lambda_2 = -5$, $\vec{u}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

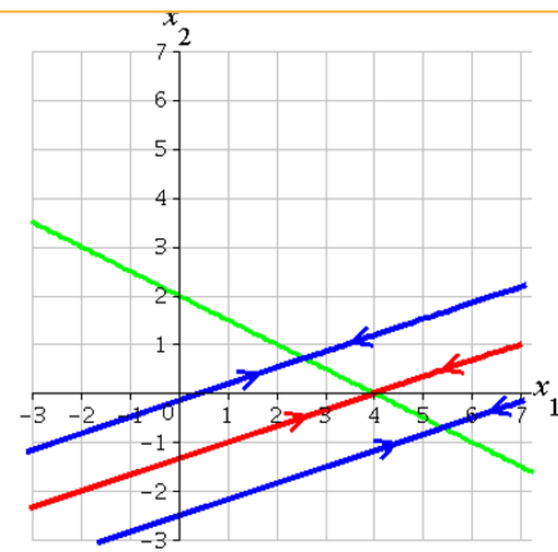
Gen. Sol's of $\frac{d\vec{y}}{dt} = A\vec{y}$:

$$\vec{y}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Gen. Sol's of $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$:

$$\boxed{\vec{x}(t) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$





Example Solve $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 7 \\ -5 \\ 14 \end{bmatrix}$

Solution $\dfrac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$.

- Find equilibria: $A\vec{x} + \vec{b} = \vec{0} \Leftrightarrow A\vec{x} = -\vec{b} \Leftrightarrow \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} -7 \\ 5 \\ -14 \end{bmatrix}$.

  Either row reduce, or $\vec{x} = -A^{-1}\vec{b}$: $\bullet\bullet\bullet\bullet \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ (an equilibrium)

- Set $\vec{y} = \vec{x} - \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ (transient solution)

  $\dfrac{d\vec{x}}{dt} = A\vec{x} + \vec{b} \Leftrightarrow \dfrac{d\vec{x}}{dt} = A\left(\vec{x} - \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}\right) \Leftrightarrow \dfrac{d\vec{y}}{dt} = A\vec{y}$

- Eigenvalues of A: $\det(A - \lambda I) = \bullet\bullet\bullet\bullet \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -1$.

- Eigenvectors for $\lambda_1, \lambda_2, \lambda_3$: $(A - \lambda_j I)\vec{y} = \vec{0}$, $\bullet\bullet\bullet\bullet \Rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

- Gen. Solutions of the Given System:

$$\boxed{\vec{x}(t) = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} + c_1 e^{t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}$$

Example Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 11 \\ 4 \end{bmatrix}$.

Solution. Find an equilibrium: Solve $\begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 11 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} -3x_1 - 6x_2 + 11 = 0 \\ -x_1 - 2x_2 + 4 = 0 \end{cases}$$

← The two equations are inconsistant. There are no solutions.

$\boxed{\text{There are no equilibria in this system}}$

- The method of this lecture does not work.
- Need to use the method of "variation of parameters", or the method of "undetermined coefficients".

"variation of parameters" $\Rightarrow \vec{x}(t) = \begin{bmatrix} -\frac{2}{5}t + \frac{57}{25} \\ \frac{1}{5}t + \frac{19}{25} \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

"undetermined coefficients" $\Rightarrow \vec{x}(t) = \begin{bmatrix} -\frac{2}{5}t + \frac{19}{5} \\ \frac{1}{5}t \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

the method of using equilibrium

WARNING!

The method in this file works ONLY for diff eq's of the forms:

$$\frac{d\vec{x}}{dt} = A(\vec{x} - \vec{a})$$ where $\vec{a}$ is a constant vector;

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}$$ where $\vec{b}$ is a constant vector.

If $\vec{a}$, $\vec{b}$ are nonconstant functions of t,

need to use another method: "variation of parameters".

Example $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\left(\vec{x} - \begin{bmatrix} t \\ t^2 \end{bmatrix}\right)$

and Example $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} \cos t \\ 2 \end{bmatrix}$

can be solved by "variation of parameters", but cannot be solved by the method in this file.