Summary

Some general properties of analytic functions were derived, one of which was the fundamental theorem of algebra that asserts every polynomial of degree \( n \) has precisely \( n \) zeros, though these need not all be distinct. The maximum/minimum modulus theorem for analytic functions was also proved, showing that the maximum and minimum values of the modulus of a nonconstant analytic function defined in a domain \( D \) must occur on the boundary of \( D \). A corresponding theorem for harmonic functions was also proved.

EXERCISES 14.4

5. Let \( P_n(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \) be a complex polynomial, and \( \Gamma \) be a positively oriented circle with its center at the origin. Show that

\[
\frac{1}{2\pi i} \sum_{k=0}^{n} \int_{\Gamma} \frac{P_n(z)}{z^{k+1}} \, dz = \sum_{k=0}^{n} a_k.
\]

6. Let \( f(z) \) be analytic inside and on the circle \( \Gamma \) defined by \( |z| = R \), and let \( z_0 = re^{i\theta} \), with \( 0 < r < R \), be a point inside the circle. Show that the point \( Z = \frac{z_0}{r} \) lies outside the circle \( \Gamma \), so that

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-z_0} \, dz = 0.
\]

By differentiating this expression and the expression for \( f(z_0) \) determined by the Cauchy integral formula, show that

\[
f(re^{i\theta}) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z} \left( \frac{z_0}{r} \right) \frac{dz}{z-z_0} f(z) \, dz.
\]

7. By setting \( z_0 = re^{i\phi} \) and \( z = Re^{i\psi} \) in the result of Exercise 2, show that

\[
f(re^{i\phi}) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(R^2 - r^2)}{R^2 - 2rR\cos(\psi - \theta) + r^2} f(Re^{i\psi}) \, d\psi.
\]

Write \( f(re^{i\phi}) = u(r, \theta) + iv(r, \theta) \) in the preceding result and derive the Poisson integral formula for a disc,

\[
u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(R^2 - r^2)u(R, \psi)}{R^2 - 2rR\cos(\psi - \theta) + r^2} \, d\psi.
\]

This formula determines the value of the harmonic function \( u = \text{Re}(f(z)) \) at any point \((r, \theta)\) inside the disc in terms of the prescribed values of \( u \) on the boundary \( \Gamma \) of the disc. The specification of \( u \) on the boundary of a domain in which \( u \) is harmonic constitutes what is called a Dirichlet problem for Laplace's equation. This formula determines, for example, the steady state electrostatic potential in a long cavity with a circular cross-section of radius \( R \), on the walls of which the potential \( u(R, \psi) = f(\psi) \). As the steady state two-dimensional temperature distribution in a long metal rod of circular cross-section of radius \( R \) is also a solution of Laplace's equation, this same formula determines the temperature distribution in the rod when its surface is at a temperature \( u(R, \psi) = f(\psi) \).
4. By setting $u(R, \psi) = M$ in the Poisson integral formula for a disc given in Exercise 3, and using the result

$$\int_0^{2\pi} \frac{dt}{1 + a \cos t} = \frac{2\pi}{\sqrt{1 - a^2}} \quad \text{for} \quad a^2 < 1$$

that can be established by the method of Example 14.13, show that when $u(R, \psi) = M$ (constant) on the boundary of the disc, it must follow that $u(r, \theta) \equiv M$ throughout the disc.

5. Let domain $D$ be the interior of the positively oriented contour $C$ comprising the semicircle $C_R$ of radius $R$ in the upper half-plane with its center at the origin, and the segment of the real axis from $-R$ to $R$. If $z_0$ is an interior point of $D$, explain why

$$f(z_0) = \frac{1}{2\pi I} \int_C \frac{f(z)}{z - z_0} \, dz$$

Set $z_0 = x_0 + iy_0$ and difference these results to show that

$$f(z_0) = \frac{y_0}{\pi} \int_{x_R}^{x_L} \frac{f(x)}{(x - x_0)^2 + y_0^2} \, dx + \frac{x_0}{\pi} \int_{y_R}^{y_L} \frac{f(z_0)}{(z_0 - z)^2} \, dz.$$

6. Using the notation of Exercise 5, and writing $z = z_0 + (z - z_0)$ and $z = z_0 + (z - z_0)$, show that

$$R + |z_0|^2 \leq |z - z_0| \leq |z - z_0|.$$ 

Deduce from this that if $|f(z)| \leq K$ in the upper half plane, then

$$\int_{x_R}^{x_L} \frac{f(x)}{(x - x_0)^2 + y_0^2} \, dx \leq \frac{K |y_0| R}{|z_0|^2}.$$ 

By taking the limit of the result of Exercise 5 as $R \to \infty$ and using the result from this exercise, deduce that

$$f(z_0) = \frac{y_0}{\pi} \int_{x_R}^{x_L} \frac{f(x)}{(x - x_0)^2 + y_0^2} \, dx.$$ 

Then, by setting $f(z) = u(x, y) + iv(x, y)$ and equating the real parts of the equation, show that

$$u(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{u(x, 0)}{(x - x_0)^2 + y_0^2} \, dx.$$ 

This result is the Poisson integral formula for a half-plane, and it determines the harmonic function $u(x_0, y_0)$ at points $(x_0, y_0)$ in the upper half-plane in terms of a prescribed function $u(x, 0)$ on the real axis. The function $u(x, 0)$ is called a Dirichlet boundary condition for the two-dimensional boundary value problem for Laplace's equation. This formula can be used to determine the steady state temperature distribution $u(x, y)$ in a thermally conducting half-plane when the temperature on the plane bounding surface is $T(x, 0) = u(x, 0)$, with $u(x)$ a given function. A similar interpretation applies when the formula is used to determine the steady state electrostatic potential $u(x, y)$ in a half-space when the potential on the plane bounding surface is $u(x, 0) = T(x)$.

7. Let $P_n(z)$ be the complex polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$, and $n \geq 1$. Justify the assertion in the proof of the fundamental theorem of algebra that if $Q(z) = 1/P_n(z)$, then $\lim_{|z| \to \infty} |Q(z)| = 0$.

8. Given that $z = 1 + 2i$ is a root of the polynomial $z^2 + 2z^2 + 12z + 56 = 0$ with real coefficients, use the deflation method described in the proof of the fundamental theorem of algebra to find the remaining roots.

9. Verify the maximum/minimum principle for the function $f(z) = e^z$ in the domain $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, and place bounds on $\Re z$ inside the given domain.

10. Verify the maximum/minimum principle for the function $f(z) = \cos z$ in the domain $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and place bounds on $\Re z$ inside the given domain.

In Exercises 11 through 14 place bounds on the function $u(x, y)$ inside the given domain.

11. $u(x, y) = x + 2x^2 - 2y^2$ in the domain $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

12. $u(x, y) = e^y (\cos y + x \sin y)$ in the domain $0 \leq x \leq 1$, $-\pi/2 \leq y \leq \pi/2$.

13. $u(x, y) = e^y (\cos y + x \sin y)$ in the domain $0 \leq x \leq 1$, $-\pi/2 \leq y \leq \pi/2$.

14. $u(x, y) = e^y (\cos^2 y + x \sin^2 y)$ in the domain $-1 \leq x \leq 1$, $0 \leq y \leq \pi/2$.

15. Show by Rouche's theorem that $P(z) = 3z^3 + 5z + 1$ has one zero in the disc $|z| \leq 1$ and three zeros in the annulus $1 \leq |z| \leq 2$.

16. Use Rouche's theorem to find the number of zeros of $P(z) = z^3 - 4z + 1$ contained in (a) the disc $|z| \leq 1$, (b) the annulus $1 \leq |z| \leq 2$, and (c) the annulus $2 \leq |z| \leq 3$.

17. Use the geometrical interpretation of the restricted argument principle to show that $f(z) = z^3 + 3z + 1$ has no zeros in $|z| \leq 1$, one zero in $|z| = 1$, and two zeros in $|z| > 1$.

18. Given that $f(z) = z \exp(z) - 2z^2 + iz + 3i$, use the geometrical interpretation of the restricted argument principle to determine the number of zeros of $f(z)$ in (a) the disc $|z| \leq 1$, (b) the annulus $1 \leq |z| \leq 2$, (c) the annulus $2 \leq |z| \leq 3$, and (d) the annulus $3 \leq |z| \leq 4$.

Project 14.2

The Numerical Solution of Analytic Functions

This project involves the development of various numerical methods for finding zeros of analytic functions. The methods are independent, and the student is encouraged to work on as many as possible.

1. Let $z = x + iy$ be a complex number. If $z$ is a root of a polynomial $P(z)$, then $P(z) = 0$. Thus, a root is a solution of the equation $P(z) = 0$, and the equations $P(z) = 0$ and $P(z) = 0$ are equivalent. Use this fact to develop a new method for finding zeros of analytic functions.

2. Consider the function $P(z) = z^4 + 1$. Find all of the zeros of $P(z)$.

3. Find the zeros of $P(z) = z^5 + 2z^3 + 3z^2 + 4z + 5$.

4. Find the zeros of $P(z) = z^6 + 3z^5 + 5z^4 + 7z^3 + 9z^2 + 11z + 13$.

5. Find the zeros of $P(z) = z^7 + 4z^6 + 9z^5 + 16z^4 + 25z^3 + 36z^2 + 49z + 64$.

6. Find the zeros of $P(z) = z^8 + 6z^7 + 15z^6 + 24z^5 + 30z^4 + 36z^3 + 36z^2 + 36z + 36$. 

7. Find the zeros of $P(z) = z^9 + 8z^8 + 24z^7 + 50z^6 + 85z^5 + 120z^4 + 150z^3 + 150z^2 + 120z + 80$.