and so $\det Q = \pm 1$. If $\det Q = +1$, rotation. If $\det Q = -1$, rotation plus reflection in general.

Result (ii) follows from the fact that if $Q_1$ and $Q_2$ are two $n \times n$ orthogonal matrices, then $(Q_1Q_2)^T Q_1 Q_2 = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$, and the result is established.

The proof of Result (iii) is similar to the proof of (i) in Theorem 4.3. If $Q$ is taking the complex conjugate of $Q = \lambda x$, giving $Q \bar{x} = \bar{\lambda x}$, so taking the transpose of this we find that $\bar{x}^T \bar{Q} = \bar{\lambda x}^T$. Forming the product of these two results $\bar{x}^T Q^T \bar{x} = \bar{\lambda x}^T \bar{x}$, but $Q^T Q = I$, so $\bar{x}^T \bar{x} = \lambda \bar{x}^T \bar{x}$, showing that $\lambda \bar{x} = 1$. Result follows from this last result because $\lambda \bar{x} = |\lambda|^2 = 1$.

Finally, Result (iv) follows from the definition of an orthogonal matrix, because $QQ^T = I$, and if $u_i$ is the $i$th row of $Q$ and $v_j$ is the $j$th column of $Q^T$ (the $j$th column of $Q$), then $u_i v_j = 0$ for $i \neq j$, and $u_i v_j = 1$ for $i = j$, confirming that the vectors form an orthonormal set.

Summary

After definition of the eigenvalues of an $n \times n$ matrix $A$ in terms of its characteristic polynomial, the associated eigenvectors were defined. An eigenvalue that is repeated $r$ times was said to have the algebraic multiplicity $r$, and the set of all eigenvalues of $A$ is called the spectrum of $A$. The spectral radius of $A$ was defined in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the number $R = \max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$, and the linear independence of the set of all eigenvectors was established. The most frequently used method of normalizing eigenvectors was introduced, and examples were worked showing how to determine eigenvectors once the eigenvalues are known.

A simple test was given to check the sum of all eigenvalues, and the Gershgorin circle theorem was proved that determines a region inside which all eigenvalues lie, though the region determined in this manner is far from optimal. Inner product, norm, and systems of orthogonal and orthonormal vectors were introduced, and the important eigenvalue and eigenvector properties of symmetric matrices and orthogonal matrices were derived.

EXERCISES 4.1

In Exercises 1 through 8, find the characteristic polynomial of the given matrix.

1. \[
\begin{bmatrix}
2 & 1 & 3 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
-1 & 0 & 1 \\
3 & 2 & 1 \\
1 & 2 & 3
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
3 & -2 & 2 \\
6 & -4 & 6 \\
2 & -1 & 3
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
0 & 1 & -2 \\
2 & -1 & 2 \\
2 & -2 & 4
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
2 & 1 & 3 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
4 & 1 & -1 \\
1 & 0 & 2 \\
-1 & 1 & 2
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
3 & -1 & 1 \\
4 & -1 & 4 \\
2 & -1 & 4
\end{bmatrix}
\]

15. \[
\begin{bmatrix}
-5 & 8 & 1 \\
-3 & 6 & 1 \\
6 & -8 & 0
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 & 0 & 2 \\
-1 & 1 & -1 \\
0 & 2 & 1
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 \\
1 & -3 & 3 & 0 \\
-1 & 2 & -1 & -1
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
-3 & 2 & 2 \\
4 & -1 & 4 \\
8 & -4 & 7
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
-1 & 0 & -2 \\
-1 & 2 & -1 \\
4 & 0 & 5
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
3 & 1 & 1 \\
-2 & 2 & 1 \\
1 & -1 & 2
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
-1 & 1 & 0 & 1 \\
-1 & 2 & -1 & 1 \\
5 & -3 & 4 & -5 \\
3 & -2 & 3 & -3
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
3 & -2 & 4 \\
-4 & 5 & -4 \\
-4 & 4 & -5
\end{bmatrix}
\]

17. \[
\begin{bmatrix}
-1 & 0 & 2 \\
-1 & 2 & 0 \\
-1 & 0 & 2
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
-5 & 4 & -1 \\
-3 & 2 & -1 \\
6 & -4 & 2
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
6 & 0 & 4 \\
3 & 1 & 3 \\
-8 & 0 & -6
\end{bmatrix}
\]

In Exercises 9 through 24 find the eigenvalues and eigenvectors of the given matrix.


22. \[
\begin{bmatrix}
3 & 0 & 1 \\
2 & 1 & 1 \\
-2 & 0 & 0
\end{bmatrix}
\]

23. \[
\begin{bmatrix}
-1 & -1 & 1 & 0 \\
1 & 1 & 1 & -1 \\
1 & 3 & -1 & -1 \\
-2 & 2 & -2 & 1
\end{bmatrix}
\]

24. \[
\begin{bmatrix}
0 & 1 & 0 & -1 \\
1 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 \\
-3 & 3 & 0 & 2
\end{bmatrix}
\]

Prove that the eigenvalues of upper and lower triangular matrices are equal to the elements on the leading diagonal. Show by example that, unlike the case of diagonal matrices, an eigenvalue of an upper or lower triangular matrix with algebraic multiplicity \( r \) has fewer than \( r \) eigenvectors.

Apply the Gerschgorin circle theorem to one or more of the matrices in Exercises 9 through 24 to verify that the eigenvalues lie within or on the circles determined by the theorem.

It can be shown that all the zeros of the polynomial

\[ p_n(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_n \lambda^n, \quad a_n \neq 0, \]

lie in the circle

\[ |\lambda| < 1 + \max \left| \frac{a_k}{a_n} \right|, \quad k = 0, 1, 2, \ldots, n - 1. \]

Verify this result by applying it to one or more of the characteristic equations associated with the matrices in Exercises 9 through 24.

The Routh–Hurwitz stability criterion

Let the real polynomial \( p_n(\lambda) \) be given by

\[ p_n(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n, \]

and form the determinants

\[ \Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \end{vmatrix}, \quad \ldots, \]

\[ \Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-2} \\ 1 & a_2 & a_4 & \cdots & a_{2n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} \]

with \( a_k = 0 \) for \( k > n \).

Then, \( \Delta_k > 0 \) for \( k = 1, 2, \ldots, n \), if and only if every zero of \( p_n(\lambda) \) has a negative real part.

28. (a) Numerical computation shows that the matrix

\[ A = \begin{bmatrix} -2 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 4 & 2 \end{bmatrix} \]

has the eigenvalues \( 5.7238, -1.3619 + 1.9328i, \) and \(-1.3619 - 1.9328i \). Apply the Routh–Hurwitz stability criterion to confirm that not every zero of the characteristic polynomial has a negative real part.

(b) Numerical computation shows that the matrix

\[ A = \begin{bmatrix} -2 & -2 & -2 \\ 3 & -1 & 0 \\ -4 & 0 & -3 \end{bmatrix} \]

has the eigenvalues \(-5.4873, -0.2563 - 1.4564i, \) and \(-0.2563 + 1.4564i \). Apply the Routh–Hurwitz stability criterion to confirm that every zero of the characteristic polynomial has a negative real part.

An \( n \times n \) matrix \( A \) is said to be similar to an \( n \times n \) matrix \( B \) if there exists a nonsingular \( n \times n \) matrix \( M \) such that \( B = M^{-1}AM \). The relationship between \( A \) and \( B \) is said to constitute a similarity transformation between the two matrices.

29. If \( A \) and \( B \) are similar, show that \( \det A = \det B \), and by substituting \( B = M^{-1}AM \) in \( \det B \) and expanding the result, show that similar matrices have the same eigenvalues.

30. Verify the result of Exercise 29 by direct calculation by using

\[ A = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 0 & -1 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \]

to show that both \( A \) and \( B \) have the eigenvalues \(-1, 2, \) and \( 3 \).

31. Let the \( n \times n \) elementary matrix \( E \) be obtained from the unit matrix \( I \) by interchanging its \( i \)th and \( j \)th rows (columns). By considering the product \( EQ \), where \( Q \) is an \( n \times n \) orthogonal matrix, prove that an orthogonal matrix remains orthogonal when its rows (columns) are interchanged.
4.3 Special Matrices with Complex Elements

In the previous section it was seen that one way in which matrices with complex elements can occur is when the eigenvectors of an arbitrary \(n \times n\) matrix are used to construct a diagonalizing matrix. This is not the only reason for considering \(n \times n\) matrices with complex elements, because the following three special types of matrices arise naturally in applications of mathematics to physics and engineering, and elsewhere.

Hermitian, skew-Hermitian, and unitary matrices

Let \(A = [a_{ij}]\) be an \(n \times n\) matrix with possibly complex elements. Then:

- \(A\) is called an Hermitian matrix if \(A^\dagger = A\), so that \(a_{kj} = a_{jk}\);
- \(A\) is called a skew-Hermitian matrix if \(A^\dagger = -A\), so that \(a_{kj} = -a_{jk}\);
- \(U\) is called a unitary matrix if \(U^\dagger = U^{-1}\).

The basic properties of these three types of matrices follow almost directly from their definitions.

Basic Properties of Hermitian, Skew-Hermitian, and Unitary Matrices

1. The elements on the leading diagonal of an Hermitian matrix are real, because \(\bar{a}_{ii} = a_{ii}\), and this is only possible if \(a_{ii}\) is real.

2. The elements on the leading diagonal of a skew-Hermitian matrix are either purely imaginary or 0. This follows from the fact that \(\bar{a}_{ii} = -a_{ii}\), so the real part of \(a_{ii}\) must equal its negative, and this is only possible if \(a_{ii}\) is purely imaginary or 0.
The eigenvectors of a unitary matrix. The rows and columns of a unitary matrix each form a unitary system of vectors.

Proof. By definition the $n \times n$ matrix $U$ is unitary if $U^T = U^{-1}$, so that $U^T U = I$. The element in the $i$th row and $j$th column of $U$ is the inner product $x_i \cdot x_j = \bar{x}_i^T x_j$, where $x_i$ and $x_j$ are the $i$th and $j$th columns of $U$. Consequently,

$$\bar{x}_i^T x_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

showing that the columns of $U$ form a unitary system. The rows also form a unitary system, because taking the transpose of $U^T U$ we find that $(U^T U)^T = U^T U = I^T = I$. 

Summary

Matrices with complex elements arise in a variety of different applications, and from among these matrices, the most important are Hermitian, skew-Hermitian, and unitary matrices. Hermitian and skew-Hermitian matrices are the complex analogues of real symmetric and skew-symmetric matrices, respectively, and unitary matrices are the complex analogue of real orthogonal matrices. This section derived and illustrated by means of examples the most important properties of these matrices, and then introduced the inner product and norm of matrices with complex elements.

EXERCISES 4.3

1. In Exercises 1 through 4 write the given matrix as the sum of a Hermitian and a skew-Hermitian matrix.

   $1. \begin{bmatrix} 1+i & 3+i & 3+2i \\ -1+3i & 2 & 4+i \\ -3-2i & 2+3i & 4+2i \end{bmatrix}$

   $2. \begin{bmatrix} 0 & 3+i & 1+2i \\ 1-3i & 1+i & 2 \\ 1+4i & -2i & 3 \end{bmatrix}$

   $3. \begin{bmatrix} 4-2i & 1+i & 2+2i \\ -1-3i & 1+2i & 4 \\ 0 & 2 & 0 \end{bmatrix}$

   $4. \begin{bmatrix} 3+i & 4-i & 5+2i \\ 2+i & 1+2i & 2 \\ -1 & 2i & 4-i \end{bmatrix}$

2. In Exercises 5 through 8 find the eigenvalues of the Hermitian matrices and hence confirm the result of Theorem 4.3(a) that they are real.

   $5. \begin{bmatrix} 1 & 2-i \\ 2+i & 2 \end{bmatrix}$

   $6. \begin{bmatrix} 2 & 2+2i \\ 1-2i & 3 \end{bmatrix}$

   $7. \begin{bmatrix} 3 & 2-3i \\ 2+3i & 1 \end{bmatrix}$

   $8. \begin{bmatrix} -4 & 2-2i \\ 2+2i & 3 \end{bmatrix}$

3. In Exercises 9 through 12 find the eigenvalues of the skew-Hermitian matrices and hence confirm the result of Theorem 4.3(b) that they are purely imaginary.

   $9. \begin{bmatrix} i & 3+i \\ -3+i & 2i \end{bmatrix}$

   $10. \begin{bmatrix} 3i & 2-i \\ -2-i & 0 \end{bmatrix}$

   $11. \begin{bmatrix} 0 & 3+2i \\ -3+2i & 0 \end{bmatrix}$

   $12. \begin{bmatrix} 4i & 2+3i \\ -2+3i & i \end{bmatrix}$

   $13. \text{Show the following matrix is unitary:}$

   $\begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

   $14. \begin{bmatrix} (i-1)/\sqrt{2} & (1-i)/\sqrt{2} \\ (i-1)/\sqrt{2} & (i-1)/\sqrt{2} \end{bmatrix}$

   $15. \begin{bmatrix} (1+i)/\sqrt{2} & -(1+i)/\sqrt{2} \\ (1+i)/\sqrt{2} & (1+i)/\sqrt{2} \end{bmatrix}$

4. In Exercises 14 and 15 show the matrices are unitary, find their eigenvalues and eigenvectors, and confirm that the eigenvalues all lie on the unit circle.
(c) $P(x)$ is indefinite if at least one eigenvalue is opposite in sign to the other. In this case, depending on the choice of $x$, $P(x)$ may be either positive or negative.

**EXAMPLE 4.21**

The following are examples of different types of standard forms associated with a $3 \times 3$ matrix:

1. $x_1^2 + 2x_2^2 + 5x_3^2$ is positive definite;
2. $-(2x_1^2 + 7x_2^2 + 4x_3^2)$ is negative definite;
3. $4x_1^2 + 3x_3^2$ is positive semidefinite (it is positive, but irrespective of the value of $x_2 \neq 0$ it can vanish when $x \neq 0$);
4. $-(2x_1^2 + x_2^2)$ is negative semidefinite (it is negative, but irrespective of the value of $x_2 \neq 0$ it can vanish when $x \neq 0$);
5. $3x_1^2 - 2x_2^2 + x_3^2$ is indefinite (it can be positive or negative).

Further, and more detailed, information relating to the material in Sections 4.3 and 4.4 is to be found in the appropriate chapters of references [2.1] and [2.5] to [2.7].

**Summary**

A real quadratic form involving the $n$ real variables $x_1, x_2, \ldots, x_n$ is a homogeneous polynomial of degree two in these variables. Such forms arise in many different ways, one of which occurs in optimization problems where a reduction to a sum of squares simplifies the task of finding an optimum least squares solution. In this section it was shown that a real quadratic form arises when studying the mechanics of solid bodies, since the set of principal axes $O(x_1, x_2, x_3)$ is used to simplify the description of the body in terms of its inertia about each of the three axes. The reduction of a quadratic form to a sum of squares both simplifies the analysis of its properties and also enables it to be classified as being positive or negative definite, semipositive or seminegative, or of indefinite type, one of which classifications have important implications in applications.

**EXERCISES 4.4**

In Exercises 1 through 6 find the symmetric matrix $A$ that is associated with the given quadratic form.

1. $x_1^2 + 4x_1x_3 - 6x_2x_3 + 3x_3^2 - 2x_2^2$.
2. $5x_1^2 - 2x_2^2 - 5x_3^2 - 4x_1x_3$.
3. $-2x_1^2 + 3x_2^2 - 2x_1x_3 + 4x_2x_3$.
4. $x_1^2 + 3x_2^2 - 2x_1x_2 + 4x_2x_4 - 2x_3x_4 + x_3^2 + 6x_2^2$.
5. $3x_1^2 - 4x_1x_2 - 6x_2x_3 - 2x_3x_4 + 2x_2^2 + 8x_3^2$.
6. $x_1^2 + x_2^2 + 4x_2^2 - 3x_2x_3 - x_1x_3 + 4x_3x_4 + 2x_3x_4$.

In Exercises 7 through 10 write down the quadratic form associated with the given matrix.

7. $[2 4 4 1 2 1 4 2 -1 2 0 1 2 3]$.
8. $[1 -3 2 1 -3 2 0 0 2 2 0 -3 0 1 2 0 4]$.

9. $\begin{bmatrix} 0 & 2 & -4 \\ 2 & 3 & 1 \\ -4 & 1 & 2 \\ 2 & 0 & 1 \\ 7 \end{bmatrix}$.
10. $\begin{bmatrix} 1 & -2 & 4 & 3 \\ -2 & 3 & 1 & 2 \\ 4 & 1 & 5 & 0 \\ 3 & 2 & 0 & 3 \end{bmatrix}$.

In Exercises 11 through 18 use hand computation to reduce the quadratic form to its standard form, and use the result to classify it. Confirm the reduction by using computer algebra.

11. $(5/2)x_1^2 + x_1x_3 + x_2^2 + (5/2)x_3^2$.
12. $4x_1^2 + x_2^2 + 2x_1x_3 + x_2^2$.
13. $4x_1^2 + 4x_2^2 + 2x_1x_3 + 4x_3^2$.
14. $(3/2)x_1^2 - x_1x_3 + x_2^2 + (3/2)x_3^2$.
15. $(3/2)x_1^2 + x_1x_3 - x_2^2 + (3/2)x_3^2$.
16. $(1/2)x_1^2 + x_1x_3 + 2x_2^2 + (1/2)x_3^2$.
17. $2x_1^2 + x_2^2 - 4x_1x_3 + x_3^2$.
18. $2x_1^2 + 2x_2^2 + 2x_1x_3 + 2x_3^2$.

In Exercises 19 through 24 use computer algebra to reduce the quadratic form on the left to its standard form. Use the result to identify the conic section described by the equation as a circle, an ellipse, or a hyperbola.

19. $3x_1^2 - 6x_1x_2 + 9x_2^2 = 3$.
20. $8x_1^2 - x_2^2 + 20x_1x_2 = 12$. 
25. $x_1^2 + 8x_1x_2 + x_2^2 + 3x_1 - 2x_2$.
26. $x_1^2 - 8x_1x_2 + x_2^2 + 2x_1 + 3x_2$.
27. $-2x_1^2 + 4x_1x_2 + x_2^2 + 4x_1 - x_2$.
28. $(8/5)x_1^2 - (8/5)x_1x_2 + (2/5)x_2^2 + 2x_1 + 4x_2$.
29. $(35/17)x_1^2 + (8/17)x_1x_2 + (50/17)x_2^2 + 4x_2$.
30. By using the definitions of a symmetric and a skew-symmetric matrix, generalize the definition of a quadratic form by proving that the quadratic form associated with any real $n \times n$ matrix $A$ can be written $x^T B x$, where $B$ is the symmetric part of $A$.

### 4.5 The Matrix Exponential

It is shown in Chapter 6 that the matrix exponential can be used when solving systems of linear first order differential equations. As this approach uses matrix diagonalization when determining what is called the matrix exponential involving an arbitrary $n \times n$ diagonalizable matrix, it is convenient to introduce the matrix exponential in this chapter.

To motivate what is to follow, we notice that the first order homogeneous linear differential equation

$$dx/dt = ax \quad (a = \text{constant})$$

has the general solution

$$x = ce^{at}$$

where $c$ is an arbitrary constant.

Let us now consider the system of $n$ linear first order homogeneous differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{align*}
\]

(27)

Setting

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

allows the system of differential equations in (27) to be written in the matrix form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

(28)

where $d\mathbf{x}/dt = [dx_1/dt, dx_2/dt, \ldots, dx_n/dt]^T$ (see Section 3.2(d)).
so that

\[
\int e^{At} dt = \begin{bmatrix}
e^{t^2/3} + c_1 & 0 & 0 \\
0 & -e^{-2t^2/2} + c_2 & 0 \\
0 & 0 & e^{t^2/4} + c_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
e^{t^2/3} & 0 & 0 \\
0 & -e^{-2t^2/2} & 0 \\
0 & 0 & e^{t^2/4}
\end{bmatrix} + \begin{bmatrix}
c_1 & 0 & 0 \\
0 & c_2 & 0 \\
0 & 0 & c_3
\end{bmatrix}
\]

where \(c_1, c_2,\) and \(c_3\) are arbitrary constants.

Applications of the matrix exponential to ordinary differential equations are to be found in reference [3.15].

**Summary**

The matrix exponential \(e^{At}\) arises as the natural extension of the exponential function when solving a system of linear first order constant coefficient differential equations in the matrix form \(dx/dt = Ax\). This section has described how \(e^{At}\) can be calculated in simple cases and shown that \(e^{A}e^{B} = e^{A+B}\) if, and only if, \(AB = BA\). A different way of finding \(e^{At}\) using the Laplace transform is given later in Section 7.3(b).

**EXERCISES 4.5**

1. Given that

\[
A = \begin{bmatrix}0 & 3 & 1 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

show that it is nilpotent and find the smallest power for which \(A^n = 0\).

2. Given that

\[
A = \begin{bmatrix}0 & 1 & 2 & 2 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

find \(e^{At}\).

3. Given that

\[
A = \begin{bmatrix}0 & 2 \\
0 & 0,\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}0 & 0 \\
3 & 0,\end{bmatrix}
\]

show that \(A\) and \(B\) do not commute, and by finding \(e^{At}, e^{Bt}\), and \(e^{(A+B)t}\), verify that \(e^{At}e^{Bt} \neq e^{(A+B)t}\).

4. \(A = \begin{bmatrix}0 & 1 \\
1 & 0,\end{bmatrix}\)

7. \(A = \begin{bmatrix}-2 & 2 \\
2 & 1,\end{bmatrix}\)

5. \(A = \begin{bmatrix}m & 0 \\
0 & n,\end{bmatrix}\)

8. \(A = \begin{bmatrix}3 & -2 & 2 \\
6 & -4 & 6 \\
2 & -1 & 3,\end{bmatrix}\)

\(\bigcirc\) 6. \(A = \begin{bmatrix}0 & -c \\
c & 0,\end{bmatrix}\)

9. \(A = \begin{bmatrix}0 & 1 & -2 \\
2 & -1 & 2 \\
2 & -2 & 4,\end{bmatrix}\)

10. By considering the definition of \(e^{At}\), show, provided the square matrices \(A\) and \(B\) commute, that

\[Ae^{Bt} = e^{Bt}A.\]

\(\bigcirc\) 11. By considering the definition of \(e^{At}\), show that

\[\int e^{-At}dt = -A^{-1}e^{-At} + C = e^{-At}A^{-1} + C,\]

where \(C\) is an arbitrary constant matrix that is conformable for addition with \(A\).

12. Show that if the square matrices \(A\) and \(B\) commute, then the binomial theorem takes the form

\[(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}.\]