so \( C_1 = 13/4 \) and \( C_2 = -5/4 \). Substituting these values for \( C_1 \) and \( C_2 \) in the general solution gives the solution of the initial value problem

\[
y(x) = \left( \frac{13}{4} + \frac{1}{2}x \right) e^{-x} - \frac{5}{4} e^{-3x}.
\]

**Summary**

The determination of particular integrals for nonhomogeneous equations is important, and the method of undetermined coefficients that was described in this section is the simplest method by which they can be found. The method is only applicable to nonhomogeneous terms formed by a sum of polynomials, exponentials, trigonometric functions, and certain of their combinations. It depends for its success on recognizing the general form of function that, when substituted into the left of the differential equation, produces terms of the type found in the nonhomogeneous term on the right. The method involves substituting a linear combination of such terms with arbitrary constant multipliers (the undetermined coefficients) into the left of the equation and matching the constants so the terms that result are identical to the terms on the right.

### EXERCISES 6.4

Find the general solutions of the following differential equations:

1. \( y'' - 2y' + 3y = 4 + x + 4e^{3x} \).
2. \( y'' - 4y' + 4y = 2 - \sin 3x \).
3. \( y'' - 2y' + y = 5 + x^2 e^x \).
4. \( y'' + 4y' + 3y = 3x^2 + 2e^{2x} \).
5. \( y'' - 4y' + 4y = \sin x - 2 \cos x \).
6. \( y'' + 5y' = \sin x \).
7. \( y'' - 2y' + 2y = 1 + x + e^{-x} \).
8. \( y'' - 5y' + 6y = 3\sin x + 5x + x^2 \).
9. \( y'' + 2y' = 2 - 4e^x \).
10. \( y'' + 2y = \sin x \).
11. \( y'' - 5y' + 12y = x + e^{3x} + e^{-x} \).
12. \( y'' - 4y' + 5y = 5 + 2e^{2x} \).
13. \( y'' - 2y' + 8y = 3x \cos 4x \).
14. \( y'' - 2y' - 15y = 3 + 2x \sin x \).
15. \( y'' + 9y = 2 \cos 3x + \sin 3x \).
16. \( y'' + 4y' + 5y = 3e^{-x} + 4e^{-2x} \).

17. \( y'' + 3y' + 2y = x^2 + 3e^{-2x} \).
18. \( y'' + y' + 3y - 5y = 4e^{-x} \).
19. \( y'' + 4y' + 5y = e^{-2x} \sin x \).
20. \( y'' + 4y' + 5y = x^2 - e^{-2x} \cos x \).

In Exercises 21 through 28 solve the initial value problems. Where the characteristic equation is of degree 3, at least one root is an integer and can be found by inspection.

21. \( y'' + 6y' + 13y = e^{-3x} \cos x \), with \( y(0) = 2 \), \( y'(0) = 1 \).
22. \( y'' - 4y' + 5y = e^{-x} \cos x \), with \( y(0) = 0 \), \( y'(0) = 2 \).
23. \( y'' + 9y = 7 + 2 \sin 3x - 4 \cos 3x \), with \( y(0) = -1 \), \( y'(0) = 1 \).
24. \( y'' + 4y' + 5y = x + \sin x \), with \( y(0) = -1 \), \( y'(0) = 0 \).
25. \( y'' - 2y' + 5y = 1 + e^{-x} \), with \( y(0) = 2 \), \( y'(0) = 1 \).
26. \( y'' + 4y' + 5y = 2 + e^{-2x} \sin x \), with \( y(0) = 0 \), \( y'(0) = 0 \).
27. \( y'' + y' - 2y = 3 + 2 \cos x \), with \( y(0) = 0 \), \( y'(0) = 1 \), \( y''(0) = -1 \).
28. \( y'' + y' - y' - y = 2 + e^{-x} \), with \( y(0) = 1 \), \( y'(0) = 1 \), \( y''(0) = 0 \).

### 6.5 Cauchy–Euler Equation

**Cauchy–Euler equation**

One of the simplest linear variable coefficient differential equations is the homogeneous second order Cauchy–Euler equation, whose standard form is

\[
x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = 0.
\]  \[\text{(55)}\]
Summary

The Cauchy–Euler equation is the simplest linear variable coefficient equation for which a closed form analytical solution can be found. The solution is obtained by recognizing that it must be of the form \( y(x) = Ax^m \) and finding the permissible values of \( m \).

EXERCISES 6.5

Find the general solutions of the following Cauchy–Euler equations.

1. \( x^2y'' + 3xy' - 3y = 0 \).  
2. \( x^2y'' + 3xy' + 5y = 0 \).  
3. \( x^2y'' + 5xy' + 9y = 0 \).  
4. \( x^2y'' - 3xy' - 5y = 0 \).  
5. \( x^2y'' + 3xy' - 8y = 0 \).  
6. \( x^2y'' + 2xy' + 4y = 0 \).  
7. \( x^2y'' + 6xy' + 4y = 0 \).  
8. \( x^2y'' + xy' + 4y = 0 \).  
9. \( x^2y'' + 4xy' + 4y = 0 \).  
10. \( x^2y'' + 3xy' + 6y = 0 \).

II. With the change of variable \( x = e^t \), we find using the chain rule that

\[
\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right).
\]

Use these results to show that this change of variable transforms a Cauchy–Euler equation into a constant coefficient equation, and solve Exercise 3 by this method.

12. Use the substitution \( y(x) = Ax^m \) to solve the third order Cauchy–Euler equation

\[
x^3y''' - 3x^2y'' + 6xy' - 6y = 0.
\]

13. Use the substitution of Exercise 11 to solve the Cauchy–Euler equation in Exercise 12.

14. Express \( dy/dt \), \( d^2y/dt^2 \), and \( d^3y/dt^3 \) in terms of \( dy/dt \), \( d^2y/dt^2 \), and \( d^3y/dt^3 \) if \( ax + b = e^t \). Use the substitution to show that the general solution of

\[
(2x + 3)^3y''' + 3(2x + 3)y'' - 6y = 0
\]

is

\[
y(x) = C_1(2x + 3) + C_2(2x + 3)^{1/2} + C_3(2x + 3)^{3/2}
\]

for \( x > 0 \).

6.6 Variation of Parameters and the Green’s Function

Variation of Parameters

The method of variation of parameters, perhaps more properly called variation of constants, is a powerful method used to find a particular integral of a linear differential equation once its complementary function is known. In what follows the method will be developed for a general linear second order variable coefficient differential equation, though it is easily extended to include linear variable coefficient differential equations of any order.

As linear constant coefficient equations are a special case of variable coefficient equations, the method enables particular integrals to be found for all linear equations. The method also has the advantage that no special cases arise due to the nonhomogeneous term being included in the complementary function.

Consider the general linear second order differential equation

\[
\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x), \quad (62)
\]

defined on some interval \( \alpha \leq x \leq \beta \) over which \( a(x) \), \( b(x) \), and \( f(x) \) are defined and continuous. Let \( y_1(x) \) and \( y_2(x) \) be two known linearly independent solutions...
EXERCISES 6.6

1. \( y'' + 2y' - 2y = xe^x \)
2. \( y'' + 3y' + 6y = x^2 e^{3x} \)
3. \( y'' + 5y' + 6y = x^2 e^{-2x} \)
4. \( y'' + 4y' + 4y = x \sin x \)
5. \( y'' + 2y' + 2e^x \)
6. \( y'' + 4y' + 5y = e^{-2x} \sin x \)
7. \( y'' + 4y' + 5y = x e^{-2x} \cos x \)

In Exercises 1 through 13 find the general solution.

8. \( y'' - 4y' + 4y = e^{2x} / x \)
9. \( y'' + 16y = x^2 e^x \)
10. \( y'' + 16y = \sec x \)
11. \( y'' + 3y' + 2y = 3 / (1 + e^x) \)
12. \( y'' + y = \tan x \)
13. \( y'' + y = \sec^2 x \)

In Exercises 14 through 18 verify that the functions \( y_1(x) \) and \( y_2(x) \) are linearly independent solutions of homogeneous form of the stated differential equation, and use them to find a particular integral and a general solution of the given equation.

14. \( x^2y'' - 4xy' + 6y = 2x + \ln x \), where \( y_1(x) = x^2 \) and \( y_2(x) = x^{3} \).
15. \( x^2y'' + 3xy' - 3y = \sqrt{x} \), where \( y_1(x) = x \) and \( y_2(x) = x^{3} \).
16. \( x^2y'' + 3xy' - 8y = 2 \ln x \), where \( y_1(x) = x^2 \) and \( y_2(x) = x^{-4} \).
17. \( (1 - x^2)y'' - xy' + 4y = x \), where \( y_1(x) = 2x^2 - 1 \) and \( y_2(x) = x(x^2 - 1)^{1/2} \).
18. \( (1 - x^2)y'' - 2y' + 1 \), where \( y_1(x) = 1 \) and \( y_2(x) = x + 2 \ln (x - 1) \).

In Exercises 19 through 22 use result (76) to solve the stated initial value problem.

19. \( x^2y'' - 3xy' + 3y = 2x^3 \ln x \), with \( y(1) = 0 \) and \( y'(1) = 0 \).
20. \( y'' + 3y' + 6y = x e^{-2x} \), with \( y(1) = 0 \) and \( y'(1) = 0 \).
21. \( y'' + y = 2 \sec^2 x \), with \( y(0) = 0 \) and \( y'(0) = 0 \).
22. \( y'' + 4y' + 5y = x \), with \( y(0) = 0 \) and \( y'(0) = 0 \).

In Exercises 23 through 26 find the Green's function for the given differential equation, subject to the associated homogeneous boundary conditions.

23. \( y'' = f(x) \), with \( y(0) = 0 \) and \( y'(0) = 0 \).
24. \( y'' = f(x) \), with \( y(0) = 0 \) and \( y'(0) = 0 \).
25. \( y'' + \lambda^2 y = f(x) \), with \( y(0) = 0 \) and \( y'(0) = 0 \).
26. \( y'' + \lambda^2 y = f(x) \), with \( y(0) = 0 \) and \( y'(1) = 0 \).

In Exercises 27 through 30 solve the given boundary value problem by means of a suitable Green's function.

27. \( x^2y'' + xy' - y = x^2 e^{-x} \), with \( y(1) = 0 \) and \( y(2) = 0 \).
28. \( x^2y'' + 2xy' - 2y = x^2 \), with \( y(1) = 0 \) and \( y(2) = 0 \).
29. \( x^2y'' - 3xy' + 3y = x^2 \ln x \), with \( y(1) = 0 \) and \( y(2) = 0 \).
30. \( x^2y'' - 3xy' = x^2 \), with \( y(1) = 0 \) and \( y(2) = 0 \).

6.7 Finding a Second Linearly Independent Solution from a Known Solution: The Reduction of Order Method

In working with homogeneous linear second order variable coefficient equations, it can happen that one solution \( y_1(x) \) is known and it is necessary to find a second linearly independent solution \( y_2(x) \). The method we now describe, called the reduction of order method, involves seeking a second solution of the form

\[
y_2(x) = u(x)y_1(x),
\]  

where the function \( u(x) \) is to be determined. Provided \( u(x) \) is not constant, the solutions \( y_1(x) \) and \( y_2(x) \) will be linearly independent, because \( y_1(x) \) and \( y_2(x) \) will not be proportional.

The method will be developed using the homogeneous second order variable coefficient equation in the standard form

\[
\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0.
\]
Summary

It is often the case that one solution of a linear second order variable-coefficient homogeneous variable-coefficient equation can be found, often by inspection, though a second linearly independent solution cannot be found in similar fashion. This section showed how a known solution can be used to find a second linearly independent solution. It was shown that the second linearly independent solution of the original second order equation is determined in terms of a first order equation, and it is this feature that has caused the approach to be called the reduction of order method.

EXERCISES 6.7

In the following exercises, verify that \( y_1(x) \) is a solution of the given differential equation and use it to find a second linearly independent solution.

1. \( y'' - 5y' - 14y = 0 \) with \( y_1(x) = e^{7x} \).
2. \( y'' + 4y = 0 \) with \( y_1(x) = \sin 2x \).
3. \( y'' + 4y' + 5y = 0 \) with \( y_1(x) = e^{-x} \cos x \).
4. \( x^2y'' + 3xy' + y = 0 \) with \( y_1(x) = 1/x \).
5. \( x^2y'' - xy' + y = 0 \) with \( y_1(x) = x \).
6. \( x^2y'' + xy' - y = 0 \), with \( y_1(x) = \cos \ln x \).
7. \( xy'' + 2y' + xy = 0 \), with \( y_1(x) = \sin x/x \).
8. \( x^2y'' + xy' + (x^2 - 1/4)y = 0 \), with \( y_1(x) = \sin x/\sqrt{x} \).
9. \( x^2(\ln x - 1)y'' - xy' + y = 0 \), with \( y_1(x) = x \).
10. \((1 - x \cot x)y'' - xy' + y = 0 \), with \( y_1(x) = x \).

(Hint: When finding \( \int -a(x)dx \), make the substitution \( u = \sin x - x \cos x \), and in the final integral make the substitution \( v = \sin x/x \).)

6.8 Reduction to the Standard Form \( u'' + f(x)u = 0 \)

When studying the properties of second order variable-coefficient equations it is sometimes advantageous to reduce the equation

\[ y'' + a(x)y' + b(x)y = 0 \]  

to the standard form for a second order equation

\[ u'' + f(x)u = 0, \]  

from which the first derivative term \( u' \) is missing. This reduction has many uses, one of which occurs in Section 8.6 when we derive the analytical form of Bessel functions of fractional order.

To accomplish the reduction we seek a solution of (94) of the form

\[ y(x) = u(x)v(x), \]

and then try to choose \( v(x) \) so the first derivative term in \( u \) vanishes. Differentiation of \( y = uv \) gives \( y' = u'v + uv' \) and \( y'' = u''v + 2u'v' + uv'' \), so substitution in equation (94) gives

\[ u''v + (2v' + av)u' + (v'' + av' + bv)u = 0. \]

This result shows that the first derivative term \( u' \) will vanish if \( v(x) \) is such that

\[ 2v' + av = 0. \]