Initial Value Problem

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + a_{11} \frac{\partial u_1}{\partial x} + \cdots + a_{1n} \frac{\partial u_n}{\partial x} &= 0 & x \in \mathbb{R}, t > 0, & \quad [\text{PDE } 1] \\
\vdots & & \vdots & \\
\frac{\partial u_n}{\partial t} + a_{n1} \frac{\partial u_1}{\partial x} + \cdots + a_{nn} \frac{\partial u_n}{\partial x} &= 0 & x \in \mathbb{R}, t > 0, & \quad [\text{PDE } n] \\
\end{align*}
\]

\[
\begin{align*}
u_1(x, 0) &= g_1(x) & x \in \mathbb{R}, & \quad [\text{IC } 1] \\
\vdots & & \vdots & \\
\end{align*}
\]

\[
\begin{align*}
u_n(x, 0) &= g_n(x) & x \in \mathbb{R}, & \quad [\text{IC } n] \\
\end{align*}
\]

(0.1)

where \(a_{ij}\) are given constants, and \(g_i(x)\) are given functions. We want to find \(u_1(x, t)\) and \(u_n(x, t)\).

Matrix Form

Using the vector and matrix notations, the above system can be rewritten in the matrix form.

Denote

\[
\mathbf{\tilde{u}}(x, t) = \begin{bmatrix} u_1(x, t) \\ \vdots \\ u_n(x, t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{\tilde{g}}(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix}.
\]

Problem (0.1) is equivalent to:

\[
\begin{align*}
\mathbf{\tilde{u}}_t + A\mathbf{\tilde{u}}_x &= 0 & x \in \mathbb{R}, t > 0, & \quad [\text{PDE}] \\
\mathbf{\tilde{u}}(x, 0) &= \mathbf{\tilde{g}}(x) & x \in \mathbb{R}, & \quad [\text{IC}] \\
\end{align*}
\]

(0.2)

Solution Method

If the equations are decoupled, in other words, if \(u_j\) does not appear in [PDE \(i\)] for \(i \neq j\), the problem can be easily solved. Notice that this is the case where \(A\) is a diagonal matrix.

This also suggests the approach to the general case, namely, try to convert the system into decoupled equations by diagonalizing the matrix \(A\).

Step 1: Diagonalization of Matrix \(A\). Suppose that the matrix \(A\) is diagonalizable. Let \(\lambda_1, \ldots, \lambda_n\) are the eigenvalues of the matrix \(A\) and \(\mathbf{\tilde{w}}_1, \ldots, \mathbf{\tilde{w}}_n\) are the corresponding eigenvectors that are linearly independent:

\[
A\mathbf{\tilde{w}}_1 = \lambda_1 \mathbf{\tilde{w}}_1, \ldots, A\mathbf{\tilde{w}}_n = \lambda_n \mathbf{\tilde{w}}_n \quad \lambda_1 \neq 0, \ldots, \lambda_n \neq 0.
\]
A converting matrix $P$ can be chosen as

$$P = \begin{bmatrix} \omega_1 & \cdots & \omega_n \end{bmatrix}.$$  

This does the diagonalization as follows:

$$P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$  

**Step 2: Decoupled Equations.** Setting $\bar{\phi} = P\phi$ with $\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}$. Equation (0.2) gives

$$\frac{\partial \bar{\phi}}{\partial t} + \Lambda \frac{\partial \bar{\phi}}{\partial x} = 0 \quad x \in \mathbb{R}, t > 0, \quad (0.3)$$

$$\bar{\phi}(x, 0) = P^{-1}\bar{u}(x, 0) = P^{-1}\bar{g}(x) \quad x \in \mathbb{R}, \quad (0.4)$$

or, equivalently,

$$\begin{bmatrix} \phi_1(x, 0) \\ \vdots \\ \phi_n(x, 0) \end{bmatrix} = P^{-1} \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix}, \quad x \in \mathbb{R}. \quad \text{[Decoupled Equations]} \quad (0.5)$$

Now we can solve the equations in this system separately to get solutions $\phi_1(x, t), \cdots, \phi_n(x, t)$.

**Step 3: Solution Formula.**

$$\bar{u}(x, t) = P\bar{\phi}(x, t).$$

**Example**

Consider

$$\begin{cases} u_t + u_x - 10v_x = 0 & x \in \mathbb{R}, t > 0, \\ v_t - 5u_x - 4v_x = 0 & x \in \mathbb{R}, t > 0, \\ u(x, 0) = \sin x & x \in \mathbb{R}, \\ v(x, 0) = \cos x & x \in \mathbb{R}. \end{cases}$$

**STEP 1:** The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -10 \\ -5 & -4 \end{bmatrix}$$
are $\lambda_1 = 6, \lambda_2 = -9$ with the corresponding eigenvectors

$$\tilde{w}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \tilde{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. $$

So a converting matrix $P$ is

$$P = [\tilde{w}_1 \ \tilde{w}_2] = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

diagonalizes $A$:

$$P^{-1}AP = \Lambda = \begin{bmatrix} 6 & 0 \\ 0 & -9 \end{bmatrix}.$$

**STEP 2:** Define

$$\begin{bmatrix} u \\ v \end{bmatrix} = P \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}. $$

The decoupled equations are

$$\frac{\partial}{\partial t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & -9 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = 0 \quad x \in \mathbb{R}, t > 0,$$

$$\begin{bmatrix} \phi(x, 0) \\ \psi(x, 0) \end{bmatrix} = P^{-1} \begin{bmatrix} u(x, 0) \\ v(x, 0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \sin x - \frac{1}{3} \cos x \\ \frac{1}{3} \sin x + \frac{2}{3} \cos x \end{bmatrix} \quad x \in \mathbb{R}.$$

Or, equivalently,

$$\phi_t + 6\phi_x = 0, \quad \phi(x, 0) = \frac{1}{3} \sin x - \frac{1}{3} \cos x,$$

$$\psi_t - 9\psi_x = 0, \quad \psi(x, 0) = \frac{1}{3} \sin x + \frac{2}{3} \cos x.$$

The solution of the decoupled system is given by

$$\phi(x, t) = \frac{1}{3} \sin(x - 6t) - \frac{1}{3} \cos(x - 6t),$$

$$\psi(x, t) = \frac{1}{3} \sin(x + 9t) + \frac{2}{3} \cos(x + 9t).$$

**STEP 3:** The solution to the original problem is

$$\begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = P \begin{bmatrix} \phi(x, t) \\ \psi(x, t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \sin(x - 6t) - \frac{1}{3} \cos(x - 6t) \\ \frac{1}{3} \sin(x + 9t) + \frac{2}{3} \cos(x + 9t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \sin(x - 6t) - \frac{2}{3} \cos(x - 6t) + \frac{1}{3} \sin(x + 9t) + \frac{2}{3} \cos(x + 9t) \\ -\frac{1}{3} \sin(x - 6t) + \frac{1}{3} \cos(x - 6t) + \frac{1}{3} \sin(x + 9t) + \frac{2}{3} \cos(x + 9t) \end{bmatrix}. $$
Exercises

[1] (a) Solve \( u_t - 2u_x + v_x = 0, v_t - 4u_x + 3v_x = 0, u(x, 0) = g(x), v(x, 0) = h(x). \)

(b) Consider the initial data

\[
u(x, 0) = g(x) = \begin{cases} 0 & |x| \geq 3, \\ 9 - x^2 & |x| < 3, \end{cases}
\]

\[
v(x, 0) = h(x) \equiv 0. \]

Graph the solutions \( u(x, t) \) and \( v(x, t) \) vs \( x \) at times \( t = 0, 1, 2, 3, 4, 5. \)

[2] (a) Solve \( u_t + 5u_x - 3v_x = 0, v_t + u_x + v_x = 0, u(x, 0) = g(x), v(x, 0) = h(x). \)

(b) Consider the initial data \( u(x, 0) = g(x) \equiv 0 \) and

\[
v(x, 0) = h(x) = \begin{cases} 0 & |x| \geq 3, \\ 9 - x^2 & |x| < 3. \end{cases}
\]

Graph the solutions \( u(x, t) \) and \( v(x, t) \) vs \( x \) at times \( t = 0, 1, 2, 3, 4, 5. \)

ANSWERS

[1] (a) \( u(x, t) = -\frac{1}{3} g(x - 2t) + \frac{1}{3} h(x - 2t) + \frac{4}{3} g(x + t) - \frac{1}{3} h(x + t), \)

\( v(x, t) = -\frac{4}{3} g(x - 2t) + \frac{4}{3} h(x - 2t) + \frac{4}{3} g(x + t) - \frac{1}{3} h(x + t). \)

(b) Graphs skipped. Try to observe that there are two waves, one moving to the right with speed 2, and the other moving to the left with speed 1.

[2] (a) \( u(x, t) = \frac{3}{2} g(x - 4t) - \frac{3}{2} h(x - 4t) - \frac{1}{2} g(x - 2t) + \frac{3}{2} h(x - 2t), \)

\( v(x, t) = \frac{1}{2} g(x - 4t) - \frac{1}{2} h(x - 4t) - \frac{1}{2} g(x - 2t) + \frac{3}{2} h(x - 2t). \)

(b) Graphs skipped. Try to observe that there are two waves; both waves move to the right with constant speeds, but one is faster than the other. The fast wave has speed 4, and the slower one moves with speed 2.