One Dimensional Wave Equation

[1] Consider the initial-boundary value problem for a finite vibrating string under the gravity with two fixed ends:

\[ u_{tt} = \frac{T_0}{\rho} u_{xx} - g \quad 0 < x < L, t > 0, \]

\[ u(0, t) = a, \quad u(L, t) = b \quad t > 0, \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad 0 \leq x \leq L. \]

(a) Show that the time-independent solution of this boundary value problem is

\[ \phi(x) = \frac{\rho g}{2T_0} x(x - L) + \frac{b - a}{L} x + a. \]

and sketch the graph of \( \phi \) vs \( x \).

(b) Show that the transient solution \( v(x, t) = u(x, t) - \phi(x) \) satisfies the following initial-boundary value problem, in which both PDE and boundary conditions are homogeneous:

\[ v_{tt} = \frac{T_0}{\rho} v_{xx} \quad 0 < x < L, t > 0, \]

\[ v(0, t) = 0, \quad v(L, t) = 0 \quad t > 0, \]

\[ v(x, 0) = f(x) - \phi(x), \quad v_t(x, 0) = g(x) \quad 0 \leq x \leq L. \]

[2] Rederive the wave equation for a vibrating string, when a distributed vertical force \( F(x, t) \) is acting on the string at position \( x \) and time \( t \). (Keep all other assumptions as we did in the class.) Show that the partial differential equation becomes:

\[ u_{tt} = \frac{T_0}{\rho} u_{xx} + \frac{1}{\rho} F(x, t). \]

[3] Verify that the function

\[ u(x, t) = \frac{1}{2} e^{-(x-ct)^2} + e^{-(x+ct)^2} \]

satisfies

\[ u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, t \in \mathbb{R}. \]

Graph \( u(x, t) \) vs \( x \) for times \( t = -5/c, -4/c, \ldots, 5/c. \) (If you have animation software, produce the animation of \( u(x, t) \) vs \( x \), for \( -5/c \leq t \leq 5/c. \))

[4] Verify that the function

\[ u(x, t) = \frac{1}{2} e^{-(x-ct)^2} - e^{-(x+ct)^2} \]

satisfies

\[ u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, t \in \mathbb{R}. \]

Graph \( u(x, t) \) vs \( x \) for times \( t = -5/c, -4/c, \ldots, 5/c. \) (If you have animation software, produce the animation of \( u(x, t) \) vs \( x \), for \( -5/c \leq t \leq 5/c. \))