Problem 1
Find the least positive integer $n$ so that the $n$th degree Taylor Polynomial $P_n(x)$ in $x$ approximates $\sin 0.3$ to within 0.01.

Solution: The $2n + 1$ reminder of $f(x) = \sin x$ is

$$R_{2n+1}(x) = \frac{f^{2n+3}(c)}{(2n+3)!}x^{2n+3}$$

We want $|R_{2n+1}(0.3)| < 0.01$. Since $|\sin x^{(n)}| \leq 1$ for all $n$ and $x$, we only need to find the least $n$ such that $0.3^{2n+3} < 0.01$. It’s easy to see $n = 0$ is the least integer, and $2n + 1 = 1$. So the 1st degree Taylor Polynomial of $\sin x$ approximates $\sin 0.3$ to within 0.01.

Problem 2
Find $\lim_{x \to +\infty}(1 + e^{2x})^{1/x}$.

Solution: This is the $\infty^0$ form, so we take the natural log first.

$$\lim_{x \to +\infty} \ln(1 + e^{2x})^{1/x} = \lim_{x \to +\infty} \frac{1}{x} \ln(1 + e^{2x})$$

$$= \lim_{x \to +\infty} \frac{1}{x} \left( \frac{2e^{2x}}{1 + e^{2x}} \right)$$

$$= \lim_{x \to +\infty} \frac{2e^{2x}}{x(e^{2x} + 1)}$$

(applying L’Hospital’s Rule)

$$= \lim_{x \to +\infty} \frac{4e^{2x}}{2xe^{2x}}$$

(applying L’Hospital’s Rule again)

$$= 2$$

So $\lim_{x \to +\infty}(1 + e^{2x})^{1/x} = e^2$. 