Oscillations driven by external forces

Frequency response

Resonance
Forced Vibration in Spring-Mass

Assume that there are three forces only:

- The restoring force exerted by the spring: \(-ky\);  
- The damping force: \(-\gamma \frac{dy}{dt}\);  
- The external force: \(F(t)\).

Here, \(k\) is the spring constant and \(\gamma\) is the damping coefficient.

The Derivation of the Diff Equation

\[(\text{mass})(\text{acceleration}) = (\text{the net force on the mass}),\]

\[m \frac{d^2y}{dt^2} = -ky - \gamma \frac{dy}{dt} + F(t),\]

\[m \frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + ky = F(t)\]
The differential equation is given by:

\[
\frac{1}{2} \frac{d^2y}{dt^2} + 8y = F(t)
\]

The solutions are:

\[
F(t) = F_0 \cos(\omega t)
\]

\[
y_p(t) = A \cos(\omega t)
\]

**Question:**

How does \( A \) depend on \( F_0 \)?

![Graphs showing the relationship between \( F_0 \) and \( A \)]
\[
\frac{1}{2} \frac{d^2y}{dt^2} + 8y = F(t)
\]

\[F(t) = F_0 \cos(\omega t)\]

\[y_p(t) = A \cos(\omega t)\]

**Question:** How does \(A\) depend on \(F_0\)?

**Answer:** 
\(A \propto F_0\). A larger force induces a larger motion.
\[
\frac{1}{2} \frac{d^2y}{dt^2} + 8y = F(t)
\]

\[F(t) = 10 \cos(\omega t)\]

\[y_p(t) = A \cos(\omega t)\]

**Question:** How does \(A\) depend on \(\omega\)?
\[ \frac{1}{2} \frac{d^2 y}{dt^2} + 8y = F(t) \]

\[ F(t) = 10 \cos(\omega t) \]

\[ y_p(t) = A \cos(\omega t) \]

**Question:** How does \( A \) depend on \( \omega \)?

**Frequency Response**

- \( \omega = 1 \), External Force: \( F(t) \) vs \( t \)
- \( \omega = 1 \), Induced Motion: \( y_p(t) \) vs \( t \)
- \( \omega = 3.5 \), External Force: \( F(t) \) vs \( t \)
- \( \omega = 3.5 \), Induced Motion: \( y_p(t) \) vs \( t \)
- \( \omega = 5 \), External Force: \( F(t) \) vs \( t \)
- \( \omega = 5 \), Induced Motion: \( y_p(t) \) vs \( t \)
\[ \frac{1}{2} \frac{d^2y}{dt^2} + 8y = F(t) \]

**Frequency Response**

\[ F(t) = 10 \cos(\omega t) \]

\[ y_p(t) = A \cos(\omega t) \]

**Question:**

How does \( A \) depend on \( \omega \)?

The Natural Frequency:

\[ \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4 \]
\[ \frac{1}{2} \frac{d^2y}{dt^2} + 8y = F(t) \]

**Frequency Response**

\[ F(t) = 10 \cos(\omega t) \]

\[ y_p(t) = A \cos(\omega t) \]

**Question:** How does \( A \) depend on \( \omega \)?

**Resonance**

The Natural Frequency:

\[ \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4 \]
The normalized gain \( \frac{|G(i\omega)|}{|G(0)|} \) vs \( \omega \)

### Un-Damped Spring-Mass System

\[
m \frac{d^2 y}{dt^2} + ky = F(t)
\]

<table>
<thead>
<tr>
<th>( F(t) )</th>
<th>( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_0 \cos(\omega t) )</td>
<td>( \frac{F_0}{m} G(i\omega) \cos(\omega t) )</td>
</tr>
<tr>
<td>( F_0 \sin(\omega t) )</td>
<td>( \frac{F_0}{m} G(i\omega) \sin(\omega t) )</td>
</tr>
<tr>
<td>( F_0 e^{i\omega t} )</td>
<td>( \frac{F_0}{m} G(i\omega) e^{i\omega t} )</td>
</tr>
</tbody>
</table>

### The Frequency Response Curve when \( m = \frac{1}{2}, k = 8 \)

The Natural Frequency:

\[
\omega_0 = \sqrt{\frac{k}{m}}
\]

The Frequency Response:

\[
G(i\omega) = \frac{1}{(i\omega)^2 + \omega_0^2} = \frac{1}{-\omega^2 + \omega_0^2}
\]

At the natural frequency \( \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4 \),
the gain diverges.
**Damped Spring-Mass System**

\[
m \frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + ky = F(t)
\]

<table>
<thead>
<tr>
<th>(F(t))</th>
<th>(y_p(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_0 e^{i\omega t})</td>
<td>(\frac{F_0}{m} G(i\omega) e^{i\omega t})</td>
</tr>
<tr>
<td>(F_0 \cos(\omega t))</td>
<td>(\text{Re}\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right])</td>
</tr>
<tr>
<td>(F_0 \sin(\omega t))</td>
<td>(\text{Im}\left[\frac{F_0}{m} G(i\omega) e^{i\omega t}\right])</td>
</tr>
</tbody>
</table>

**The Frequency Response Curve when** \(m = \frac{1}{2}, \gamma = 1, k = 8\)

The natural frequency:
\[
\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/2}} = 4.
\]

The maximal gain is at:
\[
\omega_{max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}} = \sqrt{16 - \frac{1}{2(1/2)^2}} = \sqrt{14}.
\]
**Damped Spring-Mass System**

\[
m \frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + ky = F(t)
\]

<table>
<thead>
<tr>
<th>(F(t))</th>
<th>(y_p(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_0 e^{i\omega t})</td>
<td>(\frac{F_0}{m} G(i\omega) e^{i\omega t})</td>
</tr>
<tr>
<td>(F(t) = F_0 \cos(\omega t))</td>
<td>(y_p(t) = \text{Re} \left[ \frac{F_0}{m} G(i\omega) e^{i\omega t} \right])</td>
</tr>
<tr>
<td>(F(t) = F_0 \sin(\omega t))</td>
<td>(y_p(t) = \text{Im} \left[ \frac{F_0}{m} G(i\omega) e^{i\omega t} \right])</td>
</tr>
</tbody>
</table>

The Frequency Response Curve when \(m = \frac{1}{2}, \gamma = 3, k = 8\)

The normalized gain \(|G(i\omega)|/|G(0)|\) vs \(\omega\)

The Natural Frequency:

\[\omega_0 = \sqrt{\frac{k}{m}}\]

The Frequency Response:

\[
G(i\omega) = \frac{1}{(i\omega)^2 + \frac{\gamma}{m} (i\omega) + \omega_0^2}
\]

\[
= \frac{1}{-\omega^2 + \omega_0^2 + \frac{\gamma\omega}{m} i}
\]

The natural frequency:

\[\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{\frac{1}{2}}} = 4.\]

The maximal gain is at:

\[\omega_{max} = 0.\]