Nonhomogeneous 1-D Heat Equation

Duhamel’s Principle on Infinite Bar

Objective: Solve the initial value problem for a nonhomogeneous heat equation with zero initial condition:

\[
\begin{cases}
    u_t - ku_{xx} = p(x, t) & -\infty < x < \infty, t > 0, \\
    u(x, 0) = 0 & -\infty < x < \infty.
\end{cases}
\]

An Auxiliary Problem: For every fixed \( s > 0 \), consider a homogeneous heat equation for \( t > s \), with \( p(x, s) \) as the initial data at time \( t = s \):

\[
\begin{cases}
    u_t(x, t; s) - ku_{xx}(x, t; s) = 0 & -\infty < x < \infty, t > s, \\
    u(x, s; s) = p(x, s) & -\infty < x < \infty.
\end{cases}
\]

The solution \( u(x, t; s) \) of this auxiliary problem (parametrized by \( s \)) is given by

\[
u(x, t; s) = \int_{-\infty}^{\infty} G(x - \xi, t - s)p(\xi, s)d\xi \quad -\infty < x < \infty, t > s,
\]

where \( G(x, t) \) is the heat kernel: \( G(x, t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)} \).

Duhamel’s Principle: The solution of (*) is given by

\[
u(x, t) = \int_{0}^{t} u(x, t; s)ds.
\]

Thus, we have

\[
u(x, t) = \int_{s=0}^{t} \int_{\xi=-\infty}^{\infty} G(x - \xi, t - s)p(\xi, s)d\xi ds
\]

\[
= \int_{s=0}^{t} \int_{\xi=-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t - s)}}e^{-\frac{(x-\xi)^2}{4k(t-s)}}p(\xi, s)d\xi ds \quad -\infty < x < \infty, t > 0.
\]
Duhamel’s Principle on Finite Bar

Objective: Solve the initial boundary value problem for a nonhomogeneous heat equation, with homogeneous boundary conditions and zero initial data:

\[
\begin{aligned}
&u_t - ku_{xx} = p(x, t) \\
&0 < x < L, t > 0, \\
&u(0, t) = 0, u(L, t) = 0 \\
&t > 0, \\
&u(x, 0) = 0 \\
&0 \leq x \leq L.
\end{aligned}
\]  

An Auxiliary Problem: For every fixed \( s > 0 \), consider a homogeneous heat equation for \( t > s \), with the same homogeneous boundary conditions and with \( p(x, s) \) as the initial data at time \( t = s \):

\[
\begin{aligned}
&u_t(x, t; s) - ku_{xx}(x, t; s) = 0 \\
&0 < x < L, t > s, \\
&u(0, t; s) = 0, u(L, t; s) = 0, \\
&t > s, \\
&u(x, s; s) = p(x, s) \\
&0 \leq x \leq L.
\end{aligned}
\]

The solution \( u(x, t; s) \) of this auxiliary problem (parametrized by \( s \)) is given by

\[
u(x, t; s) = \int_0^t G(x, t - s; \xi)p(\xi, s)d\xi \quad 0 \leq x \leq L, t > s,
\]

where \( G(x, t; \xi) \) is the Green’s function:

\[
G(x, t; \xi) = \sum_{n=-\infty}^{\infty} \left[ G(x - 2nL - \xi, t) - G(x - 2nL + \xi, t) \right]
\]

\[
= \frac{1}{\sqrt{4\pi kt}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2nL-\xi)^2/(4kt)} - e^{-(x-2nL+\xi)^2/(4kt)} \right].
\]

Duhamel’s Principle: The solution of (**) is given by

\[
u(x, t) = \int_0^t u(x, t; s)ds.
\]

Thus, we have

\[
u(x, t) = \int_0^t \int_{\xi=0}^{L} G(x, t - s; \xi)p(\xi, s)d\xi ds
\]

\[
= \int_0^t \int_{\xi=0}^{L} \frac{1}{\sqrt{4\pi kt - s}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2nL-\xi)^2/(4k(t-s))} - e^{-(x-2nL+\xi)^2/(4k(t-s))} \right] p(\xi, s)d\xi ds,
\]

for \( 0 \leq x \leq L, t > 0 \).

Another expression using Fourier Series: We can also use Fourier series to solve the above auxiliary problem:

\[
u(x, t; s) = \sum_{n=1}^{\infty} b_n(s) \sin(n\pi x/L)e^{-k(n\pi/L)^2(t-s)} \quad 0 \leq x \leq L, t > s,
\]
where the coefficients $b_n(s)$ are determined by the initial data of the auxiliary problem:

$$b_n(s) = \frac{2}{L} \int_0^L p(x, s) \sin(n\pi x/L) dx.$$  

Then, by Duhamel’s Principle, the solution of (**) is given by

$$u(x, t) = \int_0^t \int_0^t u(x, t; s) ds$$

$$= \sum_{n=1}^{\infty} \sin(n\pi x/L) \int_0^t b_n(s) e^{-k(n\pi/L)^2(t-s)} ds,$$

for $0 \leq x \leq L, t > 0$. 
EXERCISES

[1] Solve the above problem on infinite bar (*) when \( p(x, t) = \delta(x - ct) \), with \( c = \) constant.

[2] Solve the above problem on finite bar (**) when \( p(x, t) = x \sin t \), using the Fourier series expression.

[3] Find the solution formula for
\[
\begin{align*}
&\begin{align*}
&u_t - ku_{xx} = p(x, t) \\
&u_x(0, t) = 0, u_x(L, t) = 0 \\
&u(x, 0) = 0
\end{align*} \\
&0 < x < L, t > 0, \\
&t > 0, \\
&0 \leq x \leq L.
\end{align*}
\]

[4] Find the solution formula for
\[
\begin{align*}
&\begin{align*}
&u_t - ku_{xx} = p(x, t) \\
&u(0, t) = 0 \\
&u(x, 0) = 0
\end{align*} \\
x > 0, t > 0, \\
t > 0, \\
x \geq 0.
\end{align*}
\]

(See next page for the answers)
ANSWERS

[1] \( u(x, t) = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-ct)^2}{4k(t-s)}} ds \), or, equivalently, \( u(x, t) = \int_0^t \frac{1}{\sqrt{4\pi k\tau}} e^{-\frac{(x-ct+cr)^2}{4k\tau}} d\tau \)

[2] \( u(x, t) = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi \left(k^2(n\pi/L)^4 + 1 \right)} \left(-\cos t + k(n\pi/L)^2 \sin t + e^{-k(n\pi/L)^2 t} \right) \sin \left( \frac{n\pi x}{L} \right) \)

[3] \( u(x, t) = \int_{s=0}^t \int_{\xi=0}^L \frac{1}{\sqrt{4\pi k(t-s)}} \sum_{n=-\infty}^{\infty} \left[ e^{-\frac{(x-2nL-\xi)^2}{4k(t-s)}} + e^{-\frac{(x-2nL+\xi)^2}{4k(t-s)}} \right] p(\xi, s) d\xi ds \),

or, equivalently,

\( u(x, t) = \int_0^t a_0(s) ds + \sum_{n=1}^{\infty} \cos \left( \frac{n\pi x}{L} \right) \int_0^t a_n(s) e^{k \left( \frac{n\pi x}{L} \right)^2 (t-s)} ds \),

where \( a_0(s) = \frac{1}{L} \int_0^L p(x, s) dx \), \( a_n(s) = \frac{2}{L} \int_0^L p(x, s) \cos \left( \frac{n\pi x}{L} \right) dx \).

[4] \( u(x, t) = \int_{s=0}^t \int_{\xi=0}^\infty \frac{1}{\sqrt{4\pi k(t-s)}} \left[ e^{-\frac{(x-\xi)^2}{4k(t-s)}} - e^{-\frac{(x+\xi)^2}{4k(t-s)}} \right] p(\xi, s) d\xi ds \)