Scalar First Order Linear PDEs with Several Space Variables

\[ p(x,t) \frac{\partial u}{\partial t} + \sum_{i=1}^{n} q_i(x,t) \frac{\partial u}{\partial x_i} + r(x,t)u = f(x,t) \]

Initial Value Problem

\[
\begin{align*}
p(x,t) \frac{\partial u}{\partial t} + \sum_{i=1}^{n} q_i(x,t) \frac{\partial u}{\partial x_i} + r(x,t)u &= f(x,t) \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, t > 0, \\
u(x,0) &= g(x) \quad x \in \mathbb{R}^n, \quad \text{[I.C.]} \end{align*}
\]

where \( p(x,t), q_i(x,t), r(x,t), f(x,t) \) and \( g(x) \) are given functions. We want to find \( u(x_1, \ldots, x_n, t) \).

Solution Method

Step 1: Characteristic Curves. Solve the following ODE system of \( n+1 \) equations:

\[
\begin{align*}
\frac{dt}{ds} &= p(x,t), \quad t(0) = 0, \\
\frac{dx}{ds} &= q(x,t), \quad x(0) = \xi = (\xi_1, \ldots, \xi_n).
\end{align*}
\]

Denote the solution by \( t = t(s, \xi), \quad x = x(s, \xi) \).

Step 2: Inverse Function. From

\[ t = t(s, \xi), \quad x = x(s, \xi), \]

compute the inverse function

\[ s = s(x,t), \quad \xi = \xi(x,t). \]

Step 3: Follow the Characteristics. Solve the ODE:

\[
\frac{dU}{ds} + r(x,t)U = f(x,t), \quad U(0) = g(\xi).
\]

Denote the solution by \( U = U(s, \xi) \).

Step 4: Solution Formula.

\[ u(x,t) = U(s(x,t), \xi(x,t)). \]
Example 1
Consider
\[
\begin{align*}
8u_t + 9u_x + 2u_y &= 0 & x, y &\in \mathbb{R}, t > 0, & \text{[PDE]} \\
\quad u(x, y, 0) &= e^{-x^2-3y^2} & x, y &\in \mathbb{R}, & \text{[I.C.]} 
\end{align*}
\]

STEP 1:
\[
\begin{align*}
\frac{dt}{ds} &= 8, & t(0) &= 0, \\
\frac{dx}{ds} &= 9, & x(0) &= \xi, \\
\frac{dx}{ds} &= 2, & y(0) &= \eta.
\end{align*}
\]
The parameterized form of the characteristics is
\[
t = 8s, \quad x = \xi + 9s, \quad y = \eta + 2s.
\]
They are straight lines.

STEP 2: The inverse function is
\[
s = t/8, \quad \xi = x - 9t/8, \quad \eta = y - 2t/8 = y - t/4.
\]

STEP 3:
\[
\frac{dU}{ds} = 0, \quad U(0) = e^{-\xi^2-3\eta^2} \quad \implies \quad U = e^{-\xi^2-3\eta^2}.
\]
This means the solution remains constant on every characteristic line.

STEP 4: The solution is
\[
u(x, t) = e^{-(x-9t/8)^2-3(y-t/4)^2}.
\]

3-D graph of \(u(x, y, t)\) vs \((x, y)\) at \(t = 0\).
The graph of \(u(x, y, t)\) vs \((x, y)\) at \(t = 3\).
Exercises

[1] Solve \(2u_t + 5u_x + 3u_y = e^{-3t}\), \(u(x, y, 0) = 0\).

[2] Solve \(u_t + u_x - u_y + 5u_z = 0\), \(u(x, y, z, 0) = \sin x \sin y \sin z\).

[3] Solve \(u_t + u_x + u_y + u = 0\), \(u(x, y, 0) = x^2 - y^2\).

[4] Give the solution formula for the problem

\[u_t + au_x + bu_y + cu_z + ku = 0, \quad u(x, y, z, 0) = g(x, y, z)\]

Here, \(a, b, c, k\) are given constants and \(g(x, y, z)\) is a given function.

Answers

[1] \(u(x, t) = \frac{1}{6} - \frac{1}{6}e^{-3t}\)

[2] \(u(x, t) = \sin(x - t) \sin(y + t) \sin(z - 5t)\)

[3] \(u(x, t) = e^{-t} \left[(x - t)^2 - (y - t)^2\right]\)

[4] \(u(x, t) = e^{-kt}g(x - at, y - bt, z - ct)\)