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Nonequilibrium steady states for a class of particle systems

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Received 7 September 2013, revised 21 October 2013

Accepted for publication 15 December 2013

Published 25 February 2014

Recommended by C Liverani

Abstract

This paper contains rigorous results on nonequilibrium steady states for a class of particle systems coupled to unequal heat baths. These stochastic models are derived from the mechanical chains studied by Eckmann and Young by randomizing certain quantities while retaining other features of the model. Our results include the existence and uniqueness of nonequilibrium steady states, their relation to Lebesgue measure, tail bounds on total energy and number of particles in the system, and exponential convergence to steady states from suitable initial conditions.

Keywords: nonequilibrium steady states, particle systems, energy exchange, Lyapunov functions, coupling

Mathematics Subject Classification: 37Hxx, 82C05, 93E03

(Some figures may appear in colour only in the online journal)

1. Introduction

While much of the existing dynamical systems theory focuses on systems defined by a single equation or map, real-world systems seldom operate in isolation; they are interconnected and the dynamics of each system are modified constantly by interactions with the external world. Under stationarity assumptions on external forces, it is important to ask whether or not a system will eventually equilibrate with its surroundings? And, will it tend to a steady state? Mathematically, this is equivalent to asking if, starting from an arbitrary initial condition, the evolution of the system will, as time goes to infinity, be described by a unique invariant probability distribution on the set of all possible states? Here, ‘invariance’ is not with respect to the system’s internal dynamics, but it is with respect to the combined action of its internal dynamics and the sum total of all the external forces acting on it.

Questions of this type have not received a great deal of attention and we believe the subject can benefit from many more examples. In this paper we report on some results for a model from *nonequilibrium statistical mechanics*, which is an excellent source of out-of-equilibrium models. Although, given the nature of our model, the discussion here will take on a statistical mechanics flavour, similar issues arise in dynamical systems and we propose to view these questions in the more general context of *steady state problems for open dynamical systems*.

Model and results. Our starting point is the class of mechanical chains studied in [4]. In these models, particles move about in a 1D array of interconnected cells. Although they exchange energy with one another, they do not do it directly but they do it instead via rotating discs with which the particles collide. The two ends of the chain are coupled to unequal heat baths. Particles at characteristic energies are emitted by the baths at characteristic rates, and are reabsorbed into the baths at later times. Bypassing mathematical issues, such as existence and uniqueness of steady states, the authors of [4] showed, via a combination of analysis and simulations, that these models have very reasonable physical properties. We would have liked to provide rigorous proofs for some of the ‘facts’ taken for granted in [4], but they are currently out of reach: the dynamics of this model are basically deterministic, and there are few tools available for analysing statistical properties of deterministic systems except for those with fairly strong hyperbolic properties (or certain other special qualities). The models in [4] are not in these categories.

This paper rigorously treats some of the issues bypassed in [4] for a *stochastic* version of the models considered there. Specifically, we randomize certain quantities while retaining the main features of the model. For example, a particle loses memory of its precise location within a cell. This leads naturally to random interaction times and random repartitioning of energy upon collisions. Modifications of this kind turn the deterministic dynamics within the chain into Markovian dynamics. Similar ideas were already proposed in [4]. For the resulting stochastic models, we prove, under an additional technical condition, the existence and uniqueness of nonequilibrium steady states for arbitrary bath temperatures and injection rates. We provide statistical information on tail bounds for total energy and particle numbers, as well as the relation of the invariant measure to Lebesgue measure. We also prove our results on exponential rates of correlation decay.

Main issues. On the conceptual level, a major question for us is: what prevents the system from gradually heating up? As energy and number of particles can be unboundedly large, in order for the system to settle down to a steady state the average inflow and outflow of energy and particles must be balanced. In our model, inflow is constant and is regulated by heat baths. Outflow, on the other hand, is the product of the dynamics within the chain for which we have no direct control. In a nutshell, we deduce that there is a balance between inflow and outflow by showing that the process resembles a diffusion. That, however, is not entirely straightforward to prove since the events governing particle movements and energy exchanges are highly interdependent—this is so even in the stochastic model, which has inherited much of the character of the deterministic dynamics.

A second potentially serious issue is the accumulation of large numbers of (inactive) low-energy particles in the system. This problem is of a very different nature, and we do not deal with it here: we avoid it by imposing a strictly positive lower bound on interaction rates. Aside from turning the model in [4] into a stochastic model, this is the only additional technical assumption we have imposed.

Method of proofs and related works. For problems of this type, it is natural to attempt either a spectral or a probabilistic approach. We have opted to use the latter. Our proof is a combination of Lyapunov functions and coupling ideas [13]. While these methods are by now standard, each application has context-specific issues, as we have discussed. We believe this body of ideas can be used more widely in nonequilibrium statistical mechanics (see next paragraph) and in dynamical systems in general.

Among the mathematical models of heat conduction in the literature, there are surprisingly few rigorous results on nonequilibrium steady states. The following are the only ones we know of. First, there are fairly complete results on chains of anharmonic oscillators driven by stochastic heat baths. Under certain assumptions, these models can be described by stochastic differential equations, and results that include the existence and uniqueness of nonequilibrium steady states have been shown [3], as have exponential convergence and other statistical properties [16, 17]; see also [1]. A second group consists of results for Hamiltonian models similar to those in [4], with additional assumptions or special features to make the problem more tractable (as noted earlier, Hamiltonian models are much harder), they include: [2, 20], which prove ergodicity of the invariant measure assuming existence; [9], which proves existence and uniqueness for a model in which all energy exchanges are exclusively with ‘thermostats’ (or heat baths); and, [21], which treats a model with special geometry. A third group of results we know of consists of [10] and variants of this model [11, 15]. As with the present study, models in this third group are stochastic versions of models with mechanical origin. Papers [10] and [15] went beyond existence and uniqueness to prove *local thermal equilibrium*, which is an important property of the invariant measure that gives meaning to the idea of ‘local temperature’. Indeed, existence and uniqueness are only the very first step, one that opens the door to further understanding of steady state dynamics.

Paper organization. In section 2, we give a description of our model, and compare its equilibrium measure to that of the mechanical model from which it is derived. Section 3 contains the statement of results. The lower bounds on energy and particle outflow (see the main issues paragraph above) are treated in section 4. Section 5 proves the contractivity of the Markov operator, giving immediately some of the desired results for discrete-time chains. The various loose ends are tied together in section 6.

2. Model description

This section introduces the stochastic model that is considered in the rest of this paper. It is organized as follows. Section 2.1 contains a description of the class of mechanical models that serve as starting point of the present study. In section 2.2 we build stochastic models out of these mechanical models by imposing memory losses of certain kinds. To show that we have retained crucial features of the original systems, section 3 will compare equilibrium measures of the two models. In section 2.4, we introduce a further technical assumption needed for the analysis in later sections.

The stochastic models introduced in section 2.2, modified by the addition of the technical assumption in section 2.4, are the objects of our investigation in the rest of this paper.

2.1. Mechanical models with rotating discs and moving particles

The mechanical models considered here are very close to those studied in the second half of [4], modified slightly to enhance memory loss in the local dynamics. It consists of a 1D chain of identical *cells* each one of which is a bounded domain $\Gamma_i \subset \mathbb{R}^2$, $i = 1, 2, \dots, N$,

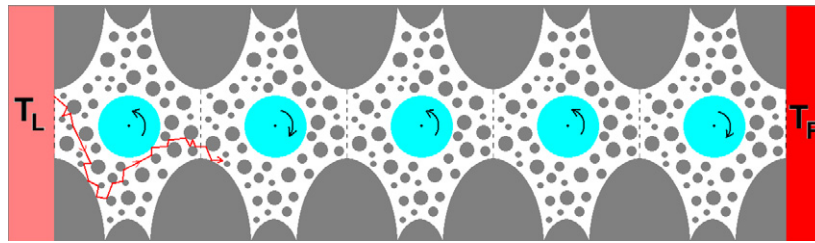


Figure 1. A mechanical model that serves as starting point for the stochastic model studied in this paper. A row of identical cells coupled to unequal heat baths (red) at the two ends. Each cell contains a freely rotating disc (cyan) with fixed centre. Particles move about in the domain $\cup_i \Gamma_i$ (white) bouncing off the outer walls of the domain and the many scatterers (grey).

with piecewise C^3 boundary. The boundary of Γ_i , $\partial\Gamma_i$, is the union of (i) its outer walls with two segments $\partial\Gamma_i^L$ on the left wall and $\partial\Gamma_i^R$ on the right wall removed, (ii) a large number of pairwise disjoint simple closed curves bounding convex domains that serve as *scatterers* for the particles, and (iii) a circle the interior of which is occupied by a *rotating disc* D_i . (Item (ii) is absent in [4].) For $i = 1, \dots, N - 1$, $\partial\Gamma_i^R$ is identified with $\partial\Gamma_{i+1}^L$, forming a passageway between the two cells. There are two *heat baths*, one attached to $\partial\Gamma_1^L$, and the other to $\partial\Gamma_N^R$. An example of this general lay-out is shown in figure 1.

Particles are injected into $\cup_{i=1}^N \Gamma_i$ by the baths, entering through either $\partial\Gamma_1^L$ or $\partial\Gamma_N^R$, and particles in $\cup_{i=1}^N \Gamma_i$ exit the system upon reaching the same two openings. While in $\cup_{i=1}^N \Gamma_i$, they pass freely from cell to cell, moving about with uniform motion until they collide with $\cup_i \partial\Gamma_i$, at which time the dynamics are defined by the rules below. The particles do not interact directly with one another.

The rules governing particle injection and collisions are as follows:

- (i) The temperatures of the left and right baths are T_L and T_R , and particles are injected into the system with exponential rates ρ_L and ρ_R respectively. The points of entry of particles from the left bath are uniformly distributed on $\partial\Gamma_1^L$, and the velocities of the entering particles have density

$$\frac{2\beta_L^{3/2}}{\sqrt{\pi}} e^{-\beta_L |v|^2} |v| |\sin(\phi)| dv \tag{2.1}$$

where $\beta_L = T_L^{-1}$, v is a vector pointing into Γ_1 , and ϕ is the angle v makes with the segment representing the opening. An analogous description holds for particles entering from the right. The points of entry, initial velocities, and injection times for each bath are i.i.d., and they are independent for the two baths.

- (ii) When a particle collides with the outer walls of $\cup_i \Gamma_i$ or with a scatterer, it is reflected without energy exchange. More precisely, if $v = (v^t, v^n)$ are the tangential and normal components of its velocity relatively to $\partial\Gamma_i$ just before the collision, and $v' = ((v^t)', (v^n)')$ its velocity immediately after the collision, then

$$(v^n)' = -v^n; \quad (v^t)' = v^t.$$

- (iii) When a particle collides with a rotating disc whose angular velocity is ω , the rule of updating is

$$(v^n)' = -v^n; \quad (v^t)' = \omega; \quad \omega' = v^t, \tag{2.2}$$

where $v = (v^t, v^n)$ is as above.

Notice that both total energy and momentum tangential to the disc are conserved in (iii). Equation (2.2) is the usual rule of energy and angular momentum conservation for a particular choice of parameters (involving the mass of the particle and radius and moment of inertia of the rotating disc). We have chosen to work with these parameters because the formulas have an especially simple form. Conceptually, the general case (see [12]) should be no different.

2.2. From deterministic to stochastic dynamics

As noted in the Introduction, Hamiltonian models are considerably harder to work with than stochastic ones. We present in this section a class of stochastic models obtained by replacing the deterministic dynamics in the mechanical models described in section 2.1 by Markovian dynamics. Although the resulting stochastic models are, mathematically, not equivalent to the original mechanical models, they capture many of the characteristics of the mechanical models, as we will show.

We first discuss informally how we plan to convert the model in section 2.1 into a stochastic one, with the precise description to follow. The main simplification is to *forget the geometry within the cells*, including

- (i) locations of particles within cells,
- (ii) their directions of travel, and
- (iii) the directions of rotation of the discs.

(i) and (ii) imply that at any one moment in time, we take note only of the cell number, that is, the i in Γ_i , in which a particle resides, without distinguishing between different physical locations within Γ_i , and we record the energy of the particle, hence its speed, without paying attention to its direction of travel. Likewise, the disc as a geometric object ceases to exist. It is represented instead by a single quantity, namely its energy, or ω^2 . These modifications will require new rules for particle injection, cell-to-cell movements of particles and their interactions with discs. We *assume* the following:

- (a) A particle in cell i carrying energy x will, at a random time given by an exponential distribution whose mean is proportional to \sqrt{x} , either jump or exchange energy with the ‘disc’ in its cell.
- (b) If it jumps, the particle will appear instantaneously in cell $i \pm 1$ with equal probability.
- (c) In the case of an exchange, its energy and that of the ‘disc’ at site i will be pooled together and repartitioned in a way that respects (2.2); see below.

Collisions of particles with outer walls of cells and with scatterers are not by themselves registered as events in this stochastic model. They are used, however, in the following justification of (a) and (b). The exponential distribution is used because in a chaotic system with strong hyperbolicity, the time to reach a pre-specified region starting from a random location is known to often have exponentially small tails (see e.g. [22]). In the models in section 2.1, the many convex scatterers produce a deterministic system with strong hyperbolicity. A more intuitive way to say it is that after colliding many times with the scatterers, it is not unreasonable to assume that a particle has lost memory of its initial location and direction of travel (while its energy is unchanged). That the rates of the clocks are proportional to \sqrt{x} comes from the fact that, in the mechanical model, a particle traveling the same trajectory but moving twice as fast will arrive at its destination (be it a collision with the disc or a passageway between cells) in half the time, hence, the \sqrt{x} scaling.

Precise description of stochastic model. Formally, our stochastic model is a (continuous time) Markov jump process taking values in a set of the form

$$\Omega = \prod_{i=1}^N \Omega^i, \quad \Omega^i = \cup_{k=0}^\infty \Omega_k^i,$$

where

$$\Omega_k^i = \{(s_i, \{x_1^i, \dots, x_k^i\}) \mid s_i, x_j^i > 0\}.$$

Here $i = 1, \dots, N$ are *sites* corresponding to the cells in the mechanical model. Each Ω^i represents the set of all possible states of the i th site, and Ω_k^i represents those states with exactly k particles at site i . The number of particles can be anywhere from zero to infinity—including both zero and infinity, at least *a priori*—and $\cup_{k=0}^\infty \Omega_k^i$ above should be interpreted as the disjoint union of $\Omega_0^i, \Omega_1^i, \dots, \Omega_\infty^i$. Focusing on one site at a time, s_i is the *stored energy* at site i , representing the energy of the disc in the i th cell, and $x_j^i, j = 1, \dots, k$, are the *particle energies* at that site. We have used curly brackets to denote *unordered* lists. As usual, for purposes of studying invariant measures or steady states, it is necessary to regard particles as indistinguishable (if particles were named, one would have to deal with their exits and re-entries into the system).

Note that $\Omega_k^i = \mathbb{R}_+ \times ((\mathbb{R}_+)^k / \sim)$ where $\mathbb{R}_+ = [0, \infty)$ and \sim is the following equivalence relation: if (x_1, \dots, x_k) and (y_1, \dots, y_k) are elements of \mathbb{R}^k , then $(x_1, \dots, x_k) \sim (y_1, \dots, y_k)$ if and only if the two k -tuples are permutations of one another. There is a natural reference measure Λ on Ω , namely $\Lambda = \prod_1^N \Lambda_i$ where Λ_i is the measure on Ω^i whose restriction to Ω_k^i is λ_{k+1} / \sim , the quotient of the Lebesgue measure on $\mathbb{R} \times \mathbb{R}^k$ by \sim .

The state of the system at time t , then, is defined by a random variable $x(t)$ of the form

$$x = ((s_1, \{x_1^1, \dots, x_{k_1}^1\}), (s_2, \{x_1^2, \dots, x_{k_2}^2\}), \dots, (s_N, \{x_1^N, \dots, x_{k_N}^N\}))$$

where k_1, k_2, \dots, k_N take values in $\mathbb{N} \cup \{0, \infty\} = \{0, 1, \dots, \infty\}$.

As in the mechanical model, we assume there are two heat baths located at sites 0 and $N + 1$, and let (T_L, ρ_L) and (T_R, ρ_R) denote the temperatures and injection rates of the left and right baths respectively. Particles with *i.i.d.* energy distributed with law

$$\frac{2\beta_L^{3/2}}{\sqrt{\pi}} \sqrt{x} e^{-\beta_L x}, \quad \beta_L = T_L^{-1},$$

are emitted by the left bath at the exponential rate of ρ_L ; emitted particles appear instantaneously in site 1. The Gamma distribution above is in accordance with (2.1), integrated over ϕ . Notice that the mean of this distribution is $\frac{3}{2}T_L$ and not T_L , as in the mechanical model. An analogous description applies to the right bath.

We now turn to the updating of $x(t)$ due to activity within the chain. Associated with each particle in the system is an exponential clock which, at any one moment in time, rings at rate $(1 + m)S\sqrt{x}$ where x is the energy of the particle at that moment, and m and S are system constants. When the clock of a particle rings, it ‘jumps’ with probability $\frac{1}{1+m}$, and ‘mixes’, that is, exchanges energy with the stored energy at its site, with probability $\frac{m}{1+m}$. The precise rules for jumping and mixing for a particle at site i carrying energy x_j^i follow.

If this particle jumps, it goes to site $i + 1$ or $i - 1$ with equal probability. Instantaneously k_i is decreased by 1, $k_{i\pm 1}$ is increased by 1, and the value x_j^i is moved to the list associated with sites $i \pm 1$ as the case may be. For $i = 1$, jumping to $i - 1$ means that the particle exits the system, and similarly for $i = N$ when the particle jumps to $i + 1$.

If mixing occurs, and the stored energy at site i is s_i , then, in accordance with (2.2), s_i and x_j^i are instantaneously updated to s_i' and $(x_j^i)'$ with

$$(s_i', (x_j^i)') = (x_j^i \cos^2 \theta, s_i + x_j^i \sin^2 \theta),$$

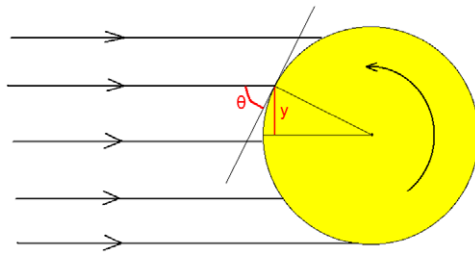


Figure 2. Distribution of angles between approaching trajectories and rotating discs.

where θ is randomly drawn from a distribution that should reflect the distribution of angles between the approaching trajectory and the tangent line in the mechanical model. To determine this distribution, assume for definiteness that the disc is bounded by the unit circle C centred at 0 in the (x, y) -plane, and consider approaching trajectories given by the family of parallel lines $\{\ell_a = \{y \equiv a\}, a \in (-1, 1)\}$ as shown in figure 2. We assume $a \in (-1, 1)$ is uniformly distributed, in accordance with Liouville measure. This gives rise to a distribution for θ , namely the angles these lines make with C . By the rotational symmetry of the disc, the distribution so obtained is the desired distribution.

More precisely, suppose in figure 2 that at height y , the slope of the tangent line makes an angle of θ with the horizontal. Then, the slope of this tangent line, which is equal to $|\frac{dx}{dy}|$ where $x = \sqrt{1 - y^2}$, is $y/\sqrt{1 - y^2}$; equivalently $\cos \theta = y$. This leads to the following rule of updating: when a particle carrying energy x interacts with stored energy s , the result is

$$(s', x') = (xu^2, s + x(1 - u^2)), \quad u \in (0, 1) \text{ uniformly distributed.} \quad (2.3)$$

The description of our stochastic model is now complete.

2.3. Invariant measures at equilibrium

We remark that the models proposed in section 2.2 are *not* the random-halves models studied in the first half of [4]. We have chosen interactions that more closely mimic those in the mechanical model. As a step toward justifying our choices, we compare here the energy and particle density distributions of the models in sections 2.1 and 2.2 when the system is in equilibrium, that is, when the two heat baths are equal: $T_L = T_R := T$, and $\rho_L = \rho_R := \rho$. It is easy to check that the invariant measure given in proposition 4.1 of [4], is invariant for the mechanical models described in section 2.1; the shape of the domain and the addition of the scatterers are immaterial.

The following proposition gives an invariant measure in the equilibrium case for the stochastic models defined in section 2.2.

Proposition 2.1. *For $T_L = T_R = T$ and $\rho_L = \rho_R = \rho$, the N -fold product*

$$v_N^{T,\rho} := v^{T,\rho} \times \cdots \times v^{T,\rho},$$

where the i th copy of $v^{T,\rho}$ is the probability measure on Ω^i characterized by (1) and (2) below, is left invariant by the dynamics. In particular, $v^{T,\rho}$ is itself an invariant measure for the case $N = 1$ (and we omit the i indicating cell number in the conditions below).

(1) The number of particles present is a Poisson random variable of mean

$$\frac{4}{S\sqrt{\pi}} \cdot \frac{\rho}{\sqrt{T}}$$

(2) The conditional probability density of $v^{T,\rho}$ on Ω_k is $c_k \sigma_k$ where

$$\sigma_k(s, \{x_1, \dots, x_k\}) = \frac{1}{\sqrt{s}} e^{-\beta(s+x_1+\dots+x_k)},$$

$\beta = T^{-1}$ and c_k is the normalizing constant.

Proposition 2.1 is a special case of proposition 2.2, a proof of which will be given.

After performing the coordinate changes $s = \omega^2$ and $x = |v|^2$, we observe that the measure in proposition 2.1 has the same form as the invariant measure of the mechanical model given in [4], section 4.2. The constants are different but have similar interpretations.

That these two invariant measures have the same scalings with respect to T and ρ implies that they have the same statistics in terms of energies and particle densities. Specifically, it follows from proposition 2.1 that the following hold when our stochastic system is in equilibrium with $T_L = T_R = T$ and $\rho_L = \rho_R = \rho$:

- (i) The mean stored energy at each site is equal to $T/2$;
- (ii) The mean per particle energy in the system is equal to T (not to be confused with the mean per particle energy at collisions or jumps, which is $\frac{3}{2}T$); and,
- (iii) The mean number of particles per site is $\propto \rho/\sqrt{T}$.

These statistics are identical to those in the mechanical model, providing further justification for the choices made.

2.4. A technical assumption

A technical hurdle that we encountered in our analysis of the stochastic model in section 2.2 is that, when the interaction rates can be arbitrarily close to zero, large numbers of low-energy particles can in principle linger at a site for arbitrarily long without jumping to adjacent sites or exchanging energy with the disc. This leads to many problems. To simplify the situation, we impose a strictly positive lower bound on clock rates. For example, for fixed $\epsilon > 0$, we may take the clock rate for a particle with energy x to be $(1+m)f(x)$ where $f(x) = S \max\{\sqrt{x}, \epsilon\}$. More generally, we assume

(*) the clock rate for a particle with energy x is given by $(1+m)f(x)$ where f is measurable with $\inf_{x \in (0, \infty)} (1+m)f(x) := f_0 > 0$.

Proposition 2.2. Under assumption (*), the statement of proposition 2.1 continues to hold with the mean of the Poisson random variable in (1) replaced by

$$\frac{4}{\sqrt{\pi}} \beta^{3/2} \rho \cdot \left(\int_0^\infty \frac{\sqrt{x}}{f(x)} e^{-\beta x} dx \right)$$

and the density in (2) replaced by

$$\sigma_k(s, \{x_1, \dots, x_k\}) = \left(\prod_{i=1}^k \frac{\sqrt{x_i}}{f(x_i)} \right) \frac{1}{\sqrt{s}} e^{-\beta(s+x_1+\dots+x_k)}.$$

We remark that the result in proposition 2.2 is in fact valid for any f provided the integral in (1) is finite, and if $f(x) = S \max\{\sqrt{x}, \epsilon\}$, then the invariant measure above converges to that in proposition 2.1 as $\epsilon \rightarrow 0$. A proof of proposition 2.2 is given in the appendix.

The rest of this paper will exclusively focus on the stochastic model defined in section 2.2 with interaction rate given by (*).

The Markov chain generated by this stochastic model is denoted by Φ_t . Let P^t be the transition probability kernel of Φ_t . That is to say, for any $t \geq 0$ and $x \in \Omega$, $P^t(x, \cdot)$ is a probability measure on Ω , and given any Borel measurable set $A \subset \Omega$, $P^t(\cdot, A)$ is a measurable function on Ω . The left and right operators generated by P^t are written

$$(P^t \xi)(x) = \int_{\Omega} P^t(x, dy) \xi(y)$$

for a measurable function ξ on Ω , and

$$(\mu P^t)(A) = \int_{\Omega} P^t(x, A) \mu(dx)$$

for a probability measure μ on Ω . We also use the notation $\mu(\xi) = \int_{\Omega} \xi(x) \mu(dx)$.

3. Main Results

3.1. Statement of main results

We introduce two functions, M and E , on Ω , the first giving the total number of particles and the second the total energy of a state $x \in \Omega$. In the notation of section 2.2,

$$M(x) = \sum_{i=1}^N k_i \quad \text{and} \quad E(x) = \sum_{i=1}^N \left(\sum_{j=1}^{k_i} x_j^i + s_i \right).$$

In the statement of results below, the clock rate f is fixed throughout. System size N , which we assume to be large, is also fixed in each statement—although from time to time we point out the scaling with N .

Theorem 1 (Existence and uniqueness of invariant measure). *Given (T_L, ρ_L) and (T_R, ρ_R) , the Markov chain Φ_t admits a unique invariant probability measure π .*

When $(T_L, \rho_L) = (T_R, \rho_R)$, that is, when the system is in equilibrium, theorem 1 asserts that the invariant probability measure given explicitly by proposition 2.2 is unique, hence ergodic. When $(T_L, \rho_L) \neq (T_R, \rho_R)$, that is, when the system is out of equilibrium, theorem 1 asserts the existence of a unique (hence ergodic) *nonequilibrium steady state* π . It is easy to check that in the nonequilibrium case π is *not* a product measure and does not appear to have a form that can be written down explicitly.

Theorem 2 (Properties of invariant measure). *The measure π in theorem 1 has the following properties:*

- (a) *It is absolutely continuous with respect to Λ on Ω with a strictly positive density.*
- (b) *There are constants $C, \alpha > 0, \alpha = O(N^{-2})$, such that*

$$\pi(\{x \mid M(x) \geq n\}) \leq C e^{-\alpha n} \quad \text{and} \quad \pi(\{x \mid E(x) \geq x\}) \leq C e^{-\alpha x}$$

for all n and x .

The tail distributions in theorem 2(b) follow from our choice of function space, equivalently space of measures, on which to apply the Markov operator. We introduce these spaces and norms systematically. Let (X, \mathcal{A}) be a measurable space, and let $W : X \rightarrow [1, \infty)$ be a measurable function on X . We define the *W-weighted supremum norm* of a measurable function $\xi : X \rightarrow \mathbb{R}$ to be

$$\|\xi\|_W = \sup_{x \in X} \frac{|\xi(x)|}{W(x)},$$

and let $\|\xi\| := \sup_{x \in X} |\xi(x)|$ be the sup norm of ξ . We also define the W -weighted variation norm of a signed measure μ on (X, \mathcal{A}) to be

$$\|\mu\|_W = \int_X W(x)|\mu|(dx),$$

where $|\mu|$ is the total variation of μ . Let $L_W(X)$ denote the set of all Borel probability measures μ on (X, \mathcal{A}) with $\mu(W) < \infty$.

Theorem 3 (Contraction of Markov operator). *There exists $\alpha = \alpha(N, T_L, T_R, \rho_L, \rho_R) > 0$ such that if $V : \Omega \rightarrow [1, \infty)$ is given by*

$$V(x) = e^{\alpha E(x)} + e^{\alpha M(x)}, \tag{3.1}$$

then the following hold:

- (a) If $\mu \in L_V(\Omega)$, then $\mu P^t \in L_V(\Omega)$ for all $t > 0$.
- (b) There exist constants $c > 0$ and $r \in (0, 1)$ such that for all $\mu_1, \mu_2 \in L_V(\Omega)$, we have

$$\|\mu_1 P^t - \mu_2 P^t\|_V \leq c \|\mu_1 - \mu_2\|_V r^t \quad \text{for all } t > 0.$$
- (c) The measure π above is the unique fixed point of $\mu \mapsto \mu P^t$ in $L_V(\Omega)$ for all $t > 0$.

The results of the rates of mixing and convergence to steady state now follow readily.

Corollary 4 (Exponential convergence to steady state and correlation decay). *Given (T_L, ρ_L) and (T_R, ρ_R) , the Markov chain Φ_t has the following properties.*

- (a) For every $x \in \Omega$,

$$\|P^t(x, \cdot) - \pi\|_V \leq cV(x)r^t$$

where c and r are as in theorem 3.

- (b) Let $\mu \in L_V(\Omega)$, and let ξ and ζ be measurable functions on (Ω, \mathcal{A}) with $\|\xi\| < \infty$ and $\|\zeta\|_V < \infty$. Then, the covariance

$$C_{\xi, \zeta}^\mu(t) := \int_X (P^t \zeta)(x)\xi(x)\mu(dx) - \int_X (P^t \zeta)(x)\mu(dx) \int_X \xi(x)\mu(dx)$$

decays exponentially, with

$$|C_{\xi, \zeta}^\mu(t)| \leq \|\xi\| \|\zeta\|_V \|\mu\|_V \cdot 2cr^t \quad \text{for all } t > 0.$$

We remark that the results in corollary 4 are new even in the equilibrium case. We do not, however, have efficient bounds on how the rate of mixing in an N -chain scales with N . Control, such as that in [5, 18], would be relevant for further studies.

3.2. Method of proof

The proof of theorem 3 uses the idea that unique ergodicity follows from (i) the existence of ‘small sets’ that are visited infinitely often together with (ii) Doeblin’s condition on a compact part of the phase space. To implement these ideas, one generally constructs a Lyapunov function; exponential mixing follows if the average value of this function decreases sufficiently fast to the region on which coupling takes place. This body of ideas was developed by Meyn and Tweedie [13]; some of the ideas go back to Harris [7]. The exposition below follows that of Hairer and Mattingly [6], which contains an elegant formulation as well as a more direct proof.

Consider a discrete-time Markov chain Ψ_n defined on a measurable space (X, \mathcal{A}) , with transition kernel $\hat{P}(x, \cdot)$. The following two conditions are relevant.

Assumption (A1). *There exist a measurable function $W : X \rightarrow [1, \infty)$ and constants $K \geq 0$, $\gamma \in (0, 1)$ such that*

$$(\hat{P}W)(x) - W(x) \leq -\gamma W(x) + K \quad \text{for all } x \in X.$$

Assumption (A2). *Let $D = \{W \leq 2K/\gamma\}$. Then, there exist $\delta \in (0, 1)$ and a probability measure ν on (X, \mathcal{A}) such that*

$$\inf_{x \in D} \hat{P}(x, \cdot) \geq \delta \nu(\cdot).$$

Let $\|\cdot\|_W$ and $L_W(X)$ be as defined in section 3.1, just before the statement of theorem 3.

Theorem 3.1 (Theorems 1.2 and 1.3 in [6]). *Assume (A1) and (A2) hold for the Markov chain Ψ_n . Then*

- (a) $\mu \hat{P} \in L_W(X)$ for $\mu \in L_W(X)$;
- (b) there exist $\hat{r} \in (0, 1)$ and $\hat{C} > 0$ such that for all $\mu_1, \mu_2 \in L_W(X)$,

$$\|\mu_1 \hat{P}^n - \mu_2 \hat{P}^n\|_W \leq \hat{C} \hat{r}^n \|\mu_1 - \mu_2\|_W.$$

It follows that $\mu \mapsto \mu \hat{P}$ has a unique fixed point, that is, Ψ_n has a unique invariant probability measure μ_* in $L_W(X)$.

The rate \hat{r} in theorem 3.1 can be expressed explicitly in terms of the constants γ , K and δ ; see [6].

To directly apply these results, we will work with the time- T chain defined by our process for a suitable $T > 0$. That is to say, instead of the continuous-time process Φ_t , we consider the discrete-time Markov chain Φ_n^T , $n = 1, 2, \dots$, the transition probabilities of which are given by $P^{nT}(x, \cdot)$.

Two major steps in our proof are to identify a suitable Lyapunov function, and to show that it has the required properties. Once Assumptions (A1) and (A2) are verified, we obtain by application of theorem 3.1 results analogous to those in theorem 3 for the time- T chain, including the existence of a unique fixed point in the space of measures in question. While it is natural to involve total energy and number of particles in a Lyapunov function, our particular choice of V (see theorem 3) is motivated by a desire for exponential tail bounds in energy and particle distributions. The bulk of the work goes into the verification of (A1), in proving lower bounds for the amount of energy that flows out of the system in particular. This step, which is essential for preventing energy and particle build-ups within the chain, can be considered the single most important idea in our proof. Once (A1) and (A2) are checked, corollary 4 also follows.

Arguments based on theorem 3.1 do not by themselves rule out the existence of invariant measures outside of $L_V(\Omega)$, nor do they identify the measure class of π , the unique fixed point in $L_V(\Omega)$. We prove in a separate argument that any invariant measure must necessarily have a strictly positive density with respect to the reference measure Λ on Ω . It is this property that implies the uniqueness, hence ergodicity, of π as asserted in theorem 1.

Finally, we must confirm that all of the results above hold not just for the time- T chain Φ_n^T but also for the continuous time Markov chain Φ_t .

4. Outflow of energy and particles

This section contains a crucial step of the proof, concerning the mean amount of energy and number of particles that flow out of the chain per unit time.

To prove the existence of an invariant measure, we must control the amount of energy and number of particles that (i) *enter* and (ii) *leave* the chain per unit time. In this model, (i) is relatively straightforward, as the heat baths release, at known rates, particles carrying known distributions of energy into the chain. In this section, we focus on (ii), which poses the following challenge. The time that a particle exits the chain, and the amount energy it carries with it when it leaves, are influenced by the stored energies with which it interacts. These stored energies are in turn determined by their interactions with particles that pass through, and all of these events are highly interdependent.

Our strategy for obtaining lower bounds on the outflow of energy and particles is to try to extract as much independence from the situation as possible, and use worst-case scenario bounds the rest of the time. Our main results are proposition 4.5 and lemma 4.6, which are stated and proved in section 4.3.

4.1. Probabilistic preliminaries

We begin by recalling a version of the classical Chernoff bounds and some corollaries that we will need. For a reference see any basic text in probability.

Lemma 4.1. *Let $X = \sum_{i=1}^n X_i$ where X_1, \dots, X_n are independent Bernoulli random variables, and let $\mu = \mathbb{E}[X]$. Then for any $0 < \delta < 1$,*

$$\mathbb{P}[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}} \right)^\mu, \quad \mathbb{P}[X \geq (1 + \delta)\mu] \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu.$$

Corollary 4.2. *Let X be a Poisson random variable with mean μ . Then*

$$\begin{aligned} \mathbb{P}[X \geq x] &\leq \frac{e^{-\mu}(e\mu)^x}{x^x} && \text{for } x > \mu; \\ \mathbb{P}[X \leq x] &\leq \frac{e^{-\mu}(e\mu)^x}{x^x} && \text{for } x < \mu. \end{aligned}$$

We will also need the following small extension of lemma 4.1.

Lemma 4.3 ([14], theorem 3.2). *Let X_1, \dots, X_n be (possibly dependent) Bernoulli random variables taking values 0 and 1. Assume that for each i , $\mathbb{P}[X_i = 1 \mid \text{all } X_j, j \neq i] \leq p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \sum_{i=1}^n p_i$. Then, for any $0 < \delta < 1$,*

$$\mathbb{P}[X \geq (1 + \delta)\mu] \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu.$$

Proof. Let Y_1, \dots, Y_n be independent Bernoulli random variables such that $\mathbb{P}[Y_i = 1] = p_i$ and $\mathbb{P}[Y_i = 0] = 1 - p_i$, and let $Y = \sum_{i=1}^n Y_i$. In the proof of the classical Chernoff bounds for Y , one first applies Markov's inequality to e^{tY} to obtain

$$\mathbb{P}[Y > (1 + \delta)\mu] = \mathbb{P}[e^{tY} > e^{t(1+\delta)\mu}] \leq \frac{\mathbb{E}[e^{tY}]}{e^{t(1+\delta)\mu}}. \tag{4.1}$$

Then, one shows, via a computation, that

$$\frac{\mathbb{E}[e^{tY}]}{e^{t(1+\delta)\mu}} \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu. \tag{4.2}$$

Since (4.1) is equally valid for X as for Y , the lemma follows once we show $\mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{tY}]$, equivalently, $\mathbb{E}[X^k] \leq \mathbb{E}[Y^k]$ for $k = 0, 1, \dots$. Now $\mathbb{E}[X^k]$ is a finite sum of terms of the form

$\mathbb{E}[X_{i_1} X_{i_2} \cdots X_{i_k}]$. We fix (i_1, \dots, i_k) , and let $\{j_1, \dots, j_{k'}\}$ be the set of distinct indices that appear as some i_ℓ . Then

$$\begin{aligned} \mathbb{P}[X_{j_1} = X_{j_2} = \cdots = X_{j_{k'}} = 1] &= \mathbb{P}[X_{j_1} = 1 \mid X_{j_2}, \dots, X_{j_{k'}}] \mathbb{P}[X_{j_2} = 1 \mid X_{j_3}, \dots, X_{j_{k'}}] \cdots \mathbb{P}[X_{j_{k'}} = 1] \\ &\leq p_{j_1} p_{j_2} \cdots p_{j_{k'}} \\ &= \mathbb{P}[Y_{j_1} = Y_{j_2} = \cdots = Y_{j_{k'}} = 1], \end{aligned}$$

so that

$$\mathbb{E}[X_{i_1} X_{i_2} \cdots X_{i_k}] = \mathbb{P}\left[\prod_{j_\ell} X_{j_\ell} = 1\right] \leq \mathbb{P}\left[\prod_{j_\ell} Y_{j_\ell} = 1\right] = \mathbb{E}[Y_{i_1} Y_{i_2} \cdots Y_{i_k}]$$

completing the proof. □

4.2. Worst-case estimates for tagged particles

The aim of this subsection is to show that there exist a time $T_1 = O(N^2)$, a probability $p > 0$, and a proportion $\eta \in (0, 1)$, such that the following holds. Let $x_0 \in \Omega$ be an arbitrary initial state. Then for every particle in the system at time 0, ‘independently of what other particles do’, the probability that this particle will exit the system before time T_1 carrying with it a fraction $\geq \eta$ of its initial energy is $\geq p$.

Since the energies, hence jump rates, of particles are inter-dependent, our first order of business is to make precise what ‘independently of what other particles’ means. Given an initial condition $x_0 \in \Omega$ with M_0 particles in the system, and assuming for the moment that no new particles enter during the time period $[0, T)$, $0 < T \leq \infty$, we claim that the set of all sample paths on this time interval can be parametrized as follows. Let

$$\Sigma = \prod_{j=1}^{M_0} \Sigma_j, \quad \text{where } \Sigma_j = ((0, 1) \times \{J, M\} \times \{L, R\} \times (0, 1))^{\mathbb{N}}$$

parametrizes the ‘randomness’, or ‘choices’, of particle j in a way to be made precise. The space Σ is endowed with the M_0 -fold product $v^{\mathbb{N}} \times \dots \times v^{\mathbb{N}}$, where each $v^{\mathbb{N}}$ is a probability on Σ_j and v is the product of Lebesgue measure on $(0, 1)$, the $(\frac{1}{1+m}, \frac{m}{1+m})$ -probability on $\{J, M\}$ and the $(\frac{1}{2}, \frac{1}{2})$ -probability on $\{L, R\}$.

Two auxiliary functions will be used:

$$\begin{aligned} \varphi_\lambda : (0, \infty) &\rightarrow (0, 1) && \text{with } \varphi_\lambda(t) = 1 - e^{-\lambda t}, \\ \text{and } \psi_{s,x} : (0, 1) &\rightarrow (0, \infty) && \text{with } \psi_{s,x}(u) = s + x(1 - u^2). \end{aligned}$$

Here, φ_λ parametrizes the set of all ring times for a clock with rate λ in such a way that the distribution of ring times on $(0, \infty)$ is carried to Lebesgue measure on $(0, 1)$, that is, the probability of ringing on $(0, t)$ is equal to the Lebesgue measure of $(0, \varphi_\lambda(t))$. The function ψ is related to energy exchanges; see section 2.2.

Let a point $(\sigma_1, \dots, \sigma_{M_0}) \in \Sigma$ be fixed. Instead of describing the events in the order in which they occur, it is simpler to specify how $\sigma_1 = (r_n, a_n; b_n, u_n)_{n=1,2,\dots} \in \Sigma_1$ determines the choices for particle 1, assuming that other particles are treated similarly. Let $(i_1(0), x_1(0))$ denote the location and energy of particle 1 at time 0, and let $\tau_1 = \varphi_{(1+m)f(x_1(0))}^{-1}(r_1)$ be the first time its clock rings. At τ_1 , particle 1 jumps or mixes depending on whether $a_1 = J$ or M . If it jumps, then it goes left or right depending on whether $b_1 = L$ or R . If it mixes, then $x_1(0)$ and

$s_{i(0)}(\tau_1)$ are pooled together and repartitioned, with the energy of particle 1 immediately after the exchange given by $x_1(\tau_1^+) = \psi_{s_{i(0)}(\tau_1), x_1(0)}(u_1)$. We assume here that the value of $s_{i(0)}(\tau_1)$ is known; it is equal to $s_{i(0)}(0)$ if no other particle has exchanged energy with this site, and it is equal to something else defined by rules analogous to those for particle 1 if such exchanges have already taken place. To continue to describe the evolution of particle 1, we repeat the process above with initial condition $(i_1(\tau_1^+), x_1(\tau_1^+))$; the next move is then determined by $(r_2, a_2; b_2, u_2)$ together with what $s_{i(\tau_1^+)}$ may be at time $\tau_2 = \tau_1 + \varphi_{(1+m)f(x_1(\tau_1^+))}^{-1}(r_2)$, when its clock rings again. This is continued until particle 1 exits the chain, after which σ_1 becomes irrelevant.

In general, particles do enter the system after time 0, and Σ has to be enlarged to

$$\Sigma^+ = \Sigma \times \prod_{j=M_0+1}^{\infty} \Sigma_j^+, \quad \Sigma_j^+ = (\{1, N\}, (0, \infty)) \times \Sigma_j,$$

where Σ_j^+ parametrizes the initial location, energy and subsequent choices for each of the particles that enters (in the order in which they enter). The associated probability is extended to Σ^+ in an obvious way, with the factor supported on $(\{1, N\}, (0, \infty))$ reflecting bath temperatures and injection rates.

Finally, given an initial condition $x_0 \in \Omega$ and an event A that involves only particle 1, when we say A occurs with probability $\geq p$ ‘independently of all other particles’, or

$$\mathbb{P}_{x_0}[A \mid \text{all particles} \neq 1] \geq p,$$

what we mean is that for every $(\sigma_2, \sigma_3, \dots) \in \prod_{j=2}^{M_0} \Sigma_j \times \prod_{j>M_0} \Sigma_j^+$,

$$\nu^{\mathbb{N}}[\sigma_1 \in \Sigma_1 \text{ leading to } A \mid x_0; (\sigma_2, \sigma_3, \dots)] \geq p.$$

Remark. The discussion above can be summarized as follows. The trajectories of individual particles are usually expressed by $(i_j(t), x_j(t))$ where $i_j(t)$ is the location and $x_j(t)$ is the energy of particle j at time t . Seen this way, trajectories of different particles depend on one another and on conditions in the chain in a very complicated way. However, the underlying ‘choices’ or ‘decisions’ made by individual particles as expressed by σ_j above are entirely independent. The σ_j are independent for different j ; for each j the sequence of choices as indexed by n above are independent for different n , and for each n , the quantities r_n, a_n, b_n, u_n are independent of one another. It is through the process of translating these ‘decisions’ into actual trajectories that various layers of inter-dependence enter. Our proof will leverage the independence of these ‘underlying decisions’.

In the rest of this subsection, we fix $x_0 \in \Omega$, tag an arbitrary particle in the system, and refer to it as particle 1. Let us abbreviate $\mathbb{P}_{x_0}[\cdot \mid \text{all particles} \neq 1]$ as $\mathbb{P}_{x_0}^*[\cdot]$, since this notation will be used many times. Let $T_1 = 2\lceil(1+m)(N^2+1)\rceil/f_0$ where f_0 is the minimum clock rate (see section 2.4)¹.

Lemma 4.4. *The following statements pertain to particle 1, and are valid for any initial condition $x_0 \in \Omega$:*

(i) *Let W be the number of jumps before it exits the chain. Then*

$$\mathbb{P}_{x_0}^*[W \leq N^2] \geq \frac{1}{2}.$$

(ii) *Let τ be the time of exit. Then*

$$\mathbb{P}_{x_0}^*[W \leq N^2 \text{ and } \tau < T_1] \geq \frac{1}{4}(1 - (2e^{-1})^{\lceil(1+m)(N^2+1)\rceil}) := p_1.$$

¹ $\lceil x \rceil$ is the ceiling function; that is, the smallest integer $\geq x$.

(iii) Let K be the number of energy exchanges before it exits. Then

$$\mathbb{P}_{x_0}^* [W \leq N^2, \tau < T_1 \text{ and } K \leq mN^2] \geq p_1.$$

(iv) Let η be the fraction of $x_1(0)$ particle 1 carries with it as it exits the chain. Then

$$\mathbb{P}_{x_0}^* [\eta \geq e^{-3mN^2} \mid K \leq mN^2] \geq \frac{1}{2}.$$

Proof.

(i) The problem is that of a simple random walk X_0, X_1, \dots with $X_0 = m$, the initial location of particle 1, $X_{n+1} = X_n \pm 1$ with probability $(\frac{1}{2}, \frac{1}{2})$, and $W = \inf_n \{X_n = 0 \text{ or } N + 1\}$. It is well known that

$$\mathbb{E}_{X_0=m} [W] = m(N + 1 - m).$$

(See e.g. [19].) Since $m(N + 1 - m) \leq N^2/2$, it follows from the Markov inequality that $\mathbb{P}[W \leq N^2] \geq \frac{1}{2}$.

(ii) We fix throughout a sample path for all particles $\neq 1$ (in the sense discussed above), and let S_1, S_2, \dots denote the sequence of times between consecutive clock rings for particle 1. Although these random variables are not independent, when conditioned on any $(a_n, b_n, u_n)_{n=1,2,\dots}$ and S_1, \dots, S_{i-1} , each S_i is exponentially distributed with mean $\leq 1/f_0$. Let Y be the number of clock rings before time T_1 . Then

$$\mathbb{P}_{x_0}^* [Y \leq \lceil (1 + m)(N^2 + 1) \rceil \mid (a_n, b_n, u_n)_{n=1,2,\dots}] \leq \mathbb{P}[\tilde{Y} \leq \lceil (1 + m)(N^2 + 1) \rceil] \quad (4.3)$$

where \tilde{Y} is a Poisson random variable with mean $f_0 T_1 = \lceil (1 + m)(N^2 + 1) \rceil$, and corollary 4.2 gives $\mathbb{P}[\tilde{Y} \leq \lceil (1 + m)(N^2 + 1) \rceil] \leq (2e^{-1})^{\lceil (1+m)(N^2+1) \rceil}$.

To finish, let us assume for purposes of the next argument that after particle 1 enters the bath its clock continues to ring with rate f_0 and the probability of a ‘jump’ at each ring is as before (of course, in reality particle 1 is forgotten once it exits the chain). Let

$$A = \{\text{of the first } \lceil (1 + m)(N^2 + 1) \rceil \text{ clock rings, } \geq N^2 \text{ are jumps}\}.$$

Then $\mathbb{P}[A] \geq \frac{1}{2}$ because the median of binomial distribution $B(n, p)$ is less than $\lceil np \rceil$ (see [8]). The events A and $\{W \leq N^2\}$ are independent, and both are fully determined by the sequence $(a_n, b_n)_{n=1,2,\dots,(1+m)N^2}$. Thus,

$$\begin{aligned} \mathbb{P}_{x_0}^* [W \leq N^2, \tau < T_1] &\geq \mathbb{P}_{x_0}^* [W \leq N^2, Y \geq \lceil (1 + m)(N^2 + 1) \rceil, A] \\ &= \mathbb{P}_{x_0}^* [W \leq N^2] \cdot \mathbb{P}_{x_0}^* [A] \cdot \mathbb{P}_{x_0}^* \\ &\quad \times [Y \geq \lceil (1 + m)(N^2 + 1) \rceil \mid W \leq N^2, A] \\ &\geq \frac{1}{4} (1 - (2e^{-1})^{\lceil (1+m)(N^2+1) \rceil}). \end{aligned}$$

(iii) Since

$$\{W \leq N^2, \tau < T_1, K \leq mN^2\} \supset \{W \leq N^2, Y \geq \lceil (1 + m)(N^2 + 1) \rceil, A\},$$

the desired result follows immediately from the computation above.

(iv) Suppose $Z_0 = x_1(0) > 0$, and let $Z_k, k = 1, 2, \dots$, denote the energy of particle 1 after the k th energy exchange. Then, the distribution of Z_k given Z_{k-1} is $\geq (1 - u^2)Z_{k-1}$, $u \in (0, 1)$ uniformly distributed; this uses only that the stored energy relevant in the k th interaction is ≥ 0 . Thus, we have the relation

$$\ln Z_k \geq \sum_{i=1}^k U_i + \ln Z_0$$

where U_1, U_2, \dots are *i.i.d.* with mean $\int_0^1 \ln(1 - u^2) du = 2(\ln 2 - 1) \approx -1.4$. Thus

$$\mathbb{P}[(\ln Z_k - \ln Z_0) < -3k] \leq \frac{1}{2},$$

and letting K be as in (iii), we have

$$\mathbb{P}_{x_0}^*[Z_K > e^{-3mN^2} Z_0 \mid K \leq mN^2] \geq \frac{1}{2}$$

as claimed. □

4.3. Lower bounds for energy and particle outflow

Let

$$V_E(\mathbf{x}) = e^{\alpha E(\mathbf{x})} \quad \text{and} \quad V_M(\mathbf{x}) = e^{\alpha M(\mathbf{x})}$$

where $\mathbf{x} \in \Omega$ is a state of the system, $M(\mathbf{x})$ is the number of particles in state \mathbf{x} , $E(\mathbf{x})$ is total energy, including both stored energy and energy carried by the particles, and $\alpha > 0$ is a parameter to be determined. This section is concerned with lower bounds for energy and particle outflow, as measured via the functions V_E and V_M . We begin with some relevant definitions.

Given an initial condition $\mathbf{x} \in \Omega$ and a sample path ω , let $t_1 < t_2 < \dots$ be the times on the interval $[0, T)$ at which energy is expelled from the chain. Define $E_\omega^-(t)$ to be the function on $[0, T]$ given by

- (i) $E_\omega^-(0) = E(\mathbf{x})$,
- (ii) E_ω^- is piecewise constant except for downward jumps at t_i , and
- (iii) $E_\omega^-(t_i) - E_\omega^-(t_i^+) = \sigma_i$ where σ_i is the amount of energy expelled at time t_i .

Likewise, let $s_1 < s_2 < \dots$ be the times on the interval $[0, T)$ when energy is injected into the chain. The function E_ω^+ is defined analogously, with $E_\omega^+(s_i^+) - E_\omega^+(s_i) = \eta_i$ where η_i is the amount of injected energy at time s_i . That is to say, the function E_ω^- records only energy outflow along this sample path, ignoring any inflow of energy, while E_ω^+ records only energy inflow along ω , ignoring any outflow. It follows that $E_\omega(t)$, total energy in the chain at time t , is given by $E_\omega(t) = E_\omega^+(t) + E_\omega^-(t) - E(\mathbf{x})$.

Starting from \mathbf{x} , of interest is the total drop in V_E on $[0, T)$ along ω due to energy outflow; that is,

$$\Delta^- V_E(\mathbf{x}, \omega; [0, T)) := \sum_{t_i} (V_E(t_i) - V_E(t_i^+)) = \sum_{t_i} (e^{\alpha E_\omega(t_i)} - e^{\alpha E_\omega(t_i^+)}) .$$

The quantity that we are able to estimate is $\sigma_\omega := \sum_i \sigma_i$, the total amount of energy expelled on this time interval. These two quantities are related by

$$\begin{aligned} \Delta^- V_E(\mathbf{x}, \omega; [0, T)) &= \sum_{t_i} e^{\alpha(E_\omega^+(t_i) - E(\mathbf{x}))} (e^{\alpha E_\omega^-(t_i)} - e^{\alpha E_\omega^-(t_i^+)}) \\ &\geq \sum_{t_i} (e^{\alpha E_\omega^-(t_i)} - e^{\alpha E_\omega^-(t_i^+)}) \\ &= e^{\alpha E(\mathbf{x})} - e^{\alpha E_\omega^-(T)} = V_E(\mathbf{x})(1 - e^{-\alpha \sigma_\omega}), \end{aligned}$$

the inequality above following from $E_\omega^+(t) \geq E(\mathbf{x})$. Finally, we let

$$\Delta^- V_E(\mathbf{x}; [0, T)) = \int \Delta^- V_E(\mathbf{x}, \omega; [0, T)) \mathbb{P}(d\omega).$$

Quantities involving V_M are defined analogously.

Proposition 4.5. *Given $C_0 > 0$, there exist $p_0 > 0$, $\bar{E} > 0$, and $T = \text{constant} \cdot N^2$, all of which independent of α , such that, for all $x \in \Omega$ with $E(x) \geq \max\{\bar{E}, \frac{1}{2}M(x)\}$, we have*

$$\Delta^- V_E(x; [0, T]) \geq p_0(1 - e^{-\alpha C_0}) V_E(x).$$

The strategy of proof is to show that with a certain probability, there is a minimum amount of energy that is released to heat baths before time T . We write $E(x)$ as

$$E(x) = E_s(x) + E_p(x) \quad \text{where } E_s(x) = \sum_{i=1}^N s_i(x)$$

and $E_p(x)$ is the total energy carried by all of the particles in state x .

Proof. We will consider separately the following cases.

Case 1. $E_p(x) \geq E(x)/4$, and

Case 2. $E_p(x) \leq E(x)/4$,

where *case 1* is further subdivided into

Case 1a. At least one particle carries energy $\geq \sqrt{E_p(x)}$, and

Case 1b. No particle carries energy $\geq \sqrt{E_p(x)}$.

In each case we propose a scenario that will lead to a desired outcome, and estimate the probability of that scenario. For example, in *case 1a*, we will want a high-energy particle to exit in time $O(N^2)$, whereas in *case 1b*, we will seek to expel a certain fraction of the particles.

As before, let $T_1 = 2\lceil(1+m)(N^2+1)\rceil/f_0$. Notation such as τ, η etc. is as in lemma 4.4 when there is no ambiguity to which particle it refers, and we omit x in $E(x), E_p(x)$ etc.

Case 1a. We fix a particle carrying energy $\geq \sqrt{E_p}$, call it particle 1, and let \mathbb{P}_x^* be as in section 4.2. Then, by lemma 4.4(iii),(iv),

$$\mathbb{P}_x^*[\tau < T_1, \eta > e^{-3mN^2}] \geq \frac{1}{8}(1 - (2e^{-1})^{(1+m)N^2}) = \frac{1}{2}p_1.$$

It follows from $x_1(0) \geq \sqrt{E_p} \geq \sqrt{E}/2$ that if C is the amount of energy particle 1 carries with it as it exits the chain, then

$$\mathbb{P}_x[\tau < T_1, C > \frac{1}{2}e^{-3mN^2}\sqrt{E}] \geq \frac{1}{2}p_1.$$

For definiteness, we assume N is large enough that $\frac{1}{2}p_1 > \frac{1}{10}$.

Case 1b. Here $M \leq 2E \leq 8E_p$, and the maximum energy carried by any one particle at time 0 is $\sqrt{E_p}$. Let n be the number of particles carrying energy $\geq \frac{1}{16}$, and assume E is sufficiently large that $n \gg 1$. From

$$E_p \leq n\sqrt{E_p} + (M - n)\frac{1}{16} \leq n(\sqrt{E_p} - \frac{1}{16}) + \frac{1}{2}E_p,$$

we deduce that $n \geq \frac{1}{2}\sqrt{E_p} \geq \frac{1}{4}\sqrt{E}$.

Labeling these n particles as particles 1 through n , we apply again lemma 4.4(iii),(iv) to each one of them, obtaining for each

$$\mathbb{P}_x^*[\tau < T_1, \eta \geq e^{-3mN^2}] \geq \frac{1}{2}p_1.$$

Let A_i be the event that particle i either does not exit before time T_1 or exits with energy $< \frac{1}{16}e^{-3mN^2}$. Then, $\mathbb{P}_x[A_i] \leq 1 - \frac{1}{2}p_1$. Applying now lemma 4.3 to the (possibly dependent)

random variables $\mathbf{1}_{A_1}, \dots, \mathbf{1}_{A_n}$ and letting $\delta = (\frac{1}{2}p_1 - \frac{1}{20}) / (1 - \frac{1}{2}p_1)$, we obtain

$$\begin{aligned} \mathbb{P}_x \left[\sum_i \mathbf{1}_{A_i} \geq \frac{19}{20}n \right] &= \mathbb{P}_x \left[\sum_i \mathbf{1}_{A_i} \geq (1 + \delta)n \left(1 - \frac{1}{2}p_1 \right) \right] \\ &\leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^{(1 - \frac{1}{2}p_1)\frac{1}{4}\sqrt{E}} := p_2(E), \end{aligned}$$

that is, with probability $1 - p_2(E)$, $\frac{1}{20}n \geq \frac{1}{80}\sqrt{E}$ particles do as prescribed. Letting C denote the total energy expelled on the time interval $[0, T_1]$, we have

$$\mathbb{P}_x \left[C \geq \frac{1}{80}\sqrt{E} \cdot \frac{1}{16}e^{-3mN^2} \right] \geq 1 - p_2(E).$$

Observe, finally, that if $E \geq 1$, then $1 - p_2(E) \geq 1 - p_2(1) > 0$.

Case 2. An immediate consequence of $E_s \geq \frac{1}{4}E$ is that there is at least one site—call it site i_0 —with $s_{i_0} \geq \frac{1}{4N}E$. Since energy can only be carried out of the chain by particles, and there may not be any particles at site i_0 , our first order of business is to ensure that a particle reaches site i_0 within reasonable time.

Suppose for definiteness that $i_0 \leq N/2$. Consider a particle at site 1 (to be thought of as having just been emitted by the left bath) performing an unbiased random walk as in lemma 4.4(i), and let W be the number of steps to reach either 0, that is, the particle exits to the left bath, or i_0 , our desired destination. Then $\mathbb{E}[W] \leq N/2$, so that $\mathbb{P}[W \geq N^2] \leq \frac{1}{2N}$, and

$$\begin{aligned} \mathbb{P}[W < N^2, X_W = i_0] &= \mathbb{P}[W < N^2] - \mathbb{P}[W < N^2, X_W = 0] \\ &\geq \mathbb{P}[W < N^2] - \mathbb{P}[X_W = 0] \\ &\geq \left(1 - \frac{1}{2N} \right) - \frac{N-2}{N} \geq \frac{3}{2N}. \end{aligned}$$

Since walks of this kind are entirely independent of one another and of events in the chain, we have that for every N particles emitted by the left bath, the probability that at least one will reach site i_0 in $< N^2$ steps is $\geq 1 - (1 - \frac{3}{2N})^N \approx 1 - e^{-\frac{3}{2}}$.

Now, on a time interval of length $2N/\rho_L$, the left bath is expected to emit $2N$ particles, and the probability of its emitting $\geq N$ particles is estimated by corollary 4.2. Conditioned on this happening, the probability of at least one of them reaching site i_0 in $< N^2$ steps is estimated in the last paragraph, and for such a particle the (real) time it takes to reach site i_0 can be estimated as in the proof of lemma 4.4(ii). Summarizing, we have shown that if $T_2 = 2N/\rho_L + T_1$, then there exists $p_3 > 0$ (independent of N) such that

$$\mathbb{P}_x[\text{at least one particle reaches site } i_0 \text{ before time } T_2] \geq p_3.$$

We now invoke the fact that $s_{i_0} \geq \frac{1}{4N}E(x)$. Let $\tau_1 \geq 0$ be the first time that there is a particle in site i_0 . (If in state x there already is a particle at site i_0 , then $\tau_1 = 0$.) Independently of x , we have shown that $\tau_1 < T_2$ with probability $\geq p_3$. Let τ_2 be the first time a particle exchanges energy with the stored energy at site i_0 . Then with probability $\frac{m}{1+m}(1 - e^{-1})$, $\tau_2 - \tau_1 \leq f_0^{-1}$, where f_0 is the minimum clock rate.

Let us call that first particle to interact with the stored energy at site i_0 ‘particle 1’. Let χ be the amount of energy particle 1 receives in this interaction, then $\chi \geq \frac{1}{4N}E(x)$. An argument identical to that in Case 1a then tells us that with probability $\geq \frac{1}{10}$, particle 1 will exit the chain in $< T_1$ units of time after τ_2 , carrying with it an amount of energy $> e^{-3mN^2}\chi$.

To summarize, we have shown under the condition of case 2 that there exists $p_4 > 0$ (independent of N) such that if $T = T_2 + f_0^{-1} + T_1$, then with probability $\geq p_4$, an amount of energy $\geq \frac{1}{4N} e^{-3mN^2} E$ is expelled before time T .

To complete the proof, we let T be as in the last paragraph. In each of the three cases above, we have shown that Γ , the set of sample paths that follow the prescribed scenario, has probability $\geq p_0 = \min\{\frac{1}{10}, p_2(1), p_4\}$. For $\omega \notin \Gamma$, we use the estimate $\sigma_\omega \geq 0$ where σ_ω is total energy expelled (see the beginning of this subsection). For $\omega \in \Gamma$, we observe that given $C_0 > 0$, $\sigma_\omega \geq C_0$ can be arranged by taking \bar{E} large enough, namely by requiring

$$e^{-3mN^2} \min \left\{ \frac{1}{2} \sqrt{\bar{E}}, \frac{1}{16 \cdot 80} \sqrt{\bar{E}}, \frac{1}{8N} \bar{E} \right\} \geq C_0$$

and also that \bar{E} is large enough for the argument in case 1b. The desired conclusion follows from the relation between $\Delta^- V_E$ and σ_ω (see the beginning of section 4.3). \square

Similar ideas are used to prove the corresponding result for V_M .

Lemma 4.6. *Let $T = T_1$ be as in section 4.2. Given $C_0 > 0$, there exists a constant \bar{M} such that whenever $M(\mathbf{x}) \geq \bar{M}$, we have*

$$\Delta^- V_M(\mathbf{x}; [0, T]) \geq \frac{1}{2} (1 - e^{-C_0 \alpha}) V_M(\mathbf{x}).$$

Proof. For each particle, we have shown in lemma 4.4(ii) that

$$\mathbb{P}_x^*[\tau > T] \geq p_1.$$

Let A_i be the event that particle i is still in the system at time T . Lemma 4.3 applied to $\mathbf{1}_{A_1}, \dots, \mathbf{1}_{A_M}$ with $1 + \delta = \frac{4}{5(1-p_1)}$ (recall that $p_1 > \frac{1}{5}$) gives

$$\mathbb{P} \left[\sum_i^M \mathbf{1}_{A_i} > \frac{4}{5} M \right] \leq \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^{(1-p_1)M}.$$

A computation shows that for $M \geq 4$, the quantity on the right is $\leq \frac{1}{2}$. Letting $\bar{M} = \max\{4, 5C_0\}$, one obtains that if $M(\mathbf{x}) \geq \bar{M}$, then

$$\mathbb{P}_x[\text{at least } C_0 \text{ particles are released to heat baths on } [0, T]] \geq \frac{1}{2},$$

and we finish as before. \square

5. Contraction of Markov operators

The purpose of this section is to prove the contractivity of the Markov operator corresponding to the time- T chain (for suitable T) associated with our model. We do this by constructing a Lyapunov function and verifying conditions (A1) and (A2) in section 3.2. The results of this section will imply several of the assertions in our theorems for the time T -chain; they are summarized in section 5.3.

5.1. Lyapunov function and condition (A1)

Given an initial condition x and a sample path ω , we define $\Delta^+ V_E(x, \omega; [0, T])$ to be the total rise in V_E along ω on $[0, T)$ due to energy inflow. Using analogous notation in section 4.3 and reasoning analogously, we obtain

$$\begin{aligned} \Delta^+ V_E(x(\omega)) &= \sum_{s_i} e^{\alpha(E_\omega^-(s_i) - E(x))} \left(e^{\alpha E_\omega^+(s_i+)} - e^{\alpha E_\omega^+(s_i)} \right) \\ &\leq \sum_{s_i} \left(e^{\alpha E_\omega^+(s_i^+)} - e^{\alpha E_\omega^+(s_i)} \right) \\ &= e^{\alpha E_\omega^+(T)} - e^{\alpha E(x)} = V_E(x)(e^{\alpha \eta_\omega} - 1), \end{aligned}$$

where η_ω is the amount of energy injected into the system along ω on $[0, T)$. Here we have used $E_\omega^-(t) \leq E(x)$. The expected increase in V_E due to energy inflow, $\Delta^+ V_E(x; [0, T])$, thus satisfies

$$\Delta^+ V_E(x; [0, T]) \leq V_E(x)(\mathbb{E}[e^{\alpha \eta}] - 1)$$

where η is the amount of energy injected into the system.

Quantities involving V_M are defined similarly. In our model, inflow of energy and particles is controlled entirely by the heat baths and are independent of what goes on in the chain.

Lemma 5.1. For any $\alpha > 0$ and any $t > 0$,

$$\Delta^+ V_M(x; [0, T]) \leq (e^{(\alpha-1)(\rho_L+\rho_R)t} - 1)V_M(x).$$

Proof. For fixed x and $t > 0$, we let η be the number of particles injected on the time interval $[0, t)$, lumping together particles injected from the left and right baths. It follows from the additivity of Poisson random variables that η is a Poisson random variable with mean $(\rho_R + \rho_L)t$. Thus,

$$\begin{aligned} \Delta^+ V_M(x; [0, T]) &\leq V_M(x) \left(\sum_{k=0}^\infty e^{\alpha k} \frac{(\rho_L + \rho_R)^k t^k}{k!} e^{-(\rho_L+\rho_R)t} - 1 \right) \\ &= V_M(x)(e^{(\alpha-1)(\rho_L+\rho_R)t} - 1) \end{aligned}$$

completing the proof. □

Lemma 5.2. Assume $\alpha < \beta_L, \beta_R$. Then for any $t > 0$,

$$\Delta^+ V_E(x; [0, T]) \leq V_E(x) \left(e^{\rho_L t(1-\alpha T_L)^{-3/2} + \rho_R t(1-\alpha T_R)^{-3/2} - (\rho_L+\rho_R)t} - 1 \right).$$

Proof. Proceeding as in the previous proof—except that here we need to add up the energy carried into the chain by injected particles—we obtain

$$\begin{aligned} &\Delta^+ V_E(x; [0, T]) \\ &\leq V_E(x) \left\{ \left(\sum_{k_L=0}^\infty \mathbb{P}[\eta^L = k_L] \cdot \mathbb{E}[e^{\alpha \sum_{i=0}^{k_L} x_i^L}] \right) \left(\sum_{k_R=0}^\infty \mathbb{P}[\eta^R = k_R] \cdot \mathbb{E}[e^{\alpha \sum_{i=0}^{k_R} x_i^R}] \right) - 1 \right\}, \end{aligned}$$

where the quantities inside the two sets of brackets are contributions from the left and right baths, respectively: η^L is the number of particles injected from the left bath on the time interval $[0, t)$, and $x_i^L, i = 1, 2, \dots, k_L$, are the energies of the injected particles. That is to say, η^L is a

Poisson random variable with mean $\rho_L t$, while the x_i^L are independent random variables with law $\Gamma(\frac{3}{2}, \beta_L)$, that is, the distribution with probability density function

$$\frac{2\beta_L^{3/2}}{\sqrt{\pi}} \sqrt{x} e^{-\beta_L x}$$

(see section 2.2), and the x_i^L are independent of η^L . Analogous statements hold for injections from the right bath.

We need to show that the quantities inside the brackets are finite. As the sum of k independent random variables with Gamma distribution $\Gamma(a, b)$ has distribution $\Gamma(ka, b)$, and the moment generating function of $\Gamma(a, b)$ is $(1 - \frac{s}{b})^{-a}$ for $s < b$, it follows that for $\alpha < \beta_L$,

$$\mathbb{E}[e^{\alpha \sum_{i=0}^{K_L} x_i^t}] = \left(1 - \frac{\alpha}{\beta_L}\right)^{-3k_L/2}.$$

Substituting this into the computation of $(P^t V_E)^+$, we obtain that the quantity associated with the left bath is equal to $e^{\rho_L t(1-\alpha T_L)^{-\frac{3}{2}} - \rho_L t}$, proving the lemma. \square

The next result is a direct consequence of proposition 4.5 and lemmas 4.6, 5.1 and 5.2. It gives, under suitable conditions, a Lyapunov function that satisfies condition (A1). Let $T = O(N^2)$ be as in proposition 4.5, and fix such a T for the rest of the proof. For $x \in \Omega$ and $\alpha > 0$, define

$$V(x) = V^{(\alpha)}(x) = V_E(x) + V_M(x)$$

where, as before, $V_E(x) = e^{\alpha E(x)}$ and $V_M(x) = e^{\alpha M(x)}$.

Theorem 5.3. *There exist $\gamma, c > 0$ such that the following holds for every $\Gamma \geq T$. For $\alpha \leq \alpha(\Gamma) := c\Gamma^{-1}$ and $V = V^{(\alpha)}$, there exists $K = K(\Gamma, \alpha, \gamma) > 1$ such that*

$$P^\Gamma V(x) - V(x) \leq -\gamma V(x) + K$$

for every $x \in \Omega$.

Proof. Let $A = \min\{p_0, \frac{1}{2}\}$ where p_0 is as in proposition 4.5. Then by proposition 4.5 and lemma 4.6, we have that for any $C_0 > 0$ and any $\alpha > 0$, there exist $\bar{E}, \bar{M} > 0$ such that for any $t \geq T$ and any x , we have

$$\begin{aligned} \Delta^- V_E(x; [0, t]) &\geq A(1 - e^{-C_0 \alpha}) V_E(x) && \text{if } E(x) \geq \max\{\bar{E}, M(x)/2\}; \\ \Delta^- V_M(x; [0, t]) &\geq A(1 - e^{-C_0 \alpha}) V_M(x) && \text{if } M(x) \geq \bar{M}. \end{aligned}$$

Here, C_0 and α can be any positive number.

We claim next that there exists $B > 0$ such that for any $t > 0$, if both α and αt are sufficiently small depending only on T_L, T_R, ρ_L and ρ_R , then the following hold for all x :

$$\begin{aligned} \Delta^+ V_E(x; [0, t]) &\leq B\alpha t V_E(x); \\ \Delta^+ V_M(x; [0, t]) &\leq B\alpha t V_M(x). \end{aligned}$$

This follows from lemmas 5.1 and 5.2: Using the facts that for $x \ll 1$, $e^x - 1 \approx x$ and $(1 - x)^{-\frac{3}{2}} \approx 1 + \frac{3}{2}x$, one sees that the bounds in lemmas 5.1 and 5.2 are no greater than a constant times $\alpha(\rho_L + \rho_R)t V_M(x)$ and $\alpha(\rho_L T_L + \rho_R T_R)t V_E(x)$, respectively.

To prove the stated result, we let $\gamma = \frac{1}{10}A$, and let B be as above. Then for $t = \Gamma \geq T$, choose α so that (i) the above inequalities governing $\Delta^+ V_E$ and $\Delta^+ V_M$ hold and (ii) $B\Gamma\alpha < \frac{1}{4}A$. This gives the relation $\alpha = O(\Gamma^{-1})$. We then let C_0 be large enough such that $\frac{1}{2}A(1 - e^{-C_0 \alpha}) - B\Gamma\alpha > \gamma$, and use this C_0 to determine \bar{E} and \bar{M} .

Then, assuming $E(x) \geq M(x)/2$, we claim to have

$$\begin{aligned} P^\Gamma V(x) - V(x) &= [P^\Gamma V_E(x) - V_E(x)] + [P^\Gamma V_M(x) - V_M(x)] \\ &= [\Delta^+ V_E(x; [0, \Gamma]) - \Delta^- V_E(x; [0, \Gamma])] \\ &\quad + [\Delta^+ V_M(x; [0, \Gamma]) - \Delta^- V_M(x; [0, \Gamma])] \\ &\leq (B\Gamma\alpha - A(1 - e^{-C_0\alpha})) (V_E(x) + V_M(x)) + A(1 - e^{-C_0\alpha})(e^{\alpha\bar{E}} + e^{\alpha\bar{M}}) \\ &\leq -\gamma V(x) + A(1 - e^{-C_0\alpha})(e^{\alpha\bar{E}} + e^{\alpha\bar{M}}). \end{aligned}$$

The first inequality above is the only line that requires justification. We use the estimate above for $\Delta^- V_E(x; [0, t])$ when it is valid, that is, when $E(x) \geq \bar{E}$, and when $E(x) < \bar{E}$, we use

$$-\Delta^- V_E(x; [0, t]) \leq 0 \leq A(1 - e^{-C_0\alpha})(e^{\alpha\bar{E}} - V_E(x)).$$

The quantity $\Delta^- V_M(x; [0, t])$ is treated similarly.

It remains to consider the case $E(x) < M(x)/2$. Here,

$$P^\Gamma V(x) - V(x) \leq -\gamma V(x) + \frac{3}{2}A(1 - e^{-C_0\alpha})e^{\alpha\bar{M}} \quad \text{if } M(x) < \bar{M},$$

and if $M(x) \geq \bar{M}$, then

$$\begin{aligned} P^\Gamma V(x) - V(x) &\leq B\Gamma\alpha (V_E(x) + V_M(x)) - AC_0\alpha V_M(x) \\ &\leq (B\Gamma\alpha - \frac{1}{2}A(1 - e^{-C_0\alpha})) (V_E(x) + V_M(x)) \leq -\gamma V(x). \end{aligned}$$

This completes the proof with $K = \frac{3}{2}A(1 - e^{-C_0\alpha})(e^{\alpha\bar{E}} + e^{\alpha\bar{M}})$. □

5.2. Verification of (A2)

Let $\mathcal{N} \subset \Omega$ be the collection of states in which there are no particles in the chain. Recall that Λ is the reference measure on Ω , and observe that $\Lambda|_{\mathcal{N}}$ is simply Lebesgue measure on $(0, \infty)^N$. Assumption (A2) is implied directly by the following.

Lemma 5.4. *Given any constants E_0 and M_0 , denote*

$$\begin{aligned} D &= D_{E_0, M_0} = \{x \in \Omega \mid E(x) \leq E_0, M(x) \leq M_0\} \\ \text{and } B &= B_{[1,2]} = \{x \in \mathcal{N} \mid 1 \leq s_i \leq 2 \text{ for } i = 1, 2, \dots, N\}. \end{aligned}$$

Then, for each $t > 0$, there exists $\delta > 0$ depending on D, B and t (as well as other system constants), such that for every $x \in D$,

$$P^t(x, \cdot) \geq \delta \Lambda|_B(\cdot).$$

Proof. Fixing $t > 0$, we claim there exists $\delta > 0$ for which the following hold. For arbitrary (s_1^*, \dots, s_N^*) and ds with $1 \leq s_i^* < s_i^* + ds \leq 2$, let

$$A(ds) = \{x \in \mathcal{N} \mid s_i^* \leq s_i \leq s_i^* + ds\}.$$

We will argue that

$$P^t(x, A(ds)) \geq \delta(ds)^N$$

for every $x \in D$. The δ below is far from optimal, but it is sufficient for our purposes.

We consider a sequence of events described by the following.

- $F_1 = \{\text{on } (0, \frac{t}{2}), \text{ no new particle enters, and all particles present at } t = 0 \text{ exit the chain}\}$,
 For $j = 0, 1, \dots, N$, let I_j be the time interval $[\frac{t}{2} + \frac{j}{2(N+1)}, \frac{t}{2} + \frac{j+1}{2(N+1)})$.
- $F_2 = \{\text{on } I_0, \text{ exactly one particle enters from the left, carrying energy } x \in (3N, 4N)\}$,

- $F_3 = \{\text{no particle enters on } \cup_{j=1}^N I_j\}$,
- $F_{4,i} = \{\text{on } I_i, \text{ every particle at site } i \text{ exchanges energy once, then jumps to site } i + 1\}$,
- $F_{5,i} = \{s'_i \in [s_i^*, s_i^* + ds] \text{ following every energy exchange on } I_i\}$;

we further write $F_4 = F_{4,1} \dots F_{4,N}$.

Since the initial number of particles is $\leq M_0$, and clock rates are bounded away from 0, it is clear that there exists $p > 0$ independent of $x \in D$ such that, regardless of what happens in the energy exchanges, $\mathbb{P}[F_1 F_2 F_3 F_4] \geq p$.

To complete the proof, we need to show the existence of $c > 0$ (independent of $x \in D$, (s_1^*, \dots, s_N^*) and ds) for which the following holds. For each i , let G_i be the event that all goes as prescribed up to time $\frac{t}{2} + \frac{i}{2(N+1)}$, and an energy exchange occurs on I_i . Then

$$\mathbb{P}[F_{5,i} | G_i] \geq cds.$$

Conditioned on G_i , and letting $x(t)$ denote the energy of the unique particle in the system at time $t = \frac{t}{2} + \frac{i}{2(N+1)}$, we have that $x(t) \in (N, 4N + E_0)$, where E_0 is the upper bound on initial energy. Since $s'_i = x(t)u^2$ where $u \in (0, 1)$ is uniformly distributed (see section 2.2), the desired result follows. We have used here the fact that both $(N, 4N + E_0)$ and $[1, 2]$ are bounded away from zero and from infinity, with $2 < N$. □

5.3. A summary of the results thus far

Let $T = O(N^{-2})$ be fixed as before (see section 5.1). For each $\Gamma \geq T$, we let Φ_n^Γ be the time- Γ chain of Φ_t , that is, the discrete-time Markov chain whose transition probabilities are $P^{n\Gamma}$, $n = 1, 2, \dots$. Then letting

$$V(x) = e^{\alpha E(x)} + e^{\alpha M(x)}$$

where α is as in theorem 5.3, we have checked (A1) and (A2): specifically (A1) follows from theorem 5.3 and (A2) from lemma 5.4. Thus, we have established that the results of theorem 3.1 are valid for Φ_n^Γ . This implies in particular the existence of a unique invariant probability measure $\pi_\Gamma \in L_V(\Omega)$, and that

$$\|\mu P^{n\Gamma} - \pi_\Gamma\|_V \leq c\rho^n \|\mu - \pi_\Gamma\|_V \quad \text{for every } \mu \in L_V(\Omega). \tag{5.1}$$

We have further established the following tail bounds for π_Γ , namely that with α as above, there exists $C > 0$ such that

$$\pi_\Gamma(\{x | M(x) \geq n\}) \leq Ce^{-\alpha n} \quad \text{and} \quad \pi_\Gamma(\{x | E(x) \geq x\}) \leq Ce^{-\alpha x}$$

for all n and x . This is an immediate consequence of our choice of V and the fact that $\pi_\Gamma \in L_V(\Omega)$.

Remark. While the results above are valid for each $\Gamma \geq T$, we note that in our proof we have used $\alpha \leq \alpha(\Gamma) \propto \Gamma^{-1}$. Thus, our Lyapunov function $V = V^{(\alpha)}$ will vary with Γ , defining measure spaces with deteriorating tail bounds and convergence rates as $\Gamma \rightarrow \infty$.

6. Completing the proofs

To complete the proofs of theorems 1–4 as stated in section 3, the following tasks remain: (1) to pass the results in section 5.3 to the continuous-time chain Φ_t , obtaining in particular a unique invariant measure π with $\pi\Phi_t = \pi$ for all $t > 0$, and (2) to prove that π has a strictly positive density with respect to Λ . We first tackle (2), assuming the existence of π .

6.1. Absolute continuity of invariant measure

The aim of this subsection is to prove that:

Proposition 6.1. *Every invariant probability measure π of the continuous-time chain Φ_t is absolutely continuous with respect to Λ with a strictly positive density.*

For $x \in \Omega$ and $t > 0$, we may decompose $P^t(x, \cdot)$ into $P^t(x, \cdot) = \nu_\perp + \nu_{\text{abs}}$ where ν_{abs} is absolutely continuous with respect to Λ and ν_\perp is singular with respect to Λ . Notice that, unlike the situation with SDEs satisfying hypoelliptic conditions, $\nu_\perp \neq 0$ for every x and every t , because there is always a positive probability that no action takes place within time t . That is to say, the point mass δ_x is always a component of $P^t(x, \cdot)$ for every $t > 0$. (Another way to put it: these processes are not strong Feller.)

Lemma 6.2. *For every $x \in \Omega$ and every $t > 0$, ν_{abs} , the absolutely continuous component of $P^t(x, \cdot)$, is nonzero and has a strictly positive density on all of Ω .*

Proof. Given an initial condition x , and a target point

$$y = ((s_1^*, \{(y_1^1)^*, \dots, (y_{k_1}^1)^*\}), \dots, (s_N^*, \{(y_1^N)^*, \dots, (y_{k_N}^N)^*\})) \in \Omega$$

together with small numbers ds and $dy > 0$, one can show, using a scheme similar to that in lemma 5.4, that

$$P^t(x, A(ds, dy)) > \delta(ds)^N (dy)^{\sum_i k_i}.$$

Here, $A(ds, dy)$ is the collection of states with k_i particles at site i , with $s_i \in [s_i^*, s_i^* + ds]$ and $y_i^j \in [(y_i^j)^*, (y_i^j)^* + dy]$, and $\delta > 0$ is allowed to depend on the number of particles and total energy of x and y but not on ds or dy . This is a straightforward exercise for as long as no other requirements are placed on δ . □

Lemma 6.2 says that as we apply the Markov operator, any measure will acquire an absolutely continuous component. The next lemma guarantees that an absolutely continuous component cannot revert back to singularity.

Lemma 6.3. *For any finite Borel measure μ , if $\mu \ll \Lambda$, then $\mu P^t \ll \Lambda$ for any $t > 0$.*

Proof. The problem will boil down to showing that certain elementary moves carry the Λ -measure class to itself. This is what we mean by an ‘elementary move’:

- a particle enters the chain from the left, respectively, right, bath
- the j th lowest energy particle at site i jumps left, respectively, right; no change if $k_i < j$
- the j th lowest energy particle at site i exchanges energy with the site; no change if $k_i = 0$

We fix $t > 0$, and for a sequence $\ell = (m_1, \dots, m_n)$, where $n \in \mathbb{Z}^+$ and m_1, \dots, m_n are elementary moves, we let $E(\ell)$ denote the event that m_1, \dots, m_n occur on the time interval $[0, t)$ in the order specified, and these are the only elementary moves that occur. We also let $E(\emptyset)$ and $E(\infty)$ denote the event that zero or infinitely many elementary moves occur on $[0, t)$. It is easy to check $\mathbb{P}_x[E(\infty)] = 0$.

We introduce, for $E = E(\ell)$ or $E(\emptyset)$, an operator P_E which acts on the space of finite Borel measures on Ω by

$$(\mu P_E)(A) = \int P_E(x, A) \mu(dx) \quad \text{where } P_E(x, A) = \mathbb{P}[(\Phi_t \in A | \Phi_0 = x) | E],$$

and observe the following:

(1) Since

$$\mu P(\cdot) = \int \left[\left(\sum_{\ell} \mathbb{P}_x[E(\ell)] P_{E(\ell)}(x, \cdot) \right) + \mathbb{P}_x[E(\emptyset)] P_{E(\emptyset)}(x, \cdot) \right] \mu(d\mathbf{x}),$$

we have that for given $\mu \ll \Lambda$, if μ_Q , $Q = \ell$ or \emptyset , is the measure on Ω defined by

$$\frac{d\mu_Q}{d\Lambda}(x) = \mathbb{P}_x[E(Q)] \frac{d\mu}{d\Lambda}(x),$$

then

$$\mu P = \mu_{\emptyset} + \sum_{\ell} \mu_{\ell} P_{E(\ell)}. \tag{6.1}$$

Thus, it suffices to show the absolute continuity of each of the countably many measures on the right side of (6.1).

(2) For every $\ell = (m_1, \dots, m_n)$, the operator $P_{E(\ell)}$ can be decomposed into

$$P_{E(\ell)} = P_{E(m_n)} \cdots P_{E(m_2)} P_{E(m_1)}.$$

Since μ_{ℓ} is just another finite Borel measure $\ll \Lambda$, (1) and (2) above reduce the problem to the following. Given $\mu \ll \Lambda$ and an elementary move m , show that $\mu P_{E(m)} \ll \Lambda$.

This is a straightforward exercise. For example, let m be the move corresponding to the second lowest-energy particle in site 1 exiting to the left bath. Let $\xi = \frac{d\mu}{d\Lambda}$, and $\hat{\xi} = \frac{d\mu P_{E(m)}}{d\Lambda}$. Then

$$\hat{\xi}((s_1, \{x_1^1, \dots, x_{k_1}^1\}), \dots) = \begin{cases} \xi(\dots) & \text{if } k_1 = 0, \\ \xi(\dots) + \int \xi((s_1, \{x_1^1, y\}), \dots) dy & \text{if } k_1 = 1, \\ \int \xi((s_1, \{x_1^1, y, x_2^1, \dots, x_{k_1}^1\}), \dots) dy & \text{if } k_1 \geq 2. \end{cases}$$

Here, (\dots) means same as the argument of $\hat{\xi}$, and it is implied when writing $\{a, b, c, \dots\}$ that $a < b < c < \dots$. In the case of a particle jumping to an adjacent site, instead of writing down the exact formula of $\hat{\xi}$ (which will involves several cases), it is conceptually clearer to view $\mu P_{E(m)}$ as the push-forward of μ by a mapping $\mathcal{T} : \Omega \rightarrow \Omega$, and to observe that \mathcal{T} is piecewise affine on a countable number of domains, each one of which is finite-dimensional. As Λ restricted to subregions of Ω with M particles is $(N+M)$ -dimensional Lebesgue measure, our assertion follows from the fact that \mathcal{T} preserves the Lebesgue measure class. For the case of energy exchange, see the proof of proposition 2.2, which uses a similar idea. The case involving the entrance of a new particle is left to the reader. \square

Proof of proposition 6.1. Decompose π into $\pi = \pi_{\perp} + \pi_{\text{abs}}$ as before and assume, to derive a contradiction, that $\pi_{\perp} \neq 0$. For $t > 0$, $\pi_{\text{abs}} P^t \ll \Lambda$ by lemma 6.3, and $\pi_{\perp} P^t$ has an absolutely continuous component by lemma 6.2. Together these statements imply that $(\pi P^t)_{\text{abs}}(\Omega) > \pi_{\text{abs}}(\Omega)$, contradicting the invariance of π . \square

6.2. Invariant measure for Φ_t

We have shown that for every $\Gamma \geq T$ (where $T = O(N^2)$ is fixed), there exists a unique $\pi_\Gamma \in L_V(\Omega)$ for $V = V^{(\alpha)}$, $\alpha \leq \alpha(\Gamma)$, such that $\pi_\Gamma P^\Gamma = \pi_\Gamma$ (see section 5.3). We now argue that all of these π_Γ are in fact one and the same, and that it is invariant under P^t for every $t > 0$. A key ingredient of the proof the following ‘continuity at 0’ property. Let $|\nu|$ denote the variational norm of ν .

Lemma 6.4. For fixed $\Gamma \geq T$,

$$|\pi_\Gamma P^\delta - \pi_\Gamma| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Proof. Given Γ and ϵ , we will show that for all sufficiently small $\delta > 0$,

$$|(\pi_\Gamma P^\delta)(A) - \pi_\Gamma(A)| < \epsilon \quad \text{for every Borel set } A \subset \Omega. \tag{6.2}$$

First, there exist \bar{M}, \bar{E} (large) and $\delta_0 > 0$ (small) such that if $U = \{x \mid M(x) \leq \bar{M}, E(x) \leq \bar{E}\}$ then (i) $\pi_\Gamma(U) > 1 - \frac{\epsilon}{4}$, and (ii) starting from $x \in U$, the probability that no clock rings before time δ_0 is $> 1 - \frac{\epsilon}{4}$. Then, for every $A \subset \Omega$ and every $\delta \in (0, \delta_0)$,

$$\int P^\delta(x, A)\pi_\Gamma(dx) = \int_{U \cap A} + \int_{U-A} + \int_{U^c} = \pi_\Gamma(U \cap A) - a_1 + a_2 + a_3$$

where

$$\begin{aligned} a_1 &= \int_{U \cap A} (1 - P^\delta(x, A))\pi_\Gamma(dx) \leq \frac{\epsilon}{4}\pi_\Gamma(U \cap A) \leq \frac{\epsilon}{4}, \\ a_2 &= \int_{U-A} P^\delta(x, A)\pi_\Gamma(dx) \leq \frac{\epsilon}{4}\pi_\Gamma(U - A) \leq \frac{\epsilon}{4}, \\ a_3 &= \int_{U^c} P^\delta(x, A)\pi_\Gamma(dx) \leq \pi_\Gamma(U^c) \leq \frac{\epsilon}{4}. \end{aligned}$$

Further, $\pi_\Gamma(A) - \pi_\Gamma(U \cap A) \leq \pi_\Gamma(U^c) < \epsilon/4$, so $|\pi_\Gamma(A) - \pi_\Gamma(U \cap A)| \leq \epsilon/4$. The assertion in (6.2) follows. \square

Corollary 6.5. Let $\pi = \pi_T$. Then $\pi P^t = \pi$ for all $t > 0$, and $\pi \in L_V(\Omega)$ for $V = V^{(\alpha(T))}$.

Proof. First we show that $\pi_\Gamma = \pi_\Delta$ for all $\Gamma, \Delta \geq T$ such that $\Delta/\Gamma \notin \mathbb{Q}$. This relationship between Γ and Δ implies, by the density of orbits in irrational rotations, the existence of $j_n, k_n \in \mathbb{Z}^+$, $j_n, k_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $\delta_n := k_n\Delta - j_n\Gamma \downarrow 0$. Letting $\alpha = \min\{\alpha(\Gamma), \alpha(\Delta)\}$ and $V = V^{(\alpha)}$, we have, by (5.1),

$$\|\pi_\Delta - \pi_\Gamma P^{k_n\Delta}\|_V \rightarrow 0.$$

But since $\pi_\Gamma = \pi_\Gamma P^{j_n\Gamma}$, we also have

$$\|\pi_\Delta - \pi_\Gamma P^{k_n\Delta}\|_V = \|\pi_\Delta - \pi_\Gamma P^{\delta_n}\|_V \geq |\pi_\Delta - \pi_\Gamma P^{\delta_n}|.$$

This, together with lemma 6.4, implies $\pi_\Gamma = \pi_\Delta$. The case where $\Delta/\Gamma \in \mathbb{Q}$ is trivial.

Having shown that $\pi_\Gamma = \pi_T$ for all $\Gamma \geq T$, it follows immediately that for every $t > 0$, $\pi P^t = \pi P^{T+t} = \pi$. \square

Proofs of theorems 1 and 2. Corollary 6.5 gives the existence of an invariant probability measure π for the continuous-time Markov chain Φ_t . Its uniqueness (among all invariant probability measures, not just measures in $L_V(\Omega)$) follows from proposition 6.1. This proposition shows that every invariant probability measure has a strictly positive density with respect to Λ , ruling out the existence of two distinct ergodic measures. Since $\pi \in L_V(\Omega)$ for $V = V^{(\alpha(T))}$ (corollary 6.5), it has the desired tail bounds with $\alpha = \alpha(T)$. \square

6.3. Exponential convergence

Here $V = V^{(\alpha(T))}$.

Proofs of theorem 3 and corollary 4. By lemmas 5.1 and 5.2, there exists B such that for all $t < T$, $P^t V \leq BV$, so $\mu \in L_V(\Omega)$ implies $\mu P^t \in L_V(\Omega)$. For arbitrary $t > 0$, the same conclusion is reached by writing $t = nT + \Delta$ for $n \in \mathbb{Z}^+$ and $\Delta \in [0, T)$, proving theorem 3(a). To prove theorem 3(b), let $\mu_1, \mu_2 \in L_V(\Omega)$ be given. Then, for $t = nT + \Delta$ as above, using (5.1) we obtain

$$\begin{aligned} \|\mu_1 P^t - \mu_2 P^t\|_V &= \|(\mu_1 P^\Delta) P^{nT} - (\mu_2 P^\Delta) P^{nT}\|_V \\ &\leq c\rho^n \cdot \|\mu_1 P^\Delta - \mu_2 P^\Delta\|_V \\ &\leq c\rho^n \cdot (B\|\mu_1 - \mu_2\|_V). \end{aligned}$$

Corollary 4(a) is a special case of theorem 3(b), with $\mu_1 = \delta_x$ and $\mu_2 = \pi$. The right side of the asserted inequality follows from

$$\|\delta_x - \pi\|_V \leq \int V(z)(\delta_x + \pi)(dz) = V(x) + \|\pi\|_V \leq (\|\pi\|_V + 1)V(x).$$

Finally, we prove corollary 4(b). From the definition of $C_{\xi, \zeta}^\mu(t)$, we have

$$\begin{aligned} |C_{\xi, \zeta}^\mu(t)| &= \left| \int \xi(x) \left((P^t \zeta)(x) - \int (P^t \zeta)(z) \mu(dz) \right) \mu(dx) \right| \\ &\leq \|\xi\| \int \left(\int |\zeta| |\delta_x P^t - \mu P^t| \right) \mu(dx) \\ &\leq \|\xi\| \|\zeta\|_V \int \|\delta_x P^t - \mu P^t\|_V \mu(dx) \\ &\leq \|\xi\| \|\zeta\|_V c\rho^t \int \|\delta_x - \mu\|_V \mu(dx) \end{aligned}$$

by theorem 3(b), and the integral above is $\leq \int (V(x) + \|\mu\|_V) \mu(dx) = 2\|\mu\|_V$. □

Acknowledgments

LSY was supported in part by NSF Grant DMS-1101594.

Appendix: Proof of proposition 2.2

We begin with a calculation for a simplified (but relevant) situation.

Lemma 6.6. Consider the case of an isolated site, in which a single trapped particle interacts with stored energy (this particle cannot exit and no other particle can enter). Assume that mixing occurs at rate $f_m(x)$. Then, all measures with densities

$$\frac{\sqrt{x}}{f_m(x)} \frac{1}{\sqrt{s}} e^{-\beta(s+x)}, \quad \text{any } \beta > 0,$$

are invariant.

Proof. We fix an arbitrary pair $(\bar{s}, \bar{x}) \in (0, \infty)^2$. For $u \in (0, 1)$, let $F_u : (0, \infty)^2 \rightarrow (0, \infty)^2$ denote the collision map from (2.3) corresponding to u , that is,

$$F_u(s, x) = ((xu^2, s + x(1 - u^2)).$$

A first observation is that (\bar{s}, \bar{x}) has an inverse image if, and only if, $u^2 \geq \bar{s}/(\bar{s} + \bar{x})$. To see this, let (s, x) be such that $F_u(s, x) = (\bar{s}, \bar{x})$. Then $\bar{s} = xu^2$ must be $\leq (\bar{s} + \bar{x})u^2$ (since $s + x = \bar{s} + \bar{x}$), implying the lower bound on u . To prove the lemma, it suffices to show

$$f_m(\bar{x}) \cdot \left(\frac{\sqrt{\bar{x}}}{f_m(\bar{x})} \frac{1}{\sqrt{\bar{s}}} e^{-\beta(\bar{s}+\bar{x})} \right) = \int_{\sqrt{\frac{\bar{s}}{\bar{s}+\bar{x}}}}^1 f_m(x) \cdot \frac{1}{|\det DF_u(s, x)|} \left(\frac{\sqrt{x}}{f_m(x)} \frac{1}{\sqrt{s}} e^{-\beta(s+x)} \right) du, \tag{6.3}$$

for the quantities on the left and right sides correspond respectively to the probabilities of leaving the configuration (\bar{s}, \bar{x}) and arriving at this configuration in infinitesimal time. A simple calculation gives the integrand as

$$\frac{1}{u^2} \sqrt{\frac{\bar{s}}{(\bar{s} + \bar{x})u^2 - \bar{s}}} e^{-\beta(\bar{s}+\bar{x})}.$$

Writing $a^2 = \frac{\bar{s}}{\bar{s}+\bar{x}}$, the right side of (6.3) is equal to

$$ae^{-\beta(\bar{s}+\bar{x})} \int_a^1 \frac{1}{u^2 \sqrt{u^2 - a^2}} du = ae^{-\beta(\bar{s}+\bar{x})} \frac{1}{a^2} \left[\sqrt{1 - \left(\frac{a}{u}\right)^2} \right]_a^1 = \sqrt{\frac{\bar{x}}{\bar{s}}} e^{-\beta(\bar{s}+\bar{x})}. \quad \square$$

We now return to the setting of proposition 2.2. The arguments are parallel to those in the ‘random halves’ model in [4] but the quantities that appear are different because the interaction is different.

Proof of proposition 2.2. Consider first the case of $N = 1$. Omitting the superscript in Ω^1 , we have $\Omega = \Omega_k \cup \Omega_k$. Fix arbitrary $\bar{z} = (s^*, \{x_1^*, \dots, x_k^*\}) \in \Omega$, and let

$$A := A(\bar{z}) = \{(\{x_1, \dots, x_k\}, s) \in \Omega_k \mid s^* < s < s^* + \epsilon; x_i^* < x_i < x_i^* + \epsilon, i = 1, \dots, N\}$$

where ϵ is very small. We will show that $\mu(A) = (\mu P^h)(A)$ for arbitrarily small $h > 0$.

Let E_1, E_2 and E_3 denote the following events:

- i E_1 : Exactly one particle is injected into the site.
- ii E_2 : Exactly one particle leaves the site.
- iii E_3 : Exactly one particle interacts with the stored energy.

Let $P(E_i, A)$ denote the probability that event E_i occurs on the time interval $(0, h)$ leading to a state in A , and let $\hat{P}(E_i, A)$ denote the probability that E_i occurs in $(0, h)$ starting from a state in A . It suffices to show that

$$\sum_{i=1}^3 P(E_i, A) - \sum_{i=1}^3 \hat{P}(E_i, A) = o(h).$$

Since the rate of energy exchange is proportional to f , the argument in lemma 6.6 balances exactly $P(E_3, A)$ and $\hat{P}(E_3, A)$. To balance the remaining terms, let σ_k and c_k be as in the Proposition, and let $p_k = \mu(\Omega_k)$. Then

$$P(E_1, A) = 2h\rho p_{k-1} c_{k-1} \epsilon^k \sum_{i=1}^k \left(\sigma_{k-1}(\check{z}[i]) \int_{x_i^*}^{x_i^*+\epsilon} \frac{2\beta^{3/2}}{\sqrt{\pi}} \sqrt{x} e^{-\beta x} dx \right) + o(h, \epsilon^{k+1})$$

where $\check{z}[i]$ is the point in Ω_{k-1} obtained from \bar{z} by removing x_i^* from the list of particle energies in \bar{z} . The value of the integral, which is the probability that the entering particle has energy in $[x_i^*, x_i^* + \epsilon)$, is combined with $\sigma_{k-1}(\check{z}[i])$ to give

$$P(E_1, A) = 2h\rho \cdot p_{k-1} c_{k-1} \sigma_k(\bar{z}) \epsilon^{k+1} \left(\frac{2\beta^{3/2}}{\sqrt{\pi}} \sum_{i=1}^k f(x_i^*) \right) + o(h, \epsilon^{k+1}).$$

Likewise, we obtain

$$\begin{aligned}
 P(E_2, A) &= hp_{k+1}c_{k+1}\sigma_k(\bar{z})\epsilon^{k+1}\frac{\sqrt{\pi}}{2\beta^{3/2}} + o(h, \epsilon^{k+1}); \\
 \hat{P}(E_1, A) &= 2hp_k c_k \sigma_k(\bar{z})\epsilon^{k+1}\rho + o(h, \epsilon^{k+1}); \\
 \hat{P}(E_2, A) &= hp_k c_k \sigma_k(\bar{z})\epsilon^{k+1}\left(\sum_{i=1}^k f(x_i^*)\right) + o(h, \epsilon^{k+1}).
 \end{aligned}$$

Writing

$$X := \int_0^\infty \frac{\sqrt{x}}{f(x)} e^{-\beta x} dx \quad \text{and} \quad I_s := \int_0^\infty \frac{1}{\sqrt{s}} e^{-\beta s} ds,$$

and recalling that

$$\frac{c_k}{k!} X^k I_s = 1 \quad \text{and} \quad p_k = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{where} \quad \lambda = \frac{4}{\sqrt{\pi}} \beta^{3/2} X \rho,$$

we obtain that modulo terms of size $o(h, \epsilon^{k+1})$,

$$\begin{aligned}
 P(E_1, A) &= 2h\rho \cdot \frac{k}{\lambda} p_k \cdot \frac{X}{k} c_k \cdot \sigma_k(\bar{z})\epsilon^{k+1} \cdot \frac{2\beta^{3/2}}{\sqrt{\pi}} \sum_{i=1}^k f(x_i^*) = \hat{P}(E_2, A), \\
 P(E_2, A) &= h \cdot \frac{\lambda}{k+1} p_k \cdot \frac{k+1}{X} c_k \cdot \sigma_k(\bar{z})\epsilon^{k+1} \cdot \frac{\sqrt{\pi}}{2\beta^{3/2}} = \hat{P}(E_1, A).
 \end{aligned}$$

Moving on to the case of arbitrary N , we pick a point

$$\hat{z} = ((\hat{s}_1, \{\hat{x}_1^1, \dots, \hat{x}_{k_1}^1\}), (\hat{s}_2, \{\hat{x}_1^2, \dots, \hat{x}_{k_2}^2\}), \dots, (\hat{s}_N, \{\hat{x}_1^N, \dots, \hat{x}_{k_N}^N\})) \in \Omega,$$

let A be an ϵ -box at \hat{z} as before, and consider events of the type $E_{j,j\pm 1}$, meaning a particle jumps from site j to site $j \pm 1$ on the time interval $(0, h)$. We claim that $P(E_{n+1,n}, A)$, the probability that $E_{n+1,n}$ occurs resulting in a state in A , is balanced by $\hat{P}(E_{n,n+1}, A)$, the probability that $E_{n,n+1}$ occurs starting from a state in A , both being equal to

$$\frac{h}{2} \cdot \prod_{i=1}^N p_{k_i} c_{k_i} \epsilon^{k_i+1} \cdot \left(\sum_{i=1}^{k_n} f(\hat{x}_i^n)\right) + o(h, \epsilon^{\sum_{i=1}^N (k_i+1)}).$$

The computations are similar to those in the $N = 1$ case. □

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