Graphs containing topological $H$

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Abstract

Let $H$ denote the tree with six vertices two of which are adjacent and of degree three. Let $G$ be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of $G$. We characterize those $G$ that contain a topological $H$ in which $u_1, u_2$ are of degree three and $a_1, a_2, a_3, a_4$ are of degree one, which include all 5-connected graphs. This work was motivated by the Kelmans–Seymour conjecture that 5-connected nonplanar graphs contain topological $K_5$.

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1 Introduction

The work in this paper was motivated by the well known conjecture of Kelmans [7] and Seymour [14]: Every 5-connected nonplanar graph contains a topological $K_5$ (i.e., subdivision of $K_5$). Earlier, Dirac [3] conjectured an extremal function for the existence of a topological $K_5$: If $G$ is a simple graph with $n \geq 3$ vertices and at least $3n - 5$ edges then $G$ contains a topological $K_5$. This conjecture was established by Mader [13]. Kézdy and Mcguiness [8] showed that the Kelmans-Seymour conjecture, if true, implies Mader’s result. (It is easy to see that Mader’s theorem does not hold if multiple edges are allowed. However, multiple edges do not make a difference for the Kelmans-Seymour conjecture. So in this paper we will consider only simple graphs, and we delete multiple edges which result from graph operations.)

The Kelmans-Seymour conjecture is also related to the $k = 4$ case of the Hajós conjecture (see [2]) that every graph containing no topological $K_{k+1}$ is $k$-colorable. Hajós’ conjecture is false for $k \geq 6$ [2, 4] and true for $k = 1, 2, 3$, and remains open for $k = 4$ and $k = 5$.

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An approach to the Kelmans-Seymour conjecture is to study the so called rooted $K_4$ problem. Given a graph $G$ and four distinct vertices $x_1, x_2, x_3, x_4$ of $G$, when does $G$ contain a topological $K_4$ in which $x_1, x_2, x_3, x_4$ are the vertices of degree three? This problem was solved for planar graphs (see [16]), and the result was used by Aigner-Horev [1] to prove the Kelmans-Seymour conjecture for apex graphs. A different and shorter proof for the apex case was found independently by Kawarabayashi [6] and Ma, Thomas and Yu [10].

One important step in [16] is to solve the following problem for planar graphs: Let $H$ represent the tree on six vertices two of which are adjacent and of degree 3. (See Figure 1.) Let $G$ be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of $G$. When does $G$ contain a topological $H$ in which $u_1, u_2$ are of degree 3 and $a_1, a_2, a_3, a_4$ are of degree 1? We say that such a topological $H$ is rooted at $u_1, u_2, \{a_1, a_2, a_3, a_4\}$. For convenience, we use quadruple to denote $(G, u_1, u_2, A)$ where $u_1, u_2$ are distinct vertices of a graph $G$, $A \subseteq V(G) - \{u_1, u_2\}$, and $|A| = 4$. We say that $(G, u_1, u_2, A)$ is feasible if $G$ has a topological $H$ rooted at $u_1, u_2, A$.

![Fig. 1: The graph $H$.](image)

The main result of this paper is a characterization of feasible quadruples, which implies the following theorem whose proof is given after the full statement of the characterization in Section 2 (see Theorem 2.1).

**Theorem 1.1.** $(G, u_1, u_2, A)$ is feasible when $G$ is 5-connected.

The connectivity in Theorem 1.1 is tight. Let $G$ be obtained from $K_6$ by deleting the edge between two vertices $u_1, u_2$, and let $A = V(G) - \{u_1, u_2\}$; then $G$ is 4-connected and $(G, u_1, u_2, A)$ is not feasible.

In Section 2, we describe the obstructions to feasibility of quadruples (there are four types) and state the main result (Theorem 2.1). In Section 3, we consider a related problem about the existence of $k$ disjoint paths in a graph between two given sets of vertices and containing a given edge. We solve the case $k = 3$ which will be used to characterize quadruples. In Section 4, we deal with those quadruples $(G, u_1, u_2, A)$ in which $G$ admits certain cuts of size at most 3. In Section 5, we study quadruples containing critical pairs, i.e., quadruples $(G, u_1, u_2, A)$ in which there exist distinct $x, y \in V(G) - A - \{u_1, u_2\}$ such that $(G/xy, u_1, u_2, A)$ is an obstruction (where $G/xy$ is obtained from $G$ by identifying $x$ and $y$ and removing loops or multiple edges). In Section 6, we deal with the case when $G/xy$ has a certain cut of size at most 4, which reduces to the case when $G$ has a certain cut of size 5. The proof is then completed in Section 7 by finding an appropriate edge $xy \in E(G - A - \{u_1, u_2\})$ such that $\{x, y\}$ is a critical pair.
We devote the rest of this section to notation and terminology. Let $G$ be a graph. (We remind the reader that only simple graphs are considered in this paper.) By $S \subseteq G$ we mean that $S$ is a subgraph of $G$. For $S \subseteq G$, we use $G[S]$ to denote the subgraph of $G$ induced by $V(S)$. For any $x \in V(G)$ we use $N_G(x)$ to denote the neighborhood of $x$ in $G$, and for $S \subseteq G$ let $N_G(S) = \{x \in V(G) - V(S) : N_G(x) \cap V(S) \neq \emptyset\}$ and $N_G[S] = V(S) \cup N_G(S)$. When understood, the reference to $G$ may be dropped. For any $S \subseteq E(G)$, $G - S$ denotes the graph obtained from $G$ by deleting all edges in $S$. For any $S \subseteq V(G)$, $G - S$ denotes the graph obtained from $G$ by deleting $S$ and all edges of $G$ incident with $S$.

A separation in a graph $G$ consists of a pair of subgraphs $G_1, G_2$, denoted as $(G_1, G_2)$, such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$ and, for $i = 1, 2$, $V(G_i) - V(G_{3-i}) \neq \emptyset$ or $E(G_i) \neq \emptyset$. (Thus, we allow $V(G_i) - V(G_{3-i}) = \emptyset$, but if this happens we require $E(G_i) \neq \emptyset$.) The order of this separation is $|V(G_1 \cap G_2)|$, and $(G_1, G_2)$ is said to be a $k$-separation if its order is $k$. Thus, a set $S \subseteq V(G)$ is a $k$-cut (or a cut of size $k$) in $G$, where $k$ is a positive integer, if $|S| = k$ and $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = S$ and $V(G_i) - S \neq \emptyset$ for $i = 1, 2$; if $v \in V(G)$ and $\{v\}$ is a cut of $G$, then $v$ is said to be a cut vertex of $G$.

Given a path $P$ in a graph and $x, y \in V(P)$, $xPy$ denotes the subpath of $P$ between $x$ and $y$ (inclusive). We may view paths as sequences of vertices; thus if $P$ is a path between $x$ and $y$, $Q$ is a path between $y$ and $z$, and $V(P \cap Q) = \{y\}$, then $PyQ$ denotes the path $P \cup Q$. The ends of the path $P$ are the vertices of the minimum degree in $P$, and all other vertices of $P$ (if any) are its internal vertices. A path $P$ with ends $u$ and $v$ (or an $u$-$v$ path) is also said to be from $u$ to $v$ or between $u$ and $v$. Let $G$ be a graph. A collection of paths in $G$ are said to be independent if no vertex of any path in this collection is an internal vertex of any other path in the collection. A path $P$ in $G$ is said to be internally disjoint from a subgraph $Q$ of $G$ if no internal vertex of $P$ belongs to $Q$.

Let $G$ be a graph. Let $K \subseteq G$, $S \subseteq V(G)$, and $T$ a collection of $2$-element subsets of $V(K) \cup S$; then $K + (S \cup T)$ denotes the graph with vertex set $V(K) \cup S$ and edge set $E(K) \cup T$, and if $S = \emptyset$ and $T = \{(x, y)\}$ we write $K + xy$ instead of $K + \{(x, y)\}$.

## 2 Obstructions

We refer the reader to Figures 2 and 3 for intuition on the following discussions about obstructions. We will show that modulo certain separations there will be just four types of obstructions.

A quadruple $(G, u_1, u_2, A)$ is an obstruction if $G$ has subgraphs $U_1, U_2$ (called sides) and $A_i, i \in [k] := \{1, 2, \ldots, k\}$ (called middle parts), such that

1. $V(G) = V(U_1) \cup V(U_2) \cup A_{[k]}$, where $A_{[k]} = \cup_{i \in [k]} V(A_i)$,
2. $V(A_i), i \in [k]$, are vertex-disjoint,
3. $E(G - A)$ is the disjoint union of $E(U_1 - A)$, $E(U_2 - A)$, and $E(A_i - A)$ (for $i \in [k]$),
4. $V(U_1 \cap U_2) \subseteq A \subseteq A_{[k]}$, $u_1 \in V(U_1) - A_{[k]}$, and $u_2 \in V(U_2) - A_{[k]}$,
5. for any $i \in [k]$, $V(A_i) \cap A \neq \emptyset$, and either $|V(A_i) \cap V(U_1 \cup U_2)| = |V(A_i) \cap A| + 1$ or $|V(A_i)| = 1$ and $V(A_i) \subseteq V(U_1 \cup U_2) \cap A$. 


(6) if $|V(A_i)| \geq 2$, then $V(A_i) \cap V(U_1 \cup U_2) \cap A = \emptyset$ and $N_G(V(A_i) \cap A) \subseteq V(A_i)$.

Note that $V(A_i) \cap V(U_1 \cup U_2) \cap A \neq \emptyset$ iff $|V(A_i)| = 1$, in which case there is no restriction on $N(A_i)$.

To see that obstructions are not feasible, let $(G, u_1, u_2, A)$ be an obstruction, $J$ a topological $H$ in $G$ rooted at $u_1, u_2, A$, and $P$ the $u_1$-$u_2$ path in $J$. By definition, $V(P) \cap A = \emptyset$ and (in particular, by (4)) $P$ has to pass through some $A_i$ with $|V(A_i)| \geq 2$; so $|V(P) \cap V(A_i) \cap V(U_1 \cup U_2)| \geq 2$. Also $J - V(P - \{u_1, u_2\})$ contains $|V(A_i) \cap A|$ independent paths from \{u_1, u_2\} to $V(A_i) \cap A$; so $|\{V(J) - V(P)\} \cap V(A_i) \cap V(U_1 \cup U_2)| \geq |V(A_i) \cap A|$.

Thus, $|V(A_i) \cap V(U_1 \cup U_2)| \geq |V(A_i) \cap A| + 2$, contradicting (5).

![Fig. 2: Obstructions of type I and type II](image)

An obstruction $(G, u_1, u_2, A)$ is said to be of type I if $k = 3$, $|V(A_i) \cap A| = 1$ for $i = 1, 2$, $|V(A_3) \cap A| = 2$, $|V(U_i \cap A_j)| = 1$ for $(i, j) \neq (1, 3)$, and $|V(U_1 \cap A_3)| = 2$.

An obstruction $(G, u_1, u_2, A)$ is said to be of type II if $k = 2$, $|V(A_1) \cap A| = 1$, $|V(A_2) \cap A| = 3$, and for $i = 1, 2$, $|V(U_i \cap A_1)| = 1$ and $|V(U_i \cap A_2)| = 2$.

![Fig. 3: Obstructions of types III and IV](image)

An obstruction $(G, u_1, u_2, A)$ is said to be of type III if $k = 2$, $|V(A_i) \cap A| = 2$ for $i = 1, 2$, $|V(U_1 \cap A_1)| = |V(U_2 \cap A_2)| = 1$, and $|V(U_1 \cap A_2)| = |V(U_2 \cap A_1)| = 2$.

An obstruction $(G, u_1, u_2, A)$ is said to be of type IV if $k = 4$ and, for $1 \leq i \leq 4$ and $j \in \{1, 2\}$, $|V(A_i) \cap A| = |V(U_j \cap A_i)| = 1$.

**Theorem 2.1.** Let $(G, u_1, u_2, A)$ be a quadruple. Then one of the following holds:

(i) $(G, u_1, u_2, A)$ is feasible.
(ii) $G$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 2$ and for some $i \in \{1, 2\}$, $u_i \in V(K) - V(L)$ and $A \cup \{u_3-i\} \subseteq V(L)$.

(iii) $G$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K) - V(L)$, and $A \subseteq V(L)$.

(iv) $(G, u_1, u_2, A)$ is an obstruction of type I, or II, or III, or IV.

Note that (ii) implies that $(G, u_1, u_2, A)$ is not feasible, and (iii) implies that $(G, u_1, u_2, A)$ is not feasible, or when $|V(K \cap L)| = 4$ the feasibility of $(G, u_1, u_2, A)$ reduces to $(K, u_1, u_2, V(K \cap L))$.

To see that Theorem 2.1 implies Theorem 1.1, we apply Theorem 2.1 to the quadruple $(G - E(G[A]), u_1, u_2, A)$. Since $G$ is 5-connected, (ii), (iii) and (iv) of Theorem 2.1 do not hold for $(G - E(G[A]), u_1, u_2, A)$. Hence $(G - E(G[A]), u_1, u_2, A)$ is feasible. Since any topological $H$ in $G - E(G[A])$ rooted at $u_1, u_2, A$ is also a topological $H$ in $G$ rooted at $u_1, u_2, A$, we see that $(G, u_1, u_2, A)$ must be feasible.

3 Disjoint paths containing a given edge

In this section we prove a result about the existence of disjoint paths from three given vertices to three other given vertices such that a specific edge is used by one of these paths. This result will be used several times in the proof of Theorem 2.1. The problem for finding two disjoint paths between two pairs of vertices and through a given edge is equivalent to the problem for finding a cycle through three given edges. The following result is due to Lovász [9].

**Lemma 3.1** (Lovász). Let $G$ be a 3-connected graph and $e_1, e_2, e_3$ be distinct edges of $G$ not all incident with a common vertex. Then $G$ contains a cycle through $e_1, e_2, e_3$ iff $G - \{e_1, e_2, e_3\}$ is connected.

We need an easy generalization of Lemma 3.1. For a subgraph $K$ of a graph $G$, a $K$-bridge of $G$ is a subgraph of $G$ that is induced either by an edge of $G - E(K)$ with both ends in $K$, or by all edges in a component of $G - V(K)$ and all edges from that component to $K$. The $K$-bridges of the latter type are said to be nontrivial.

**Lemma 3.2.** Let $e_1, e_2, e_3$ be distinct edges of a graph $G$ not all incident with a common vertex. Then one of the following holds:

(i) $\{e_1, e_2, e_3\}$ is contained in a cycle in $G$.

(ii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 2$, $V(G_i) - V(G_{3-i}) \neq \emptyset$ for $i = 1, 2$, and $|E(G_i) \cap \{e_1, e_2, e_3\}| = 1$ for some $i \in \{1, 2\}$.

(iii) $\{e_1, e_2, e_3\}$ is contained in a component $H$ of $G$, and $H - \{e_1, e_2, e_3\}$ is not connected.

**Proof.** Suppose the assertion is false, and choose a counterexample $G,e_1, e_2, e_3$ such that $|V(G)|$ is minimum. Then $G$ is connected, or else (ii) holds or we get a smaller counterexample. Moreover, $G$ is not 3-connected, as otherwise (i) or (iii) holds by Lemma 3.1. So let $T$ be
a cut in \( G \) with \(|T| \leq 2\). Since \( G \) has at least two nontrivial \( T \)-bridges, we may assume that \( B \) is a nontrivial \( T \)-bridge of \( G \) such that \(|E(B) \cap \{e_1, e_2, e_3\}| \leq 1\). If \(|E(B) \cap \{e_1, e_2, e_3\}| = 1\) then (ii) holds. So \( E(B) \cap \{e_1, e_2, e_3\} = \emptyset \). If \(|T| = 1\) let \( G' := G - V(B - T) \), and if \(|T| = 2\) let \( G' \) be obtained from \( G - V(B - T) \) by adding an edge between the vertices in \( T \). Now by the choice of \( G, e_1, e_2, e_3 \), we see that (i) or (ii) or (iii) holds for \( G', e_1, e_2, e_3 \). It is straightforward to verify that (i) or (ii) or (iii) holds for \( G, e_1, e_2, e_3 \). \( \blacksquare \)

The following figure gives illustrations of conclusions (i) – (v) of Lemma 3.3. Note that there are three pairs of vertices \( \{v_1, v_2\}, \{w_1, w_2\} \) and \( \{a_1, a_2\} \) in the statement of Lemma 3.3. These pairs appear symmetric in the first part of the statement; however, we state the second part of the lemma according to the locations of vertices \( a_1, a_2 \), to facilitate later applications where \( \{a_1, a_2\} \) will play different roles than \( \{v_1, v_2\} \) and \( \{w_1, w_2\} \).

**Fig. 4:** The separation \((G_1, G_2)\) in Lemma 3.3.

**Lemma 3.3.** Let \( G \) be a graph and \( v_1, v_2, w_1, w_2, a_1, a_2 \in V(G) \) be distinct such that \( a_1a_2, v_1v_2, w_1w_2 \notin E(G) \). Then \( G \) has three disjoint paths with one from \( \{v_1, v_2\} \) to \( \{w_1, w_2\} \), one from \( \{v_1, v_2\} \) to \( \{a_1, a_2\} \), and another from \( \{w_1, w_2\} \) to \( \{a_1, a_2\} \), or \( G \) has a separation \((G_1, G_2)\) such that one of the following holds:

(i) \(|V(G_1 \cap G_2)| \leq 2\), \( \{a_1, a_2\} \subseteq V(G_1) \), and for some \( i \in \{1, 2\} \), \( \{v_1, v_2\} \subseteq V(G_i) \) and \( \{w_1, w_2\} \subseteq V(G_{3-i}) \).

(ii) \(|V(G_1 \cap G_2)| \leq 2\), \( \{a_1, a_2\} \subseteq V(G_1) \), and \( \{v_1, v_2, w_1, w_2\} \subseteq V(G_2) \).

(iii) \( G_1 \cap G_2 = \emptyset \), \( a_1 \in V(G_1) \), \( a_2 \in V(G_2) \), and for some \( i \in \{1, 2\} \), \( \{v_1, v_2\} \subseteq V(G_i) \) and \( \{w_1, w_2\} \subseteq V(G_{3-i}) \).

(iv) \( G_1 \cap G_2 = \emptyset \), \( a_1 \in V(G_1) \), \( a_2 \in V(G_2) \), and for \( i \in \{1, 2\} \), \(|\{v_1, v_2\} \cap V(G_i)| = |\{w_1, w_2\} \cap V(G_i)| = 1\).

(v) \( G_1 \cap G_2 = \emptyset \), \( \{a_1, a_2\} \subseteq V(G_1) \), and \(|\{v_1, v_2, w_1, w_2\} \cap V(G_1)| = 3\).

**Proof.** Let \( G' = G + \{a_1a_2, v_1v_2, w_1w_2\} \) and apply Lemma 3.2 to \( G', a_1a_2, v_1v_2, w_1w_2 \). If Lemma 3.2(i) holds, i.e., \( G' \) contains a cycle \( C \) containing \( a_1a_2, v_1v_2 \) and \( w_1w_2 \), then \( C - \{a_1a_2, v_1v_2, w_1w_2\} \) gives the desired paths in \( G \). If Lemma 3.2(ii) holds then let \((G'_1, G'_2)\) be a separation in \( G' \) such that \(|V(G'_1 \cap G'_2)| \leq 2\), \( V(G'_1) - V(G'_2) \neq \emptyset \), and \(|E(G'_1) \cap \{a_1a_2, v_1v_2, w_1w_2\}| = 1\); then (i) holds if \( \{v_1v_2, w_1w_2\} \cap E(G'_1) \neq \emptyset \), and (ii) holds if \( a_1a_2 \in E(G'_1) \). So assume that Lemma 3.2(iii) holds. Then \( G \) is the disjoint union of two graphs \( G_1 \) and \( G_2 \).
and \(G_2\), and one of the pairs \(\{a_1, a_2\}, \{v_1, v_2\}, \{w_1, w_2\}\) has one element in \(G_1\) and another in \(G_2\).

Suppose \(a_1 \in V(G_1)\) and \(a_2 \in V(G_2)\). If there exists \(i \in \{1, 2\}\) such that \(\{v_1, v_2, w_1, w_2\} \cap V(G_i) \leq 1\), then \((G_i + a_3, G_3 \setminus \{v_1, v_2, w_1, w_2\})\) shows that (ii) holds. If \(\{v_1, v_2, w_1, w_2\} \cap V(G_i) = 2\) for \(i = 1, 2\) then (iii) or (iv) holds.

So assume (by symmetry) that \(a_1, a_2, v_1 \in V(G_1)\) and \(v_2 \in V(G_2)\). If \(|\{w_1, w_2\} \cap V(G_1)| \leq 1\) then \((G_1, G_2 + \{v_1, w_1, w_2\})\) shows that (ii) holds; if \(\{w_1, w_2\} \subseteq V(G_1)\) then (v) holds.

In general one could ask the following question. Given two disjoint \(k\)-sets of vertices \(A, B\) and an edge \(e\) in a graph \(G\), when does \(G\) contain \(k\) disjoint paths from \(A\) to \(B\) and passing through \(e\)? The main result of this section is an answer to this question for \(k = 3\). Note that when (i) of Lemma 3.4 occurs, the desired paths do not exist if \(|V(G_1 \cap G_2)| \leq 2\), and the problem reduces to the smaller graphs \(G_1\) or \(G_2\) if \(|V(G_1 \cap G_2)| = 3\).

**Fig. 5:** The separations in Lemma 3.4.

**Lemma 3.4.** Let \(G\) be a graph, \(A, B \subseteq V(G)\) be disjoint, and \(e \in E(G)\) such that \(|A| = |B| = 3\) and \(V(e) \cap (A \cup B) = \emptyset\). Then \(G\) has three disjoint paths from \(A\) to \(B\) and through \(e\), or one of the following holds:

(i) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 3\), \(A \subseteq V(G_1)\), and \(B \subseteq V(G_2)\).

(ii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 1\), \(e \in E(G_1)\), and \(A \cup B \subseteq V(G_2)\).

(iii) \(G = G_1 \cup G_2 \cup G_3\) such that \(G_1 \cap G_3 = \emptyset\), \(e \in E(G_2)\), \(|V(G_i \cap G_j)| \leq 1\) for \(i = 1, 3\), \(|V(G_1) \cap A| = |V(G_1) \cap B| = 1\), and \(|V(G_3) \cap A| = |V(G_3) \cap B| = 2\).

(iv) \(G = G_1 \cup G_2 \cup G_3 \cup G_4\) such that \(e \in E(G_1), V(G_i \cap G_j) = \emptyset\) for \(2 \leq i < j \leq 4\), and \(|V(G_i \cap G_j)| = |V(G_i) \cap A| = |V(G_i) \cap B| = 1\) for \(i \in \{2, 3, 4\}\).

**Proof.** We may assume that \(A, B\) are independent sets in \(G\), as otherwise (i) holds. We may also assume that \(G\) has three disjoint paths \(P_1, P_2, P_3\) from \(A\) to \(B\), or else (i) follows from Menger’s theorem. Let \(P := \bigcup_{i=1}^{3} P_i\). We may assume that \(e \notin E(P)\) for any choice of \(P\); for, otherwise, \(G\) has three disjoint paths from \(A\) to \(B\) and through \(e\). Let \(L_P\) denote the \(P\)-bridge of \(G\) containing \(e\). We choose \(P\) (i.e., \(P_1, P_2, P_3\)) so that

1. \(L_P\) is maximal.

Let \(A = \{a_1, a_2, a_3\}\) and \(B = \{b_1, b_2, b_3\}\) such that \(P_i\) is from \(a_i\) to \(b_i\) for \(i = 1, 2, 3\). Let \(x_i, y_i \in V(P_i \cap L_P)\) (if not empty) such that \(x_i, y_i\) is maximal and \(a_i, x_i, y_i, b_i\) occur on \(P_i\)
in this order. For convenience, let $L' := L_P - V(P \cap L_P)$ and let $L_i := G[L' \cup x_iP_iy_i]$ for $i = 1, 2, 3$.

(2) If $x_i, y_i$ are defined then no $P$-bridge of $G$ intersects both $a_iP_ix_i - x_i$ and $x_iP_ib_i - x_i$, or both $a_iP_iy_i - y_i$ and $y_iP_ib_i - y_i$. For suppose $G$ has a $P$-bridge $J$ intersecting both $a_iP_ix_i - x_i$ and $x_iP_ib_i - x_i$. Then $J \not\subseteq L_P$, and $J$ contains a path $Q_i$ from some $u_i \in V(a_iP_ix_i - x_i)$ to some $v_i \in V(x_iP_ib_i - x_i)$ and internally disjoint from $P \cup L_P$. Let $P' := (P - V(P_i)) \cup a_iP_iu_iQ_i, v_iP_ib_i$.

Then the $P'$-bridge of $G$ containing $e$ contains $L_P + x_i$, contradicting (1).

(3) If $x_i, y_i$ are defined and $L_i$ has a separation $(L_{i1}, L_{i2})$ such that $|V(L_{i1} \cap L_{i2})| = 1$, $x_i, y_i \in V(L_{i1})$, and $e \in E(L_{i2})$, we choose $(L_{i1}, L_{i2})$ so that $L_{i2}$ is minimal, and let $w_i \in V(L_{i1} \cap L_{i2})$. If $x_i, y_i$ are defined and the above separation does not exist, then we may assume $x_i = y_i$; as otherwise, $L_i$ contains a path $Q_i$ from $x_i$ to $y_i$ and through $e$, and hence $(P - V(P_i)) \cup a_iP_ix_iQ_iy_iP_ib_i$ gives the desired paths. In this latter case, we set $w_i = x_i = y_i$, and let $L_{i1}$ consist of $w_i$ only, and $L_{i2} = L_i$.

(4) We may assume that $w_i, x_i, y_i$, $i = 1, 2$, are defined, and $w_1 \neq w_2$. To see this, let $I = \{i : w_i, x_i, y_i \text{ are defined}\}$. If $I = \emptyset$ then the separation $(L_P, G - L_P)$ shows that (ii) holds. So assume $I \neq \emptyset$. Thus, if (4) is not true then $|I| = 1$ or $w_i = w_j$ for all $i, j \in I$; so the separation $(\cap_{i \in I} L_{i2}, G - \cap_{i \in I} V(L_{i2} - w_i))$ shows that (iii) holds.

By (4) and by the minimality of $L_{i2}$ for $i = 1, 2$ (see (3)), $L_P - V(P - \{w_1, w_2\})$ contains a path from $w_1$ to $w_2$ through $e$ and internally disjoint from $P$; hence $L_{11}, L_{21}$ are disjoint. So for $\{i, j\} = \{1, 2\}$, $L_P$ contains a path $Q_{ij}$ from $x_i$ to $y_j$, through $e$, and internally disjoint from $P$.

(5) We may assume that no $P$-bridge of $G$ other than $L_P$ intersects both $a_1P_1y_1 - y_1$ and $x_2P_2b_2 - x_2$, or both $a_2P_2y_2 - y_2$ and $x_1P_1b_1 - x_1$. Otherwise, by symmetry assume that some $P$-bridge $J$ of $G$, $J \not\subseteq L_P$, intersects both $a_1P_1y_1 - y_1$ and $x_2P_2b_2 - x_2$. Then $J$ contains a path $Q$ from some $u \in V(a_1P_1y_1 - y_1)$ to some $v \in V(x_2P_2b_2 - x_2)$ and internally disjoint from $P \cup L_P$. Now $a_1P_1uQvP_2b_2, a_2P_2x_2Q_2y_1P_1b_1, P_3$ are three disjoint paths from $A$ to $B$ and through $e$.

Case 1. $w_3, x_3, y_3$ are defined.

Suppose $w_3 \in \{w_1, w_2\}$. Then by the same argument following (4), we may assume that for any $1 \leq i \neq j \leq 3$, $L_P$ has a path $Q_{ij}$ from $x_i$ to $y_j$, through $e$ and internally disjoint from $P$, and (5) holds for any $P_i, P_j$ with $i \neq j$. Thus, $(x_1, x_2, x_3)$ or $(y_1, y_2, y_3)$ separates $A$ from $B$ (i.e. (iii) holds); or $(x_1, x_2, x_3) = \{a_1, a_2, a_3\}$, $(y_1, y_2, y_3) = \{b_1, b_2, b_3\}$, and no $P$-bridge of $G$ other than $L_P$ contains two of $(x_1, x_2, x_3)$ or two of $(y_1, y_2, y_3)$. In the latter case, (iv) holds with $G_1 = L_{12} \cap L_{22} \cap L_{32}, L_{11} \cup P_1 \subseteq G_2, L_{21} \cup P_2 \subseteq G_4, L_{31} \cup P_3 \subseteq G_3$. Thus, by symmetry assume $w_3 = w_2$.

Hence, again by the same argument following (4), for all $\{i, j\} \neq \{2, 3\}$, $L_P$ has a path $Q_{ij}$ from $x_i$ to $y_j$ through $e$ and internally disjoint from $P$, and we may assume that

(*) no $P$-bridge of $G$ other than $L_P$ intersects both $a_1P_1y_1 - y_1$ and $(x_2P_2b_2 - x_2) \cup (x_3P_3b_3 - x_3)$, or both $x_1P_1b_1 - x_1$ and $(a_2P_2y_2 - y_2) \cup (a_3P_3y_3 - y_3)$.

If no $P$-bridge other than $L_P$ intersecting $P_1$ also intersects $P_2 \cup P_3$, then (iii) holds with $G_2 = L_{12} \cap L_{22}, L_{11} \cup P_1 \subseteq G_1, L_{21} \cup L_{31} \cup P_2 \cup P_3 \subseteq G_3$. So assume that $G$ has a path $Q$
from some $u_1 \in V(P_1)$ to some $u_2 \in V(P_2 \cup P_3)$ and internally disjoint from $P \cup L_P$. Note that if for every choice of $Q$, we have $u_1 = x_1 = y_1$ then, since $a_1 \neq b_1$, $\{u_1, a_2, a_3\}$ or $\{u_1, b_2, b_3\}$ is a cut in $G$ separating $A$ from $B$; so (i) holds. Hence, by symmetry, assume $u_1 \in V(a_1P_1y_1 - y_1)$. Then by $(\ast)$, $u_2 \in V(a_2P_2x_2 \cup a_3P_3x_3)$. By symmetry, let $u_2 \in V(a_2P_2x_2)$.

First, assume that $Q$ may be chosen so that $u_1 \in V(x_1P_1y_1 - (x_1, y_1))$. Then by $(\ast)$, $x_2 = y_2 = u_2$. Since $a_2 \neq b_2$, we may let $a_2 \neq x_2$ (by symmetry). If $\{x_1, x_2, x_3\}$ is a cut in $G$ separating $A$ from $B$ then (i) holds. So by $(2)$ and $(\ast)$, $G$ has a path $R$ internally disjoint from $L_P \cup P \cup Q$, which is from some $r \in V(a_2P_2x_2 - x_2)$ to some $s \in V(x_3P_3b_3 - x_3)$, or from some $r \in V(x_2P_2b_2 - x_2)$ to some $s \in V(a_3P_3x_3 - x_3)$. In the former case, $a_1P_1u_1Q_u_2P_2P_2b_2, a_2P_2RsP_3b_3, a_3P_3x_3Q_3y_1P_1b_1$ are disjoint paths from $A$ to $B$ and through $e$. In the latter case, $a_1P_1x_1Q_1x_3y_3P_3b_3, a_2P_2u_2Q_u_1P_1b_1, a_3P_3sRb_2P_2b_2$ are disjoint paths from $A$ to $B$ and through $e$.

Therefore, we may assume $u_1 \in V(a_1P_1x_1 - y_1)$. Thus, $Q$ implies the existence of a path $Q'$ in $G$ from some $v_2 \in V(a_2P_2x_2)$ to some $v_1 \in V(a_1P_1x_1 - y_1) \cup V(a_3P_3x_3 - x_3)$ and internally disjoint from $P \cup L_P$, and we choose $Q'$ with $v_2P_2v_2$ minimal. Let $v_3 \in P_3$ with $v_2P_2v_3$ maximal such that $v_3 = a_3$, or $G$ contains a path $R$ from $v_3$ to some $r \in V(a_1P_1x_1 - x_1) \cup V(a_2P_2v_2) - v_2$ and internally disjoint from $P \cup L_P$.

Suppose $v_3 \in V(x_2P_2b_3 - x_3)$; so $R$ is defined. By $(2)$ and $(\ast)$, $R \cap Q' = \emptyset$; and by $(\ast)$, $r \in V(a_2P_2v_2 - v_2)$. If $v_1 \in V(a_1P_1x_1 - y_1)$ then $a_1P_1v_1Q'u_2P_2P_2b_2, a_2P_2Rv_3P_3b_3, a_3P_3x_3Q_3y_1P_1b_1$ are disjoint paths from $A$ to $B$ and through $e$. So assume $v_1 \in V(a_3P_3x_3 - x_3)$. Then $P_1, a_2P_2Rv_3P_3b_3, a_3P_3v_3Q'u_2P_2b_2$ contradict the choice of $P$ (the maximality of $L_P$ in $(1)$).

Thus, $v_3 \in V(a_3P_3x_3)$. If $\{x_1, v_2, v_3\}$ is a cut in $G$ separating $A$ from $B$ then (i) holds. So by $(2)$ and $(\ast)$ and by the choices of $v_2$ and $v_3$, we may assume that there is a path $R'$ from some $s' \in V(a_3P_3v_3 - v_3)$ to some $r' \in V(v_2P_2b_2 - v_2)$ and internally disjoint from $P$. Then $R$ is defined, and by the minimality of $v_2P_2v_2, r' \in V(x_2P_2b_2 - x_2)$. So $R \cap R' = \emptyset$ by $(2)$ and $(\ast)$. If $r \in V(a_2P_2v_2 - v_2)$ then $P_1, a_2P_2Rv_3P_3b_3, a_3P_3s'R'r'P_2b_2$ contradict $(1)$; and if $r \in V(a_1P_1x_1 - x_1)$ then $a_1P_1Rv_3P_3b_3, a_2P_2x_2Q_3y_1P_1b_1, a_3P_3s'R'r'P_2b_2$ are $R'$ disjoint paths from $A$ to $B$ and through $e$.

Case 2. $w_3, x_3, y_3$ are not defined.

Let $u \in V(P_3)$ with $uP_3b_3$ minimal such that $u = a_3$ or $u$ belongs to some $P$-bridge of $G$ intersecting $(a_1P_1x_1 - x_1) \cup (a_2P_2x_2 - x_2)$. We may assume $\{x_1, x_2, u\} = \{a_1, a_2, a_3\}$. For, suppose $\{x_1, x_2, u\} \neq \{a_1, a_2, a_3\}$. We further choose $P_3$ (while fixing $P_1, P_2, L_P$) so that $uP_3b_3$ is minimal; hence no $P$-bridge of $G$ intersects both $a_3P_3u - u$ and $uP_3b_3 - u$. If $G$ has no path from $a_3P_3u - u$ to $(x_1P_1b_1 - x_1) \cup (x_2P_2b_2 - x_2)$ and internally disjoint from $P \cup L_P$, then by $(2)$, $(5)$ and the choice of $u$, $\{x_1, x_2, u\}$ is a cut in $G$ separating $A$ from $B$, and (i) holds. So assume that $G$ has a path $Q$ from some $x \in V(a_3P_3u - u)$ to some $y \in V(x_1P_1b_1 - x_1) \cup V(x_2P_2b_2 - x_2)$ and internally disjoint from $P \cup L_P$. Let $R$ be a path in $G$ from $u$ to some $z \in V(a_1P_1x_1 - x_1) \cup V(a_2P_2x_2 - x_2)$ and internally disjoint from $P \cup L_P$, and by symmetry let $z \in V(a_2P_2x_2 - x_2)$. By $(2)$ and $(5)$, $Q \cap R = \emptyset$. Since we are in Case 2, $(L_P - P) \cap (Q \cup R) = \emptyset$. If $y \in V(x_2P_2b_2 - x_2)$ then $P_1, a_2P_2RzP_3b_3, a_3P_3xQyP_2b_2$ contradict the choice of $P$ (i.e., $(1)$). So $y \in V(x_1P_1b_1 - x_1)$. Then $a_1P_1x_1Q_1x_3y_3P_2a_2P_2RzP_3b_3, a_3P_3xQyP_1b_1$ are three disjoint paths from $A$ to $B$ and through $e$.

Similarly, let $v \in V(P_3)$ with $a_3P_3v$ minimal such that $v = b_3$ or $v$ belongs to some $P$-bridge
of $G$ intersecting $(y_1 P_1 b_1 - y_1) \cup (y_2 P_2 b_2 - y_2)$, and we may assume \( \{y_1, y_2, v\} = \{b_1, b_2, b_3\} \).

If no $P$-bridge of $G$ intersecting $P_3$ also meets $P_1$ (respectively, $P_2$) then (iii) holds with $G_2 = L_{12} \cap L_{22}$, $P_2 \cap P_3 \subseteq G_3$ and $P_1 \subseteq G_1$ (respectively, $P_1 \cap P_3 \subseteq G_3$ and $P_2 \subseteq G_1$). So assume that some $P$-bridge of $G$ meets both $P_2$ and $P_3$ and some meets both $P_1$ and $P_3$.

Suppose $G$ has a $P$-bridge $J$ such that $J \cap P_i \neq \emptyset$ for $i = 1, 2, 3$. Then $J \neq L_P$ as $w_3, x_3, y_3$ are not defined. So by (5) and by symmetry, we may assume $V(J \cap P_1) = \{a_i\}$ for $i = 1, 2$. Let \( w \in V(J \cap P_3) \) with $a_3 P_3 w$ maximal. We further choose $P_3$ (while fixing $P_1, P_2, L_P$) so that $w P_3 b_3$ is as short as possible; then no $P$-bridge of $G$ intersects both $a_3 P_3 w$ and $w P_3 b_3 - w$. We may assume that $G$ has a path $Q$ from some $x \in V(a_3 P_3 w - w)$ to some $y \in V(P_1 - a_1) \cup V(P_2 - a_2)$ and internally disjoint from $P \cup L_P \cup J$; for otherwise \( \{a_1, a_2, w\} \) is a cut in $G$ separating $A$ from $B$, showing that (i) holds. By symmetry, assume $y \in V(P_2 - a_2)$. Let $Q_1$ denote a path in $J$ from $w$ to $a_1$ and internally disjoint from $P$. Then $a_1 Q_1 w P_3 b_3, Q_{21}, a_3 P_3 x Q y P_2 b_2$ are three disjoint paths from $A$ to $B$ and through $e$.

So assume that no $P$-bridge of $G$ intersects all $P_i, i = 1, 2, 3$. Suppose all $P$-bridges of $G$ intersecting both $P_3$ and $P_1 \cup P_2$ meet $P_3$ in exactly one common vertex, say $z$. Assume by symmetry that $z \neq a_3$. We may further choose $P_3$ (while fixing $P_1, P_2, L_P$) so that $z P_3 b_3$ is as short as possible. Then no $P$-bridge of $G$ intersects both $a_3 P_3 z - z$ and $z P_3 b_3 - z$. So \( \{a_1, a_2, z\} \) is a cut in $G$ separating $A$ from $B$, and (i) holds. Hence, we may assume that $G$ has distinct $P$-bridges $J_1$ and $J_2$ such that $J_1 \cap P_1 \neq \emptyset$, $J_2 \cap P_2 \neq \emptyset$, and there exist $u_i \in V(J_i \cap P_3)$, $i = 1, 2$, with $u_1 \neq u_2$. By symmetry assume that $a_3, u_1, u_2, b_3$ occur on $P_3$ in order. For $i = 1, 2$, let $Q_i$ be a path in $J_i$ from $u_i$ to some $v_i \in V(P_i)$ and internally disjoint from $P$. If $v_1 \neq a_1$ and $v_2 \neq b_2$, then $Q_{12}, a_2 P_2 v_2 Q_2 u_2 P_3 b_3, a_3 P_3 u_1 Q_1 v_1 P_1 b_1$ are three disjoint paths from $A$ to $B$ and through $e$. So by symmetry, assume $V(J_2 \cap P_2) = \{b_2\}$. By modifying $P_3$ (while fixing $P_1, P_2, L_P$) we may assume that no $P$-bridge of $G$ intersects both $a_3 P_3 u_2 - u_2$ and $u_2 P_3 b_3 - u_2$.

(3) If no $P$-bridge of $G$ intersecting $u_2 P_3 b_3 - u_2$ meets $(P_1 - b_1) \cup (P_2 - b_2)$, then $G$ has separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{b_1, b_2, u_2\}, A \subseteq V(G_1)$, and $B \subseteq V(G_2)$; so (i) holds. Hence, assume that there is a path $R$ from some $s \in V(u_2 P_3 b_3 - u_2)$ to some $t \in V(P_1 - b_1) \cup V(P_2 - b_2)$. If $t \in V(P_1 - b_1)$ then $a_1 P_1 t R s P_3 b_3, Q_{21}, a_3 P_3 u_2 Q_2 b_2$ are disjoint paths from $A$ to $B$ and through $e$. So assume $t \in V(P_2 - b_2)$. Now $P_1, a_2 P_2 t R s P_3 b_3, a_3 P_3 u_2 Q_2 b_2$ reduce this case to Case 1.

4 Separations of order three

We now use Lemma 3.4 to prove the following lemma about separations of order three.

**Lemma 4.1.** Let $(G, u_1, u_2, A)$ be a quadruple, and suppose $G$ has a separation $(U_1, U_2)$ such that $|V(U_1 \cap U_2)| \leq 3$, $V(U_1 \cap U_2) \cap A \neq \emptyset$, $u_1 \in V(U_1) - V(U_2)$, $u_2 \in V(U_2) - V(U_1)$, and $A \subseteq U_i$ for some $i \in \{1, 2\}$. Then the conclusion of Theorem 2.1 holds for $(G, u_1, u_2, A)$.

**Proof.** For convenience, we say a separation of $G$ good if it satisfies the conditions of this lemma. We may assume that for any good separation $(U_1, U_2)$, $|V(U_1 \cap U_2)| = 3$ (and let $V(U_1 \cap U_2) = \{v_1, v_2, v_3\}$) and $U_{3-i}$ has three independent paths, say $P_1, P_2, P_3$, from $u_{3-i}$ to $v_1, v_2, v_3$, respectively. For, suppose otherwise. By symmetry, let $i = 1$. If $|V(U_1 \cap U_2)| \leq 2$ let
$U_{21} = U_2$ and $U_{22} = \emptyset$, and if $|V(U_1 \cap U_2)| = 3$ let $(U_{21}, U_{22})$ be a separation in $U_2$ such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in V(U_{21}) - V(U_{22})$ and $V(U_1 \cap U_2) \subseteq V(U_{22})$. Now $(U_{21}, U_{22} \cup U_1)$ is a separation in $G$ showing that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$.

We may assume that $E(G[A]) = \emptyset$; as otherwise, it is easy to see that Theorem 2.1(iii) holds. Let $A = \{a_1, a_2, a_3, a_4\}$ and $a_1 = v_1$. We may assume that

(*) for any good separation $(U_1, U_2)$, $|V(U_1 \cap U_2)| = 3$, and $V(U_1 \cap U_2) \cap A = \{a_1\}$.

Again by symmetry, let $i = 1$. If $v_2, v_3 \in A$ then $U_1, U_2 + A, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type $IV$. So we may assume $v_3 \notin A$. Suppose $v_2 \in A$, say $v_2 = a_2$. Then, because of $P_1, P_2, P_3, G$ has a topological $H$ rooted at $u_1, u_2, A$ and if only if $U_1 - \{a_1, a_2\}$ has three independent paths from $v_1$ to $a_3, a_4, v_3$, respectively. Thus either Theorem 2.1(i) holds for $(G, u_1, u_2, A)$, or $U_1$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 4$, $a_1, a_2 \in V(U_{11} \cap U_{12})$, $u_1 \in V(U_{11}) - V(U_{12})$ and $\{a_3, a_4, v_3\} \subseteq V(U_{12})$. If $|V(U_{11} \cap U_{12})| \leq 3$ then the separation $(U_{11} \cup U_2, U_{12})$ shows that Theorem 2.1(iii) holds for $(G, u_1, u_2, A)$. So assume $|V(U_{11} \cap U_{12})| = 4$. If $a_3, a_4 \notin V(U_{11} \cap U_{12})$ then $U_{11}, a_1, \{a_2\}, U_{12} - \{a_1, a_2\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type $I$. So assume $a_3 \in V(U_{11} \cap U_{12})$. If $a_4 \notin V(U_{11} \cap U_{12})$ then $U_{11}, U_2 + a_3, a_1, a_2, a_3, U_{12} - \{a_1, a_2, a_3\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type $IV$; and if $a_4 \in V(U_{11} \cap U_{12})$ then $U_{11}, U_{12} \cup U_2, a_1, a_2, a_3, a_4$ show that $(G, u_1, u_2, A)$ is an obstruction of type $IV$. This proves (*).

We now look for paths in $U_1$ in order to form a topological $H$ in $G$. Let $U_1'$ be obtained from $(U_1 - a_1) + v_2 v_3$ by duplicating $u_1$ twice, and denote the copies of $u_1$ by $u_1', u_1''$. We apply Lemma 3.4 to $U_1', \{u_1, u_1', u_1''\}, \{a_2, a_3, a_4\}, v_2 v_3$. If $U_1'$ has three disjoint paths from $u_1, u_1', u_1''$ to $\{a_2, a_3, a_4\}$ and through $v_2 v_3$, then $(U_1 - a_1) + v_2 v_3$ has three independent paths $R_1, R_2, R_3$ from $u_1$ to $a_2, a_3, a_4$, respectively, and through $v_2 v_3$, and $\bigcup_{i=1}^3 (P_i \cup R_i) - v_2 v_3$ is a topological $H$ in $G$ rooted at $u_1, u_2, A$; so Theorem 2.1(i) holds for $(G, u_1, u_2, A)$. Hence, assume the paths $R_1, R_2, R_3$ do not exist. Then one of (i) – (iv) of Lemma 3.4 holds. Since $u_1'$ and $u_1''$ are duplicates of $u_1$, (iii) and (iv) of Lemma 3.4 do not occur here. Suppose Lemma 3.4(ii) holds. Then $U_1$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $a_1 \in V(U_{11} \cap U_{12})$, $A \cup \{u_1\} \subseteq V(U_{11})$, and $\{v_2, v_3\} \subseteq V(U_{12})$. Now the separation $(U_{12} \cup U_2, U_{11})$ shows that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$. Hence, we may assume that Lemma 3.4(ii) holds.

Thus, $U_1$ has a separation $(U_{11}, U_{12})$ such that $a_1 \in V(U_{11} \cap U_{12})$, $|V(U_{11} \cap U_{12})| \leq 4$, $u_1 \in V(U_{11}) - V(U_{12})$, and $A \subseteq V(U_{12})$. We choose $(U_{11}, U_{12})$ so that $U_{12}$ is minimal. Note that $\{v_2, v_3\} \subseteq V(U_{11})$ or $\{v_2, v_3\} \subseteq V(U_{12})$. In fact, we may assume $\{v_2, v_3\} \notin V(U_{11})$; otherwise the separation $(U_{11} \cup U_2, U_{12})$ shows that Theorem 2.1(iii) holds for $(G, u_1, u_2, A)$.

We may assume that $|V(U_{11} \cap U_{12})| = 4$ and $U_{11} - a_1$ has three independent paths $Q_1, Q_2, Q_3$ from $u_1$ to the three vertices in $V(U_{11} \cap U_{12}) - \{a_1\}$ respectively. First we may assume $|V(U_{11} \cap U_{12})| \geq 3$; otherwise the separation $(U_{11}, U_{12} \cup U_2)$ shows that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$. Moreover, we may assume $|V(U_{11} \cap U_{12})| = 4$; otherwise by (*), $V(U_{11} \cap U_{12}) \cap (A - \{a_1\}) = \emptyset$, and $U_{11}, U_2, \{a_1\}, U_{12} - a_1$ show that $(G, u_1, u_2, A)$ is an obstruction of type $II$. Now if the paths $Q_1, Q_2, Q_3$ do not exist, then $U_{11}$ has a separation $(U_{11}', U_{11}'')$ such that $|V(U_{11} \cap U_{11}'')| \leq 3$, $a_1 \in V(U_{11}' \cap U_{11}'')$, $u_1 \in V(U_{11}' - V(U_{11}''))$, and $V(U_{11} \cap U_{12}) \subseteq V(U_{11}'')$. We may assume $|V(U_{11}' \cap U_{11}'')| = 3$; otherwise $(U_{11}', U_{11}' \cup U_{12} \cup U_2)$ shows that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$. By (*), $V(U_{11}' \cap U_{11}'') \cap (A - \{a_1\}) = \emptyset$. So $U_{11}', U_2, \{a_1\}, (U_{11}' \cup U_{12}) - a_1$ show that $(G, u_1, u_2, A)$ is an obstruction of type $II$. 

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We may also assume that \( \{v_2, v_3\} \subseteq V(U_{12}) - V(U_{11}) \). Otherwise, since \( \{v_2, v_3\} \nsubseteq V(U_{11}) \), we may assume that \( v_2 \in V(U_{11} \cap U_{12}) \) and \( v_3 \nsubseteq V(U_{11} \cap U_{12}) \). By the minimality of \( U_{12} \), \( U_{12} - \{a_1, v_2\} \) has three disjoint paths from \( \{a_2, a_3, a_4\} \) to \( (V(U_{11} \cap U_{12}) - \{a_1, v_2\}) \cup \{v_3\} \). Now these paths and \( \cup_{i=1}^3 (P_i \cup Q_i) \) form a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \), and Theorem 2.1(i) holds for \((G, u_1, u_2, A)\).

If \( V(U_{11} \cap U_{12}) - \{a_1\} = \{a_2, a_3, a_4\} \), then \( U_{11}, U_{12} \cup U_2, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\} \) show that \((G, u_1, u_2, A)\) is an obstruction of type IV.

Suppose \(|(V(U_{11} \cap U_{12}) - \{a_1\}) \cap \{a_2, a_3, a_4\}| = 2\), say \( a_2 \notin V(U_{11} \cap U_{12}) \). If \( U_{12} - \{a_1, a_3, a_4\} \) has two disjoint paths from \( \{v_2, v_3\} \) to \( \{a_2, a_3, a_4\} \) \( \cup \) \( V(U_{11} \cap U_{12}) - A \), then these paths and \( \cup_{i=1}^3 (P_i \cup Q_i) \) form a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \); so Theorem 2.1(i) holds for \((G, u_1, u_2, A)\). Hence, assume that \( U_{12} \) has a separation \((S, T)\) such that \(|S \cap T| \leq 4\), \( \{a_1, a_3, a_4\} \subseteq V(S \cap T), \{a_2\} \cup V(U_{11} \cap U_{12}) \subseteq V(S) \), and \( \{v_2, v_3\} \subseteq V(T) \). If \( a_2 \in V(S) - V(T) \) then \( U_{11}, U_2 \cup T, \{a_1\}, S - \{a_1, a_3, a_4\}, \{a_3\}, \{a_4\} \) show that \((G, u_1, u_2, A)\) is an obstruction of type IV; if \( a_2 \in V(S \cap T) \) then \( U_{11} \cup S, U_2 \cup T, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\} \) show that \((G, u_1, u_2, A)\) is an obstruction of type IV.

Now suppose \(|(V(U_{11} \cap U_{12}) - \{a_1\}) \cap \{a_2, a_3, a_4\}| = 0\). Then we may apply Lemma 3.4 to \( U_{12} - a_1 + v_2 v_3, V(U_{11} \cap U_{12}) - \{a_1\}, \{a_2, a_3, a_4\}, v_2 v_3 \). If \( U_{12} = a_1 + v_2 v_3 \) has three disjoint paths from \( V(U_{11} \cap U_{12}) - \{a_1\} \) to \( \{a_2, a_3, a_4\} \) and through \( v_2 v_3 \), then deleting \( v_2 v_3 \) from the union of these paths with \( \cup_{i=1}^3 (P_i \cup Q_i) \), we obtain a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \); Theorem 2.1(i) holds for \((G, u_1, u_2, A)\). So assume that one of (i) – (iv) of Lemma 3.4 holds. If Lemma 3.4(iii) holds then \( U_{12} \) has a separation \((S, T)\) such that \( a_1 \in V(S \cap T), |V(S \cap T)| \leq 4, V(U_{11} \cap U_{12}) \subseteq V(S) \), and \( \{a_2, a_3, a_4\} \subseteq V(T) \); so \((U_{11} \cup S, T)\) contradicts the choice of \((U_{11}, U_{12})\). If Lemma 3.4(iv) holds then \( U_{12} \) has a separation \((S, T)\) such that \( a_1 \in V(S \cap T), |V(S \cap T)| \leq 2, \{v_2, v_3\} \subseteq V(T) \), and \( A \cup V(U_{11} \cap U_{12}) \subseteq V(S) \); so the separation \((U_2 \cup T, U_{11} \cup S)\) shows that Theorem 2.1(iii) holds for \((G, u_1, u_2, A)\). Now, suppose Lemma 3.4(iii) holds. Then \( U_{12} - a_1 = S_1 \cup S_2 \cup S_3 \) such that \( S_1 \cap S_3 = \emptyset, \{v_2, v_3\} \subseteq V(S_2), |V(S_i \cap S_3)| \leq 1 \) for \( i = 1, 3 \), \( |V(S_i) \cap \{a_2, a_3, a_4\}| = |V(S_i) \cap (V(U_{11} \cap U_{12}) - \{a_1\})| = 1 \), and \( |V(S_3) \cap \{a_2, a_3, a_4\}| = |V(S_3) \cap (V(U_{11} \cap U_{12}) - \{a_1\})| = 2 \). Note that \( |S_i \cap S_2| = 1 \) for \( i = 1, 3 \); as otherwise, \( U_2 \cup S_2, U_{11} \cup S_1 \cup S_3 \) is a separation in \( G \) showing that Theorem 2.1 holds for \((G, u_1, u_2, A)\). Therefore, since \( V(S_i \cap S_2) \nsubseteq A \) for \( i = 1, 3 \) (by (ii) as \( \{a_1\} \cup V((S_i \cap S_3) \cap S_2) \) separates \( u_2 \) from \( A \cup \{u_1\}) \), \( U_{11}, U_2 \cup S_2, \{a_1\}, S_i, S_3 \) show that \((G, u_1, u_2, A)\) is an obstruction of type I. Thus, we may assume that Lemma 3.4(iv) holds. Then \( U_{12} - a_1 = S_1 \cup S_2 \cup S_3 \cup \{v_2, v_3\} \subseteq V(S_1), S_i \cap S_j = \emptyset \) for \( 2 \leq i < j \leq 4 \), and \( |V(S_i) \cap \{a_2, a_3, a_4\}| = |V(S_i) \cap (V(U_{11} \cap U_{12}) - \{a_1\})| = 1 \) for \( i = 2, 3, 4 \). Let \( A_1 = \{a_1\} \), and for \( i \) in \( \{2, 3, 4\} \), let \( a_i \in V(S_i), V(A_i) = \{a_i\} \) and \( A_i' = S_i \) (when \( a_i \notin V(S_i) \)). Now \( U_{11} \cup A_2' \cup A_3' \cup A_4', U_2 \cup S_1, A_1, A_2, A_3, A_4 \) show that \((G, u_1, u_2, A)\) is an obstruction of type IV.

Thus, without loss of generality, let \( V(U_{11} \cup U_{12}) = \{a_1, a_2, b, c\} \), with \( b, c \notin A \). Note that \( U_{12} \cup U_2 \) has a separation \((S, T)\) such that \( \{a_1, a_2\} \subseteq V(S \cap T), |V(S \cap T)| \leq 4, \{a_2, a_3, b, c\} \subseteq V(S) \), and \( u_2 \in V(T) - V(S) \). (For example, \( S = U_{12} \) and \( T = U_2 + a_4 \).) Choose \((S, T)\) to maximize \( T \) with \( U_2 \subseteq T \). By (ii), \( |V(S \cap T)| = 4 \). Let \( V(S \cap T) = \{a_1, a_2, v_2', v_3\} \). Then \( T - a_4 \) has three independent paths \( Q_1', Q_2', Q_3' \) from \( u_2 \) to \( a_1, v_2', v_3', \) respectively; for otherwise, \( T \) has a separation \((T_1, T_2)\) such that \( |V(T_1 \cap T_2)| \leq 3, a_1, a_2 \in V(T_1 \cap T_2), u_2 \in V(T_2) - V(T_1) \), and \( \{v_2', v_3'\} \subseteq V(T_1) \) (since \( U_2 \subseteq T \) and because of \( P_1, P_2, P_3 \)), contradicting (ii) (with the separation \((U_{11} \cup S \cup T_1, T_2)\)).
We apply Lemma 3.3 to $S - \{a_1, a_4\}, b, c, v'_2, v'_3, a_2, a_3$ (with $a_2, a_3$ play the roles of $a_1, a_2$ there). If $S - \{a_1, a_4\}$ has three disjoint paths, with one from $\{b, c\}$ to $\{v'_2, v'_3\}$, one from $\{b, c\}$ to $\{a_2, a_3\}$, and another from $\{v'_2, v'_3\}$ to $\{a_2, a_3\}$, then these paths and $\bigcup_{i=1}^p (Q_i \cup Q'_i)$ form a topological $H$ in $G$ rooted at $u_1, u_2, A$; Theorem 2.1(i) holds for $(G, u_1, u_2, A)$. So assume that $S - \{a_1, a_4\}$ has a separation $(G_1, G_2)$ such that one of (i) – (v) of Lemma 3.3 holds. By the minimality of $U_1$ and the maximality of $T$, Lemma 3.3(i) does not occur here. If Lemma 3.3(ii) holds, then the separation $(U_{11} \cup T \cup G[G_2 + \{a_1, a_4\}], G_1 + \{a_1, a_4\})$ shows that Theorem 2.1(iii) holds for $(G, u_1, u_2, A)$. If Lemma 3.3(iii) holds, say with $\{b, c\} \subseteq V(G_1)$ and $\{v'_2, v'_3\} \subseteq V(G_2)$, then $U_{11} \cup G_1 + \{a_2, a_3\}, (T \cup G_2) + \{a_2, a_3\}, \{a_1\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV. If Lemma 3.3(iv) holds, then $U_{11}, T, \{a_1\}, \{a_4\}, G_1, G_2$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV. So assume Lemma 3.3(v) holds with $\{a_2, a_3\} \subseteq V(G_1)$. If $|\{v'_2, v'_3\} \cap V(G_1)| = 1$ then the separation $(T \cup G[G_2 + \{a_1, a_4\}], U_{11} \cup G[G_1 + \{a_1, a_4\}])$ contradicts (*). So $|\{b, c\} \cap V(G_1)| = 1$. Then $U_{11} \cup G[G_2 + \{a_1, a_4\}], T, \{a_1\}, \{a_4\}, G_1$ show that $(G, u_1, u_2, A)$ is an obstruction of type I.

**Lemma 4.2.** Let $(G, u_1, u_2, A)$ be a quadruple, and assume that $G$ has a separation $(U_1, U_2)$ such that $|V(U_1 \cap U_2)| \leq 3$, $|V(U_1)| \geq 5$, $u_1 \in V(U_1) - V(U_2)$, $A \cup \{u_2\} \subseteq V(U_2)$. Suppose Theorem 2.1 holds for all graphs of order less than $|V(G)|$. Then Theorem 2.1 holds for $(G, u_1, u_2, A)$.

**Proof.** First, we may assume that $|V(U_1 \cap U_2)| = 3$ and $U_1$ has three independent paths, say $P_1, P_2, P_3$, from $u_1$ to the three vertices in $V(U_1 \cap U_2)$. For, otherwise, $|V(U_1 \cap U_2)| \leq 2$ (in which case let $K = U_1$ and $L = \emptyset$), or $U_1$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 2$, $u_1 \in V(K) - V(L)$ and $V(U_1 \cap U_2) \subseteq V(L)$. Then the separation $(K, L \cup U_2)$ in $G$ shows that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$.

Now let $G'$ be obtained from $G$ by deleting $U_1 - u_1 - V(U_1 \cap U_2)$ and adding three edges from $u_1$ to the three vertices in $V(U_1 \cap U_2)$. By assumption, Theorem 2.1 holds for $(G', u_1, u_2, A)$.

If Theorem 2.1(i) holds for $(G', u_1, u_2, A)$ then let $T'$ be a topological $H$ in $G'$ rooted at $u_1, u_2, A$. Now $(T' - u_1) \cup P_1 \cup P_2 \cup P_3$ is a topological $H$ in $G$ rooted at $u_1, u_2, A$; so Theorem 2.1(i) holds for $(G, u_1, u_2, A)$.

Suppose Theorem 2.1(ii) holds for $(G', u_1, u_2, A)$, and let $(K, L)$ denote a separation in $G'$ such that $|V(K \cap L)| \leq 2$ and, for some $i \in \{1, 2\}$, $u_i \in V(K) - V(L)$ and $A \cup \{u_{3-i}\} \subseteq V(L)$. If $i = 1$ then the separation $((K - u_1) \cup U_1, L)$ shows that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$. So $i = 2$. If $u_1 \notin V(K \cap L)$ then the separation $(K, L - u_1) \cup U_1)$ shows that Theorem 2.1(ii) holds for $(G, u_1, u_2, A)$. So $u_1 \in V(K \cap L)$. Then the separation $(U_1 \cup K, L - u_1)$ shows that Theorem 2.1(iii) holds for $(G, u_1, u_2, A)$.

Suppose Theorem 2.1(iii) holds for $(G', u_1, u_2, A)$, and let $(K, L)$ denote a separation in $G'$ such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K) - V(L)$ and $A \subseteq V(L)$. Now the separation $((K - u_1) \cup U_1, L)$ shows that Theorem 2.1(iii) holds for $(G, u_1, u_2, A)$.

Finally, assume Theorem 2.1(iv) holds for $(G', u_1, u_2, A)$. Replacing $u_1$ with $U_1$ in that side of $(G', u_1, u_2, A)$ containing $u_1$, we see that $(G, u_1, u_2, A)$ is also an obstruction of the same type as $(G', u_1, u_2, A)$. 

5 Quadruples with critical pairs

In this section, we consider quadruples \((G,u_1,u_2,A)\) in which there exist \(x,y \in V(G) - \{u_1,u_2\} - A\) such that \((G/xy,u_1,u_2,A)\) is an obstruction, where \(G/xy\) is obtained from \(G\) by identifying \(x\) and \(y\). Such a pair \(\{x,y\}\) is said to be critical. First, we need a lemma on separations of order 4 in a hypothetical minimum counterexample to Theorem 2.1.

Lemma 5.1. Suppose \((G,u_1,u_2,A)\) is a counterexample to Theorem 2.1 with \(|V(G)|\) minimum, and assume that \(G\) has a separation \((U_1,U_2)\) such that \(V(U_1 \cap U_2) = \{w_1,w_2,w_3,w_4\}\), \(u_1 \in V(U_1) - V(U_2)\), \(u_2 \in V(U_2) - V(U_1)\), and \(A \subseteq V(U_2)\). Then for any permutation \(ijkl\) of \(\{1,2,3,4\}\),

(i) \(U_1 - w_i\) has three independent paths from \(u_1\) to \(w_i, w_j, w_k\), respectively, unless \(w_i \in N(u_1)\) and \(|N(u_1)| = 3\), and

(ii) \(U_1\) has three independent paths from \(u_1\) to \(w_i, w_j, w_k\), unless \(w_i \in N(u_1)\), \(|N(u_1)| = 3\), and \(N(w_i) \cap V(U_1) \subseteq N[u_1]\).

Proof. First, note that \(|N(u_1)| \geq 3\), or else Theorem 2.1(ii) would hold for \((G,u_1,u_2,A)\).

Suppose \(U_1 - w_i\) does not have three independent paths from \(u_1\) to \(w_i, w_j, w_k\), respectively. Then \(U_1\) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 3\), \(w_i \in V(U_{11} \cap U_{12})\), \(\{w_1,w_2,w_3,w_4\} \subseteq V(U_{12})\), and \(u_1 \in V(U_{11}) - V(U_{12})\). Note that \(|V(U_{11} \cap U_{12})| = 3\); otherwise the separation \((U_{11}, U_{12} \cup U_2)\) shows that Theorem 2.1(ii) would hold for \((G,u_1,u_2,A)\).

Now the separation \((U_{11}, U_{12} \cup U_2)\) allows us to use Lemma 4.2 to conclude that \(V(U_{11}) = \{u_1\} \cup V(U_{11} \cap U_{12})\). Hence, \(|N(u_1)| = 3\) and \(N(u_1) = V(U_{11} \cap U_{12})\) (so \(w_i \in N(u_1)\)).

Now assume that \(U_1\) does not have three independent paths from \(u_1\) to \(w_i, w_j, w_k\), respectively. Then \(U_1\) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 2\), \(w_i \in V(U_{11}) - V(U_{12})\), and \(\{w_i,w_j,w_k\} \subseteq V(U_{12})\). Note that \(w_i \in V(U_{11}) - V(U_{12})\) and \(|V(U_{11} \cap U_{12})| = 2\); otherwise the separation \((U_{11}, U_{12} \cup U_2 + w_i)\) shows that Theorem 2.1(ii) would hold for \((G,u_1,u_2,A)\). Now the separation \((U_{11}, U_{12} \cup U_2 + w_i)\) allows us to use Lemma 4.2 to conclude that \(V(U_{11}) = \{u_1,w_i\} \cup V(U_{11} \cap U_{12})\). Hence, \(|N(u_1)| = 3\), \(N(u_1) = \{w_i\} \cup V(U_{11} \cap U_{12})\), and \(N(w_i) \cap V(U_1) \subseteq N[u_1]\).

We now show, in a sequence of four lemmas, that no quadruple containing a critical pair is a minimum counterexample to Theorem 2.1.

Lemma 5.2. Suppose \((G,u_1,u_2,A)\) is a counterexample to Theorem 2.1 with \(|V(G)|\) minimum, and let \(x,y \in V(G) - A - \{u_1,u_2\}\) be distinct. Then \((G/xy,u_1,u_2,A)\) is not an obstruction of type IV.

Proof. For, suppose \((G/xy,u_1,u_2,A)\) is an obstruction of type IV, with sides \(U_1, U_2\) and middle parts \(A_1, A_2, A_3, A_4\). See Figure 6. Recall from definition of obstruction that \(V(U_1 \cap U_2) \subseteq A\). Let \(A := \{a_1,a_2,a_3,a_4\}\) such that \(a_i \in V(A_i)\) for \(1 \leq i \leq 4\). Let \(V(U_1 \cap A_i) = \{v_i\}\) and \(V(U_2 \cap A_i) = \{w_i\}\), \(1 \leq i \leq 4\), and let \(u_1 \in V(U_1) - \{v_1,v_2,v_3,v_4\}\) and \(u_2 \in V(U_2) - \{w_1,w_2,w_3,w_4\}\). By definition of obstruction, if \(|A_i| \geq 2\) then \(a_i \in V(A_i) - \{v_i,w_i\}\) and \(N(a_i) \subseteq V(A_i)\), and if \(v_i = w_i\) then \(\{v_i\} = \{w_i\} = \{a_i\} = V(A_i)\).


Let $v$ be the vertex resulting from the identification of $x$ and $y$. If $v \notin \{v_i, w_i : 1 \leq i \leq 4\}$ then $(G, u_1, v_2, A)$ is an obstruction of type IV, a contradiction. Then by symmetry assume $v = v_1$. So $|V(A_1)| \geq 2$ and $a_1 \in V(A_1) \setminus \{v_1, w_1\}$. Let $U'_1, A'_1$ be obtained from $U_1, A_1$, respectively, by unidentifying $v$ to $x$ and $y$. Note that if $xy \in E(G)$ we put $xy$ back in exactly one of $U'_1$ or $A'_1$ (it does not matter which one).

Then $A'_1$ contains disjoint paths $X, Y$ from $\{x, y\}$ to $\{a_1, w_1\}$. For, otherwise, $A'_1$ has a separation $(A_{11}, A_{12})$ such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq V(A_{11})$, and $\{a_1, w_1\} \subseteq V(A_{12})$. Now $U'_1 \cup A_{11}, U_2, A_{12}, A_2, A_3, A_4$ (when $a_1 \notin V(A_{11} \cap A_{12}) \neq \emptyset$, or $U'_1 \cup A_{11} + a_1, U_2 \cup A_{12}, \{a_1\}, A_2, A_3, A_4$ (when $a_1 \in V(A_{11} \cap A_{12})$ or $V(A_{11} \cap A_{12}) = \emptyset$), show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction.

Moreover, for each $i \in \{2, 3, 4\}$, if $a_i \notin \{v_i, w_i\}$ then $A_i - v_i$ contains a path $W'_i$ between $w_i$ and $a_i$, and $A_i - w_i$ has a path $V'_i$ between $v_i$ and $a_i$. For, suppose by symmetry that $A_i - v_i$ has no path from $w_i$ to $a_i$, then $A_i$ has a separation $(A_{i1}, A_{i2})$ such that $V(A_{i1} \cap A_{i2}) = \{v_i, a_i\}$, $a_i \in V(A_{i1})$ and $w_i \in V(A_{i2})$. Then the separation $(G - (A_{i1} - v_i), A_{i1} + (A - \{a_i\}))$ shows that Theorem $2.1(iii)$ holds, a contradiction. Let $W'_i = V'_i = A_i$ if $a_i = v_i = w_i$.

Suppose for each $i \in \{2, 3, 4\}$, $U'_1 - (A - \{v_i\})$ has three independent paths $P'^1_i, P'^2_i, P'^3_i$ from $u_1$ to $x, y, v_i$, respectively. If $U'_1 - (A \{w_2\})$ has three independent paths from $w_2$ to $w_1, w_3, w_4$, respectively, then these paths and $P'^1_2, P'^2_2, P'^3_2, X, Y, V'_2, W'_3, W'_4$ form a topological $H$ rooted at $u_1, u_2, A$, and Theorem $2.1(i)$ would hold. So such paths do not exist in $U'_2 - (A - \{w_2\})$. Then by Lemma $5.1(i)$, $w_2 \in N(w_2)$ and $|N(w_2)| = 3$. Similarly, $w_3, w_4 \in N(w_2)$. Hence by Lemma $4.1$, $w_2, w_3, w_4 \notin A$. Therefore, by Lemma $5.1(iii)$, $N(w_2) \cap V(U_2) \subseteq N[w_2]$ for $i = 2, 3, 4$. Now $G[N[w_2]] + a_1, U'_1 \cup A'_1 \cup (U_2 - \{w_2, w_3, w_4\}), \{a_1\}, A_2, A_3, A_4$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction.

Hence, we may assume by symmetry that $P'^1_2, P'^2_2, P'^3_2$ do not exist. Then $U'_1$ has a separation $(U_{11}, U_{12})$ such that $A \{v_3, v_4\} \subseteq V(U_{11} \cap U_{12})$, $|V(U_{11} \cap U_{12})| \leq |A \{v_3, v_4\}| + 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{x, y, v_2\} \subseteq V(U_{12})$. We choose $(U_{11}, U_{12})$ so that $|V(U_{11} \cap U_{12})|$ is minimum and then $U_{12}$ is minimal.

We claim that $|V(U_{11} \cap U_{12})| = |A \{v_3, v_4\}| + 2$. For, otherwise, the separation $(U_{11} + \{v_3, v_4\}, G - (U_{11} - U_{12}) + \{v_3, v_4\})$ allows us to use Lemma $4.1$ to assume $v_3, v_4 \notin A$; so $|A \{v_3, v_4\}| = 0$. Then $|V(U_{11} \cap U_{12})| = 1$ and $v_3, v_4 \in V(U_{11} - U_{12})$; else, the separation $(U_{11}, U_{12} \cup A'_1 \cup A_2 \cup A_3 \cup A_4 \cup U_2)$ shows that Theorem $2.1(ii)$ would hold. Hence, $U_{11}, U_{12}, A'_1 \cup U_{12} \cup A_2, A_3, A_4$ show that $(G, u_1, u_2, A)$ is an obstruction of type I, a contradiction.

Let $V(U_{11} \cap U_{12}) - (A \{v_3, v_4\}) = \{s_1, s_2\}$. We claim that $v_2 \notin A \{s_1, s_2\}$. For, suppose
v_2 \in A \cap \{s_1, s_2\}; so v_2 = a_2. Note that for each i \in \{3, 4\}, if v_i \notin A then, since v_2 = a_2, we must have v_i \notin V(U_12) by Lemma 4.1. So U_{11}, U_2, (U_{12} - v_2) \cup A_1, \{v_2\}, A_3, A_4 show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Then by the minimality of U_{12}, U_{12} - (A \cap \{v_2\}) contains disjoint paths S_1, S_2 from \{x, y\} to \{s_1, s_2\}. We may assume that (U_{11} + \{v_3, v_4\}) - (A \cap \{v_4\}) (or (U_{11} + \{v_3, v_4\}) - (A \cap \{v_3\})) has independent paths Q_1', Q_2', Q_3' from u_1 to s_1, s_2, v_3 (or v_4), respectively. This is true if v_3 \in \{s_1, s_2\} or v_4 \in \{s_1, s_2\}; as otherwise U_{11} + \{v_3, v_4\} has a cut of size at most two separating u_1 from \{s_1, s_2\} \cup \{v_3, v_4\}, which gives a separation showing that Theorem 2.1(ii) would hold. So we may assume v_3, v_4 \notin \{s_1, s_2\} and that the paths Q_1', Q_2', Q_3' do not exist. Then by Lemma 5.1(i), \|N(u_1)\| = 3 and v_3, v_4 \notin N(u_1). So by Lemma 4.1, v_3, v_4 \notin A. Hence, by Lemma 5.1(ii), N(v_i) \cap V(U_{11} + \{v_3, v_4\}) \subseteq N[u_1] for i = 3, 4. Now G[N[u_1]], U_2 - (U_2 \cup \{u_1, v_3, v_4\}) \cup A_1 \cup A_2 \cup A_3, A_4 show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

By symmetry, assume that Q_3' ends at v_3. Then because of S_1, S_2, we see that U_1 - (A \cap \{v_2, v_4\}) has independent paths Q_1, Q_2, Q_3 from u_1 to x, y, v_3, respectively. If U_2 - (A \cap \{w_3\}) has three independent paths from u_2 to w_1, w_2, w_4, respectively, then these paths and Q_1, Q_2, Q_3, X, Y, V_3', W_2', W_4' form a topological H in G rooted at u_1, u_2, A, and Theorem 2.1(i) would hold. So such paths do not exist. Then by Lemma 5.1(i), w_3 \in N(u_2) and \|N(u_2)\| = 3. So by Lemma 4.1, w_3 \notin A. Hence, N(w_3) \cap V(U_2) \subseteq N[u_2] by Lemma 5.1(ii). Moreover, v_3 \notin A. So v_3 \in V(U_{11} - U_{12}) because of Q_1', Q_2', Q_3'. Then v_4 \in V(U_{11} - U_{12}); for otherwise, since v_4 \notin \{s_1, s_2\} when w_4 \in A, U_{11}, G[N[w_4]], A_3, U_2 \cup A_1 \cup A_2 \cup A_1 \cup (U_2 - \{u_2, w_3\}) (and removing from the last subgraph the possible edge with both ends in N(w_2)) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

Suppose U_{11} does not contain independent paths from u_1 to s_1, s_2, v_4, respectively. Then by Lemma 5.1(ii), v_3 \in N(u_1), \|N(u_1)\| = 3 and N(v_3) \cap V(U_{11}) \subseteq N[u_1]. Hence G[N[u_1]], G[N[w_2]], U_2 - (U_2 \cup \{w_3\}) \cup A_1 \cup A_2 \cup A_4 \cup (U_2 - \{u_2, w_3\}) (and removing from the last subgraph possible edges with both ends in N(u_1) or N(u_2)) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

So let R_1, R_2, R_3 be independent paths in U_{11} from u_1 to s_1, s_2, v_4, respectively. If U_2 - (A \cap \{w_4\}) has independent paths from u_2 to w_1, w_2, w_3, respectively, then these paths, R_1, R_2, R_3, S_1, S_2, X, Y, V_1', W_1', W_2', W_3' form a topological H in G rooted at u_1, u_2, A, and Theorem 2.1(i) would hold. So such paths in U_2 - (A \cap \{w_4\}) do not exist. Then by Lemma 5.1(i), w_4 \in N(u_2), and by Lemma 4.1, w_4 \notin A. Hence by Lemma 5.1(ii), N(v_i) \cap V(U_2) \subseteq N[u_2] for i = 3, 4. Thus, U_{11}, G[N[w_4]], A_3, U_2 \cup A_1 \cup A_2 \cup (U_2 - \{u_2, w_3, w_4\}) show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

Lemma 5.3. Suppose (G, u_1, u_2, A) is a counterexample to Theorem 2.1 with \|V(G)\| minimum, and let x, y \in V(G) - A - \{u_1, u_2\} be distinct. Then (G/xy, u_1, u_2, A) is not an obstruction of type I.

Proof. Suppose (G/xy, u_1, u_2, A) is an obstruction of type I, with sides U_1, U_2 and middle parts A_1, A_2, A_3. Recall that V(U_1 \cap U_2) \subseteq A. See Figure 2. Let A := \{a_1, a_2, a_3, a_4\} such that a_i \in V(A_i) for i = 1, 2 and a_3, a_4 \in V(A_3). Let V(U_1 \cap A_i) = \{v_i\} for i = 1, 2, 3, u_1 \in V(U_1) - \{v_1, v_2, v_3, v_4\}, and u_2 \in V(U_2) - \{w_1, w_2, w_3\}. By definition of obstruction, a_3, a_4 \in V(A_3) - \{v_3, v_4, w_3\} and, for
\( i = 1,2 \), if \( |V(A_i)| \geq 2 \) then \( a_i \in V(A_i) - \{v_i, w_i\} \). Note that \( A \) is independent, as otherwise the separation \((G[A], G - E(G[A]))\) shows that Theorem 2.1(iii) would hold for \((G, u_1, u_2, A)\).

Let \( v \) denote the vertex resulting from the identification of \( x \) and \( y \). Note that \( v \in \{v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq 3\} \), for otherwise \((G, u_1, u_2, A)\) is also an obstruction of type I, a contradiction. By symmetry, it suffices to consider four cases: \( v = v_1, v = w_1, v = w_3, \) and \( v = v_4 \). See Figure 7. Before distinguishing these four cases, we make observations (1), (2) and (3) below. Let \( A'_i = A_i \) if \( v \notin V(A_i) \), and otherwise let \( A'_i \) be obtained from \( A_i \) by unidentifying \( v \) to \( x \) and \( y \). Similarly, let \( U'_i = U_i \) if \( v \notin V(U_i) \), and otherwise let \( U'_i \) be obtained from \( U_i \) by unidentifying \( v \) to \( x \) and \( y \). When \( xy \in E(G) \), we put \( xy \) back in exactly one of \( U'_i \) and \( A'_i \).

1. If \( v \in \{v_1, v_4\} \) then \( v_i, w_j \notin A \) for all \( i, j \), and \( U_2 \) has three independent paths \( W_1, W_2, W_3 \) from \( u_2 \) to \( w_1, w_2, w_3 \), respectively.

Suppose \( v \in \{v_1, v_4\} \). Note that \( v_3, v_4, w_3 \notin A \) by definition of obstruction. Also, \( w_1, w_2 \notin A \) by Lemma 4.1. Hence, \( v_2 \notin A \) by definition of obstruction. Now suppose the second part of (1) fails. Then \( U_2 \) has a separation \((U_{21}, U_{22})\) such that \( |V(U_{21} \cap U_{22})| \leq 2 \), \( w_2 \in V(U_{21}) - V(U_{22}) \), and \( \{w_1, w_2, w_3\} \subseteq V(U_{22}) \). The separation \((U_{21}, U_{22} \cup U'_1 \cup A'_1 \cup A'_2 \cup A'_3)\) shows that Theorem 2.1(ii) holds, a contradiction.

2. For \( i \in \{1,2\} \), if \( v \notin V(A_i) \) and \( |V(A_i)| \geq 2 \), then \( A_i - v \) has a path \( W'_i \) from \( w_i \) to \( a_i \) and \( A_i - v \) has a path \( V'_i \) from \( v_i \) to \( a_i \) (and if \( |V(A_i)| = 1 \) then let \( W'_i = V'_i = A_i \)).

For, suppose \( W'_i \) does not exist. Then \( A_i \) has a separation \((A_{i1}, A_{i2})\) such that \( V(A_{i1} \cap A_{i2}) = \{v_i\}, w_i \in V(A_{i1}) \) and \( a_i \in V(A_{i2}) \). Now \( (A_{i2} + A, U'_1 \cup U'_2 \cup A_{i1} \cup A'_{3-i} \cup A'_3) \) shows that Theorem 2.1(iii) holds for \((G, u_1, u_2, A)\), a contradiction. So \( W'_i \) exists. Similarly, \( V'_i \) exists.

3. For \( i \in \{3,4\} \), if \( v \notin A_3 \) then \( A_3 - v \) has disjoint paths \( Q_i, R_i \) from \( w_3, v_i \), respectively, to \( \{a_3, a_4\} \).

Otherwise, \( A_3 \) has a separation \((A_{31}, A_{32})\) such that \( |V(A_{31} \cap A_{32})| \leq 2 \), \( v_{7-i} \in V(A_{31} \cap A_{32}) \), \( \{w_3, v_i\} \subseteq V(A_{31}) \), and \( \{a_3, a_4\} \subseteq V(A_{32}) \). Now the separation \((A_{32} + \{a_1, a_2\}, U'_1 \cup U'_2 \cup A'_1 \cup A'_2 \cup A_{31})\) shows that Theorem 2.1(iii) holds for \((G, u_1, u_2, A)\), a contradiction.

![Fig. 7](image-url) (G/xy, u_1, u_2, A) is an obstruction of type I.

Case 1. \( v = v_1 \).
Note that $A'_1$ has disjoint paths $X, Y$ from $\{x, y\}$ to $\{a_1, w_1\}$. Otherwise, $A'_1$ has a separation $(A_{11}, A_{12})$ such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq V(A_{11})$ and $\{a_1, w_1\} \subseteq V(A_{12})$. Then $U'_1 \cup A_{11}, U_2, A_{12}, A_2, A_3$ (when $a_1 \notin V(A_{11} \cap A_{12}) \neq \emptyset$) or $U'_1 \cup A_{11} + a_1, U_2 \cup A_{12}, \{a_1\}, A_2, A_3$ (when $V(A_{11} \cap A_{12}) \subseteq \{a_1\}$) show that $(G, u_1, u_2, A)$ is an obstruction of type I, a contradiction.

For any $i \in \{3, 4\}$, $U'_i$ does not contain three independent paths from $u_1$ to $x, y, v_i$, respectively; as such paths and $X, Y, W_1, W_2, W_3, W'_2, Q_i, R_i$ would form a topological $H$ in $G$ rooted at $u_1, u_2, A$. Thus, $U'_i$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{x, y, v_i\} \subseteq V(U_{12})$. Choose this separation to minimize $U_{12}$.

Suppose $|V(U_{11} \cap U_{12})| \leq 1$. If $|V(U_{11} \cap U_{12})| = 0$ or $\{v_2, v_4\} \not\subseteq V(U_{11}) - V(U_{12})$, then the separation $(U_{11}, U_{12} \cup U_2 \cup A'_1 \cup A_2 \cup A_3)$ shows that Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. So $|V(U_{11} \cap U_{12})| = 1$ and $v_2, v_4 \in V(U_{11}) - V(U_{12})$. Then $U_{11}, U_2, A_2, U_{12} \cup A'_1 \cup A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type I, a contradiction.

So let $V(U_{11} \cap U_{12}) = \{s_1, s_2\}$. By the minimality of $U_{12}, U_{12} - v_3$ (when $v_3 \notin \{s_1, s_2\}$) and $U_{12}$ (when $v_3 \in \{s_1, s_2\}$) contain disjoint paths $S_1, S_2$ from $\{s_1, s_2\}$ to $\{x, y\}$.

If $v_4 \notin V(U_{11}) - V(U_{12})$, then $v_2 \in V(U_{11}) - V(U_{12})$; otherwise the separation $(U_{11}, U_{12} \cup U_2 \cup A'_1 \cup A_2 \cup A_3)$ shows that Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. But then, $U_{11}, U_2, A_2, U_{12} \cup A'_1 \cup A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type II, a contradiction.

So $v_4 \in V(U_{11}) - V(U_{12})$. Now $U_{11}$ does not contain three independent paths from $u_1$ to $s_1, s_2, v_4$, respectively; otherwise these paths and $S_1, S_2$ would form three independent paths in $U'_i$ from $u_1$ to $x, y, v_4$, respectively (which were assumed to be nonexistent in the second paragraph of Case 1). Thus, $U_{11}$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 2$, $u_1 \in V(K) - V(L)$ and $\{s_1, s_2, v_4\} \subseteq V(L)$. If $v_2 \notin V(K) - V(L)$ or $|V(K \cap L)| \leq 1$ then $(K, L \cup U_{12} \cup U_2 \cup A'_1 \cup A_2 \cup A_3)$ shows that Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. So $v_2 \in V(K) - V(L)$ and $|V(K \cap L)| = 2$. Then $K, U_2, A_2, L \cup U_{12} \cup A'_1 \cup A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type II, a contradiction. 

Case 2. $v = v_4$.

Then $A'_3$ has three disjoint paths $P_1, P_2, P_3$ from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For, otherwise, $A'_3$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 2$, $\{v_3, x, y\} \subseteq V(A_{31})$, and $\{a_3, a_4, w_3\} \subseteq V(A_{32})$. If $|V(A_{31} \cap A_{32})| \leq 1$, then the separation $(A_{32} + \{a_1, a_2\}, U'_1 \cup U_2 \cup A_1 \cup A_2 \cup A_3)$ shows that Theorem 2.1(iii) would hold for $(G, u_1, u_2, A)$. So $V(A_{31} \cap A_{32}) = 2$. If $V(A_{31} \cap A_{32}) \cap A = \emptyset$ then $U'_1 \cup A_{31}, U_2, A_1, A_2, A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type I, a contradiction. If $V(A_{31} \cap A_{32}) \cap A = \{a_i\}$ for some $i \in \{3, 4\}$ then $U'_1 \cup A_{31}, U_2, A_1, A_2, \{a_i\}, A_3 - a_i$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction. Thus $a_3, a_4 \in V(A_{31} \cap A_{32})$; so $U'_1 \cup A_{31}, U_2 \cup A_{32}, A_1, A_2, \{a_3\}, \{a_4\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction.

If $U'_1$ has three independent paths from $u_1$ to $v_3, x, y$, respectively, then these paths and $P_1, P_2, P_3, W_1, W_2, W_3, W'_1, W'_2$ would form a topological $H$ in $G$ rooted at $u_1, u_2, A$. Thus $U'_1$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{v_3, x, y\} \subseteq V(U_{12})$.

Suppose $|V(U_{11} \cap U_{12})| \leq 1$. Then $|V(U_{11} \cap U_{12})| = 1$ and $\{v_1, v_2\} \not\subseteq V(U_{11}) - V(U_{12})$; otherwise the separation $(U_{11}, U_{12} \cup U_2 \cup A_1 \cup A_2 \cup A'_3)$ shows that Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. But then the separation $(U_{11} \cup U_2 \cup A'_1 \cup A_2, U_{12} \cup A'_3 + \{a_1, a_2\})$ shows that Theorem 2.1(iii) would hold for $(G, u_1, u_2, A)$. 

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show that \((G, u)\) of type IV with sides \((|v, 3, x, y|, u_1 \in V(U_1) - V(U_2'),\) and \(u_2 \in V(U_2') - V(U_1)\). Hence, if \(xy \in E(G)\) we put it in \(U_2'\), and if \(v_3v_4 \in E(G)\) we put it in \(U_1\). We apply Lemma 3.3 to \(A_3', v_3, v_4, x, y, a_3, a_4\) (as \(G, v_1, v_2, w_1, w_2, a_1, a_2\), respectively).

Suppose \(A_3'\) has a separation \((G_1, G_2)\) such that one of \((i) - (v)\) of Lemma 3.3 holds. If Lemma 3.3(ii) holds, then the separation \((G_2 \cup U_1 \cup U_2' \cup A_1 \cup A_2, G_1 + \{a_1, a_2\})\) shows that Theorem 2.1(iii) would hold for \((G, u_1, u_2, A)\). If Lemma 3.3(iii) holds then \((U_1 \cup G_i) + \{a_3, a_4\}, (U_2' \cup G_{3-i}) + \{a_3, a_4\}, A_1, A_2, \{a_3, a_4\}\) show that \((G, u_1, u_2, A)\) would be an obstruction of type IV. If Lemma 3.3(iv) holds then \(U_1, U_2', A_1, A_2, G_1, G_2\) show that \((G, u_1, u_2, A)\) would be an obstruction of type IV. If Lemma 3.3(v) holds then \(U_1 \cup U_2' \cup U_2, G_1, A_2, G_1\) when \(\{v_3, v_4\} \cap V(G_2) \neq \emptyset\) or \(U_1, U_2' \cup G_2, G_2, A_2, G_1\) when \(\{x, y\} \cap V(G_2) \neq \emptyset\) show that \((G, u_1, u_2, A)\) would be an obstruction of type I. Thus, Lemma 3.3(iii) holds. By symmetry between \(U_1\) and \(U_2'\), assume \(\{v_3, v_4, a_3, a_4\} \subseteq V(G_1)\) and \(\{x, y\} \subseteq V(G_2)\). If \(V(G_1 \cap G_2) = \{a_3, a_4\}\) then \(U_1 \cup U_1 \cup U_2' \cup U_2, G_1, A_2, \{a_3, a_4\}\) show that \((G, u_1, u_2, A)\) would be an obstruction of type IV. If \(V(G_1 \cap G_2) = \{a_3, a_4\}\) then \(U_2, A_2, \{a_3, a_4\}\) show that \((G, u_1, u_2, A)\) would be an obstruction of type IV. If \(V(G_1 \cap G_2) = \{a_3, a_4\}\) then \(U_1 \cap U_2' \cap U_2, G_1, A_2, \{a_3, a_4\}\) show that \((G, u_1, u_2, A)\) would be an obstruction of type IV. If \(V(G_1 \cap G_2) = \{a_3, a_4\}\) then \(U_1 \cup U_1 \cup U_2' \cup U_2, G_1, A_2, \{a_3, a_4\}\) show that \((G, u_1, u_2, A)\) would be an obstruction of type IV. So \(V(G_1 \cap G_2) = \{a_3, a_4\}\) is an obstruction of type IV with sides \((U_1 + a_3)/v_3v_4, U_2' \cup G_2\) and middle parts \(A_1, A_2, \{a_3, a_4\}, (G_1 - a_3)/v_3v_4\), contradicting Lemma 5.2.

Hence by Lemma 3.3, \(A_3'\) has three disjoint paths \(P_1, P_2, P_3\), with one from \(\{v_3, v_4\}\) to \(\{x, y\}\), one from \(\{v_3, v_4\}\) to \(\{a_3, a_4\}\), and another from \(\{x, y\}\) to \(\{a_3, a_4\}\).

For some \(s \in \{1, 2\}\), \(U_1 - (A \cap \{v_3-s\})\) has three independent paths \(S_1, S_2, S_3\) from \(u_1\) to \(v_s, v_3, v_4\), respectively. For, otherwise, by Lemma 5.1(i), \(v_1, v_2 \in N(u_1)\) and \(|N(u_1)| = 3\). Then by Lemma 4.1, \(N(u_1) \cap A = \emptyset\) (in particular, \(v_1, v_2 \notin A\)). Hence by Lemma 5.1(ii), \(N(u_1) \cap V(U_1) \subseteq N[u_1]\) for \(i = 1, 2\). Now \(G[N[u_1]], U_2', A_1, A_2, A_3' \cup (U_1 - \{u_1, v_1, v_2\})\) show that \((G, u_1, u_2, A)\) is an obstruction of type I, a contradiction.

Similarly, for some \(t \in \{1, 2\}\), \(U_2' - (A \cap \{w_3-t\})\) has three independent paths \(T_1, T_2, T_3\) from \(u_2\) to \(w_t, x, y\), respectively.

If \(s\) and \(t\) may be chosen so that \(s \neq t\), then \(S_1, S_2, S_3, T_1, T_2, T_3, V_s', W_t', P_1, P_2, P_3\) form a topological \(H\) in \(G\) rooted at \((u_1, u_2, A)\), a contradiction. Thus assume \(s = t = 1\) is the only possibility. So by Lemma 5.1(i), \(u_1 \in N(u_2)\) and \(|N(u_2)| = 3\), and \(v_1 \in N(u_1)\) and \(|N(u_1)| = 3\). By Lemma 4.1, \((N(u_1) \cup N(u_2)) \cap A = \emptyset\). Hence by Lemma 5.1(ii), \(N(v_1) \cap V(U_1) \subseteq N[u_1]\), and \(N(w_1) \cap V(U_2') \subseteq N[u_2]\). Thus, \(G[N[u_1]], G[N[u_2]], A_1, A_2 \cup A_3' \cup (U_1 - \{u_1, v_1\}) \cup (U_2' - \{u_2, w_1\})\) show that \((G, u_1, u_2, A)\) is an obstruction of type II, a contradiction.
Case 4. \( v = w_1 \).

As in Case 1, we can show that \( A_1' \) has disjoint paths \( X, Y \) from \( \{x, y\} \) to \( \{a_1, v_1\} \). Note that \( A_3 - w_3 \) has disjoint paths \( S, T \) from \( \{v_3, v_4\} \) to \( \{a_3, a_4\} \). For otherwise \( A_3 \) has a separation \((A_{31}, A_{32}) \) such that \( |V(A_{31} \cap A_{32})| \leq 2 \), \( w_3 \in V(A_{31} \cap A_{32}) \), \( \{v_3, v_4\} \subseteq V(A_{31}) \) and \( \{a_3, a_4\} \subseteq V(A_{32}) \). Hence the separation \((U_1 \cup U_2' \cup A_1' \cup A_2 \cup A_{31}, A_{32} + \{a_1, a_2\}) \) shows that Theorem 2.1(iii) would hold for \((G, u_1, w_2, A)\).

We claim that for some \( s \in \{2, 3\} \), \( U_2' - (A \cap \{w_{5-s}\}) \) has three independent paths \( P_1, P_2, P_3 \) from \( u_2 \) to \( x, y, w_s \), respectively. First, assume \( w_2 = a_2 \). Then \( U_2' - w_2 \) has three independent paths from \( u_2 \) to \( x, y, w_3 \), respectively; else by Lemma 5.1(i), \( w_2 \in N(u_2) \) and \( |N(u_2)| = 3 \), allowing us to use Lemma 4.1 to obtain a contradiction. So \( w_2 \neq a_2 \). Thus, if the claim fails then by Lemma 5.1(ii), \( w_2, w_3 \in N(u_2) \), \( |N(u_2)| = 3 \), and \( N\{w_2, w_3\} \subseteq N[u_2] \). Now \( U_1, G[N(u_2)], A_1' \cup (U_2' - \{w_2, w_3\}) \), \( A_2, A_3 \) show that \((G, u_1, u_2, A)\) would be an obstruction of type I.

Suppose \( s = 2 \). If \( U_1 - (A \cap \{v_2\}) \) has three independent paths from \( u_1 \) to \( v_1, v_3, v_4 \), respectively, then these paths and \( X, Y, S, T, P_1, P_2, P_3, W_2' \) would form a topological \( H \) in \( G \) rooted at \( u_1, w_2, A \). So such paths do not exist in \( U_1 - (A \cap \{v_2\}) \). If \( v_2 = a_2 \) then by Lemma 5.1(i), \( v_2 \in N(u_1) \) and \( |N(u_1)| = 3 \), which allows us to use Lemma 4.1 to obtain a contradiction. Thus \( v_2 \neq a_2 \) (and hence \( w_2 \neq a_2 \)). Then by Lemma 5.1(ii), \( v_2 \in N(u_1) \), \( |N(u_1)| = 3 \) and \( N(v_2) \cap V(U_1) \subseteq N[u_1] \). Suppose \( U_1 \) has three independent paths \( L_1, L_2, L_3 \) from \( u_1 \) to \( v_1, v_2 \) and one of \( \{v_3, v_4\} \), say \( v_3 \). If \( U_2' \) has three independent paths from \( u_2 \) to \( x, y, w_3 \), respectively, then these paths and \( L_1, L_2, L_3, X, Y, V_2', Q_3, R_3 \) (see (3)) would form a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \). So such paths do not exist in \( U_2' \). Then by Lemma 5.1(ii), \( v_2 \in N(u_2) \), \( |N(u_2)| = 3 \) and \( N[w_2] \cap V(U_2') \subseteq N[w_2] \). Now \( G[N[u_1]], G[N[u_2]], A_2, A_1' \cup A_3 \cup (U_1 - \{u_1, v_2\}) \cup (U_2' - \{w_2, w_3\}) \) (removing from the last subgraph possible edges with both ends in \( N(u_1) - \{v_2\} \)) in \( N[u_2] \), \( v_2 \neq a_2 \) an obstruction of type II, a contradiction. So these paths \( L_1, L_2, L_3 \) do not exist in \( U_1 \). Then by Lemma 5.1(ii), \( N(u_1) = \{v_2, v_3, v_4\} \) and \( N\{v_3, v_4\} \cap V(U_1) \subseteq N[u_1] \). Moreover, \( U_1 \) has a separation \((U_1, U_1') \) such that \( V(U_1 \cap U_1') = \emptyset \), \( \{u_1, v_2, v_3, v_4\} \subseteq V(U_1') \), and \( v_1 \in V(U_1') \). Now \( U_1 + v_1, U_2 \cup U_2' \cup A_1', \{a_1\}, A_2, A_3 \) show that \((G, u_1, u_2, A)\) is an obstruction of type I, a contradiction.

Thus \( s \) cannot be 2 (so \( s = 3 \)). By Lemma 5.1(ii), \( w_3 \in N(u_2) \), \( |N(u_2)| = 3 \) and \( N[w_3] \cap V(U_2') \subseteq N[w_2] \). If for some \( i \in \{3, 4\} \), \( U_1 \) has three independent paths from \( u_1 \) to \( v_1, v_2, v_i \), respectively, then these paths and \( P_1, P_2, P_3, X, Y, V_2', Q_3, R_3 \) (see (3)) would form a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \). So no such paths exist in \( U_1 \). Hence by Lemma 5.1(ii), \( v_3, v_4 \in N(u_1) \), \( |N(u_1)| = 3 \), and \( N\{v_3, v_4\} \cap V(U_1) \subseteq N[u_1] \). Now \( G[N[u_1]], G[N[u_2]], A_2, A_1' \cup A_3 \cup (U_1 - \{u_1, v_3, v_4\}) \cup (U_2' - \{w_2, w_3\}) \) (removing from the last subgraph possible edge with both ends in \( N[u_2] - \{w_3\} \)) show that \((G, u_1, u_2, A)\) is an obstruction of type III, a contradiction.

**Lemma 5.4.** Suppose \((G, u_1, u_2, A)\) is a counterexample to Theorem 2.1 with \(|V(G)|\) minimum, and let \( x, y \in V(G) - A - \{u_1, u_2\} \) be distinct. Then \((G/xy, u_1, u_2, A)\) is not an obstruction of type II.

**Proof.** Suppose \((G/xy, u_1, u_2, A)\) is an obstruction of type II with sides \( U_1, U_2 \) and middle parts \( A_1, A_2 \). Let \( V(U_1 \cap A_1) = \{v_1\}, V(U_2 \cap A_1) = \{w_1\}, V(U_1 \cap A_2) = \{v_2, v_3\}, V(U_2 \cap A_2) = \{w_2, w_3\}\).
\{w_2, w_3\}, u_1 \in V(U_1) - \{v_1, v_2, v_3\}, and u_2 \in V(U_2) - \{w_1, w_2, w_3\}. Let A := \{a_1, a_2, a_3, a_4\} such that a_1 \in V(A_1) and a_2, a_3, a_4 \in V(A_2) - \{v_2, v_3, w_2, w_3\}. Then A is independent, else \((G[A], G - E(G[A]))\) shows that Theorem 2.1(iii) would hold for \((G, u_1, u_2, A)\).

Let v denote the vertex resulting from the identification of x and y. If v \notin \{v_i, w_i : 1 \leq i \leq 3\} then \((G, u_1, u_2, A)\) would be an obstruction of type II. So by symmetry assume v \in \{v_1, v_3\}. Then by Lemma 4.1, w_1 \notin A and, hence, v_1 \notin A. As (1) and (2) in the proof of Lemma 5.3, \(U_2\) has three independent paths \(W_1, W_2, W_3\) from \(w_2\) to \(w_1, w_2, w_3\), respectively, and if v \neq v_1 then \(A_1 - v_1\) has a path \(W'_1\) from \(v_1\) to \(a_1\).

\[\text{Case 1. } v = v_1.\]

Let \(U'_1, A'_1\) be obtained from \(U_1, A_1\), respectively, by unidentifying v to x and y. Note that \(A'_1\) has disjoint paths X, Y from \(\{x, y\}\) to \(\{a_1, w_1\}\); otherwise \(A'_1\) has a separation \((A_1, A_{12})\) such that \(|V(A_{11} \cap A_{12})| \leq 1\), \(\{x, y\} \subseteq V(A_{11})\) and \(\{a_1, w_1\} \subseteq V(A_{12})\), and hence \(U'_1 \cup A_{11}, U_2, A_{12}, A_2\) (when \(V(A_{11} \cap A_{12}) \subseteq \{a_1\}\)) or \((U'_1 \cup A_{11}) + A_1, U_2 \cup A_{12}, \{a_1\}, A_2\) (when \(V(A_{11} \cap A_{12}) \subseteq \{a_1\}\)) show that \((G, u_1, w_2, A)\) is an obstruction of type II, a contradiction.

For some \(s \in \{2, 3\}\), \(U'_1\) has three independent paths \(P_1, P_2, P_3\) from \(u_1\) to \(x, y, v_s\), respectively. Otherwise by Lemma 5.1(ii), \(v_2, v_3 \in N(u_1)\), \(|N(u_1)| = 3\) and \(N(v_2, v_3) \cap V(U'_1) \subseteq N[u_1]\). Hence \(G[N[u_1]], U_2, A'_1 \cup (U'_1 - \{u_1, v_2, v_3\}), A_2\) show that \((G, u_1, w_2, A)\) is an obstruction of type II, a contradiction.

Without loss of generality, let \(s = 2\). If \(A_2 - v_3\) has three disjoint paths from \(\{a_2, a_3, a_4\}\) to \(\{v_2, w_2, w_3\}\), then these paths and \(P_1, P_2, P_3, W_1, W_2, W_3, X, Y\) would form a topological H in \(G\) rooted at \(u_1, u_2, A\). So \(A_2\) has a separation \((A_{21}, A_{22})\) such that \(|V(A_{21} \cap A_{22})| \leq 3\), \(v_3 \in V(A_{21} \cap A_{22})\), \(\{a_2, a_3, a_4\} \subseteq V(A_{22})\), and \(\{v_2, w_2, w_3\} \subseteq V(A_{21})\). Then the separation \((A_{21} \cup U'_1 \cup A'_1 \cup U_2, A_{22} + a_1)\) shows that Theorem 2.1(iii) holds for \((G, u_1, w_2, A)\), a contradiction.

\[\text{Case 2. } v = v_3.\]

Let \(U'_1, A'_2\) be obtained from \(U_1, A_2\), respectively, by unidentifying v to x and y. We choose such \(U'_1, U_2, A'_2\) (while fixing \(A_1\)) to maximize \(U'_1 \cup U_2\) (subject to \(a_2, a_3, a_4 \in V(A'_2) - \{v_2, w_2, w_3, x, y\}\)). Then \(xy, w_2w_3 \notin E(A'_2)\).

We claim that \(U'_1\) has three independent paths \(P_1, P_2, P_3\) from \(u_1\) to \(v_2, x, y\), respectively. Otherwise, by Lemma 5.1(ii), \(v_1 \in N(u_1), |N[u_1]| = 3\), and \(N(v_1) \cap V(U'_1) \subseteq N[u_1]\). Then, \(G[N[u_1]], U_2, A_2 \cup (U'_1 - \{u_1, v_1\})\) (removing from the last subgraph the possible edge with both ends in \(N[u_1] - \{v_1\}\)) show that \((G, u_1, w_2, A)\) is an obstruction of type II, a contradiction.

If \(A'_2 := A'_2 + w_2w_3\) has three disjoint paths from \(\{a_2, a_3, a_4\}\) to \(\{v_2, x, y\}\) and through \(w_2w_3\),
then these paths (deleting \( w_2w_3 \)) and \( P_1, P_2, P_3, W_1, W_2, W_3, W'_1 \) would form a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \). So one of \((i)-(iv)\) of Lemma 3.4 holds, with \( A'_i, \{a_2, a_3, a_4\}, \{v_2, x, y\}, w_2w_3 \) as \( G, A, B, e \), respectively. We use the notation in Lemma 3.4. See Figure 5.

If Lemma 3.4\((ii)\) holds then the separation \((U_2 \cup (G_1 - w_2w_3), U'_1 \cup G_2 \cup A_1)\) shows that Theorem 2.1\((ii)\) would hold for \((G, u_1, u_2, A)\).

Suppose Lemma 3.4\((iv)\) holds. For \( i \in \{2, 3, 4\} \), if \( V(G_i \cap G_1) \cap A \neq \emptyset \) then let \( G'_i = G_i \) and \( A'_i = G_i \cap G_1 \), and otherwise let \( G'_i = \emptyset \) and \( A'_i = G_i \). Then \( U'_1 \cup G'_2 \cup G'_3 \cup G'_4, U_2 \cup G_1, A_1, A'_2, A'_3, A'_4 \) show that \((G, u_1, u_2, A)\) is an obstruction of type IV, a contradiction.

Now suppose Lemma 3.4\((iii)\) holds. If \( V(G_1 \cap G_2) = \emptyset \) or \( V(G_3 \cap G_2) = \emptyset \) then the separation \((U_2 \cup (G_2 - w_2w_3), U'_1 \cup A_1 \cup G_1 \cup G_3)\) shows that Theorem 2.1\((ii)\) would hold for \((G, u_1, u_2, A)\). If \( V(G_1 \cap G_2) \cap A \neq \emptyset \) or \( V(G_3 \cap G_2) \cap A \neq \emptyset \) then the separation \((U_2 \cup (G_2 - w_2w_3), U'_1 \cup A_1 \cup G_1 \cup G_3)\) shows that \((G, u_1, u_2, A)\) is an obstruction of type IV, a contradiction.

So Lemma 3.4\((i)\) holds. Then \( w_2w_3 \in E(G_1) \); otherwise, \( w_2w_3 \in E(G_2) \), and the separation \((A_1 \cup U'_1 \cup U_2 \cup (G_2 - w_2w_3), G_1 + a_1)\) shows that Theorem 2.1\((iii)\) would hold for \((G, u_1, u_2, A)\).

Suppose \(|V(G_1 \cap G_2)| \leq 2\). If \( V(G_1 \cap G_2) \cap A \neq \emptyset \) then the separation \((U'_1 \cup U_2 \cup (G_1 - w_2w_3) \cup A_1)\) allows us to use Lemma 4.1 to obtain a contradiction. So \(|V(G_1 \cap G_2)| = |V(G_3 \cap G_2)| = 1\) and \( V(G_2) \cap A = \emptyset \). Now \( U'_1, U_2 \cup (G_2 - w_2w_3), A_1, G_1, G_3 \) show that \((G, u_1, u_2, A)\) is an obstruction of type I, a contradiction.

Thus \(|V(G_1 \cap G_2)| = 3\). If \( V(G_1 \cap G_2) \cap A = \emptyset \) then \( U'_1 \cup U_2 \cup G_2, A_1, G_1 - w_2w_3 \) contradict the choice of \( U'_1 \cup U_2 \cup A_1 \) (the maximality of \( U'_1 \cup U_2 \)). If \( V(G_1 \cap G_2) \cap A = \{a_i\}\) for some \( i \in \{2, 3, 4\} \) then let \( V(G_1 \cap G_2) - \{a_i\} = \{v, w\} \); now \((G/vw, u_1, u_2, A)\) is an obstruction of type I with sides \((U'_1 \cup G_2)/vw, U_2 + a_i\) and middle parts \( A_1, \{a_i\}, (G_1 - a_i - w_2w_3)/vw \), contradicting Lemma 5.3. Since \( V(G_1 \cap G_2) - A \neq \emptyset \), \( V(G_1 \cap G_2) \cap A = \{a_i, a_j\}\) for some distinct \( i, j \in \{2, 3, 4\} \). Now \((G/w_2w_3, u_1, u_2, A)\) is an obstruction of type IV with sides \((U'_1 \cup G_2, (U_2 + \{a_i, a_j\})/w_2w_3, A_1, \{a_i\}, \{a_j\}, (G_1 - \{a_i, a_j\})/w_2w_3)\), contradicting Lemma 5.2.

\[\tag{3.5}\]

**Lemma 5.5.** Suppose \((G, u_1, u_2, A)\) is a counterexample to Theorem 2.1 with \(|V(G)|\) minimum, and let \( x, y \in V(G) - A - \{u_1, u_2\}\) be distinct. Then \((G/xy, u_1, u_2, A)\) is not an obstruction of type III.

**Proof.** Suppose \((G/xy, u_1, u_2, A)\) is an obstruction of type III with sides \( U_1, U_2 \) and middle parts \( A_1, A_2 \). Let \( V(U_1 \cap A_1) = \{v_1\}, V(U_1 \cap A_2) = \{v_2, v_3\}, V(U_2 \cap A_1) = \{w_1, w_2\}, V(U_2 \cap A_2) = \{w_3\}, u_1 \in V(U_1) - \{v_1, v_2, v_3\}\), and \( u_2 \in V(U_2) - \{w_1, w_2, w_3\}\). Let \( A := \{a_1, a_2, a_3, a_4\} \) such that \( a_1, a_2 \in V(A_1) - \{v_1, w_1, w_2\} \), and \( a_3, a_4 \in V(A_2) - \{v_2, v_3, w_3\} \). As before, \( A \) is independent in \( G \).

Let \( v \) denote the vertex resulting from the identification of \( x \) and \( y \). Now \( v \in \{v_i, w_i : 1 \leq i \leq 3\} \); otherwise \((G, u_1, u_2, A)\) would be an obstruction of type III. Thus by symmetry, assume \( v \in \{v_1, v_2\} \). Let \( U'_1 \) (respectively, \( A'_i \) if \( v = v_i \)) be obtained from \( U_1 \) (respectively, \( A_i \))
by unidentifying \( v \) back to \( x \) and \( y \). Let \( A'_i = A_i \) when \( v \notin A_i \). See Figure 9. We choose such \( U'_1, U_2, A'_1, A'_2 \) to maximize \( U'_1 \cup U_2 \). Thus if \( xy \in E(G) \) then \( xy \in E(U'_1) \), and if \( w_1w_2 \in E(G) \) then \( w_1w_2 \in E(U_2) \).

As (1) in the proof of Lemma 5.3, \( U_2 \) contains three independent paths \( W_1, W_2, W_3 \) from \( u_2 \) to \( w_1, w_2, w_3 \), respectively.

**Case 1.** \( v = v_2 \).

We claim that \( A'_2 \) has three disjoint paths \( Q_1, Q_2, Q_3 \) from \( \{v_3, x, y\} \) to \( \{a_3, a_4, w_3\} \). For, otherwise, \( A'_2 \) has a separation \((A_{21}, A_{22})\) such that \( |V(A_{21} \cap A_{22})| \leq 2 \), \( \{a_3, a_4, w_3\} \subseteq V(A_{22}) \) and \( \{v_3, x, y\} \subseteq V(A_{21}) \). If \( |V(A_{21} \cap A_{22})| \leq 1 \) then the separation \((A_{22} \cup A_{1} \cup U_2, \{a_1, a_2\}, U'_1 \cup A_{21})\) shows that Theorem 2.1 \((ii)\) would hold for \((G, u_1, u_2, A)\). So \( |V(A_{21} \cap A_{22})| = 2 \). If \( V(A_{21} \cap A_{22}) \cap A = \emptyset \) then \( U'_1 \cup U_2, A_1, A_{22} \) show that \((G, u_1, u_2, A)\) is an obstruction of type III, a contradiction. So \( V(A_{21} \cap A_{22}) \cap A \neq \emptyset \). Then the separation \((U'_1 \cup A_{21}, U_2 \cup A_1 \cup A_{22})\) allows us to apply Lemma 4.1 to obtain a contradiction.

Also, \( A_1 - v_1 \) contains disjoint paths \( R_1, R_2 \) from \( \{v_1, w_2\} \) to \( \{a_1, a_2\} \). For, otherwise, \( A_1 \) has a separation \((A_{11}, A_{12})\) such that \( |V(A_{11} \cap A_{12})| \leq 2 \), \( v_1 \in V(A_{11} \cap A_{12}) \), \( \{w_1, w_2\} \subseteq V(A_{11}) \) and \( \{a_1, a_2\} \subseteq V(A_{12}) \). Then the separation \((U'_1 \cup U_2 \cup A_{11} \cup A_{22}, A_{12} \cup \{a_3, a_4\})\) shows that Theorem 2.1 \((iii)\) holds for \((G, u_1, u_2, A)\), a contradiction.

If \( U'_1 \) has three independent paths from \( u_1 \) to \( v_3, x, y \), respectively, then these paths and \( Q_1, Q_2, Q_3, R_1, R_2, W_1, W_2, W_3 \) would form a topological \( H \) in \( G \) rooted at \( u_1, u_2, A \). So such paths do not exist in \( U'_1 \). By Lemma 5.1 \((ii)\), \( v_1 \in N(u_1) \), \( |N(u_1)| = 3 \) and \( N(v_1) \cap V(U'_1) \subseteq N[u_1] \). Hence, \( G[N[u_1]], U_2, A_1, A'_2 \cup (U'_1 - \{u_1, v_1\}) \) (removing from the last subgraph the possible edge with both ends in \( N(u_1) - \{v_1\} \)) show that \((G, u_1, u_2, A)\) is an obstruction of type III, a contradiction.

**Case 2.** \( v = v_1 \).

Note that for any \( i \in \{2, 3\} \), \( A_2 - v_{5-i} \) contains disjoint paths \( Q_i, R_i \) from \( \{w_3, v_i\} \) to \( \{a_3, a_4\} \). For, otherwise, \( A_2 \) has a separation \((A_{21}, A_{22})\) such that \( |V(A_{21} \cap A_{22})| \leq 2 \), \( v_{5-i} \in V(A_{21} \cap A_{22}) \), \( \{a_3, a_4\} \subseteq V(A_{21}) \) and \( \{w_3, v_i\} \subseteq V(A_{22}) \). Then \((A_{21} + \{a_1, a_2\}, U'_1 \cup U_2 \cup A_{22} \cup A'_1)\) shows that Theorem 2.1 \((iii)\) holds for \((G, u_1, u_2, A)\), a contradiction. We apply Lemma 3.3 to \( A'_1, x, y, w_1, w_2, a_1, a_2 \) (as \( G, v_1, v_2, w_1, w_2, a_1, a_2 \), respectively).

Suppose \( A'_1 \) has three disjoint paths \( P_1, P_2, P_3 \), with one from \( \{x, y\} \) to \( \{w_1, w_2\} \), one from \( \{x, y\} \) to \( \{a_1, a_2\} \), and another from \( \{w_1, w_2\} \) to \( \{a_1, a_2\} \). If for some \( i \in \{2, 3\} \), \( U'_1 \) has three
Lemma 6.1. Let \((u_1, u_2, A)\) be a quadruple in which \(N(u_1) \cap N(u_2) \subseteq A\), and there exist \(xy \in E(G - A - \{u_1, u_2\})\) and a separation \((G_1, G_2)\) in \(G\) such that

\[
\begin{align*}
(1) & \quad \{x, y\} \not\subseteq N(u_i) \text{ for } i \in \{1, 2\}, \\
(2) & \quad x, y \in V(G_1 \cap G_2), \ xy \in E(G_1), \text{ and} \\
(3) & \quad |V(G_1 \cap G_2)| = 5, u_1, u_2 \in V(G_1) - V(G_2), \text{ and } A \subseteq V(G_2).
\end{align*}
\]

See Figure 10. Quadruples satisfying (1), (2) and (3) will occur in our proof of Theorem 2.1. The aim of this section is to show that such quadruples (with additional properties (4) and (5) below) cannot be a minimum counterexample to Theorem 2.1. First, we prove a lemma about disjoint paths in \(G_2\), which will be used frequently in this section.

Lemma 6.1. Let \((u_1, u_2, A)\) be a quadruple in which \(G\) has a separation \((G_1, G_2)\) satisfying (1), (2) and (3) above. Suppose Theorem 2.1(iii) fails for \((u_1, u_2, A)\), and let \(v \in V(G_1 \cap G_2)\).

\[
\begin{align*}
(i) & \quad \text{If } v \notin A \text{ then } G_2 - v \text{ has four disjoint paths from } V(G_1 \cap G_2) - \{v\} \text{ to } A, \text{ and} \\
(ii) & \quad \text{if } v \in A \text{ and } N(v) \cap V(G_2) \neq \emptyset, \text{ } G_2 \text{ has four disjoint paths from } V(G_1 \cap G_2) - \{v\} \text{ to } A.
\end{align*}
\]

Proof. Suppose (i) fails. Then \(G_2\) has a separation \((K, L)\) such that \(v \in V(K \cap L), |V(K \cap L)| \leq 4, V(G_1 \cap G_2) \subseteq V(K), \text{ and } A \subseteq V(L)\). Hence, the separation \((G_1 \cup K, L)\) shows that \((u_1, u_2, A)\) satisfies Theorem 2.1(iii), a contradiction.

6 Separations of order five

In this section, we let \((G, u_1, u_2, A)\) be a quadruple in which \(N(u_1) \cap N(u_2) \subseteq A\), and there exist \(xy \in E(G - A - \{u_1, u_2\})\) and a separation \((G_1, G_2)\) in \(G\) such that

\[
\begin{align*}
(1) & \quad \{x, y\} \not\subseteq N(u_i) \text{ for } i \in \{1, 2\}, \\
(2) & \quad x, y \in V(G_1 \cap G_2), \ xy \in E(G_1), \text{ and} \\
(3) & \quad |V(G_1 \cap G_2)| = 5, u_1, u_2 \in V(G_1) - V(G_2), \text{ and } A \subseteq V(G_2).
\end{align*}
\]

See Figure 10. Quadruples satisfying (1), (2) and (3) will occur in our proof of Theorem 2.1. The aim of this section is to show that such quadruples (with additional properties (4) and (5) below) cannot be a minimum counterexample to Theorem 2.1. First, we prove a lemma about disjoint paths in \(G_2\), which will be used frequently in this section.

Lemma 6.1. Let \((G, u_1, u_2, A)\) be a quadruple in which \(G\) has a separation \((G_1, G_2)\) satisfying (1), (2) and (3) above. Suppose Theorem 2.1(iii) fails for \((G, u_1, u_2, A)\), and let \(v \in V(G_1 \cap G_2)\).

\[
\begin{align*}
(i) & \quad \text{If } v \notin A \text{ then } G_2 - v \text{ has four disjoint paths from } V(G_1 \cap G_2) - \{v\} \text{ to } A, \text{ and} \\
(ii) & \quad \text{if } v \in A \text{ and } N(v) \cap V(G_2) \neq \emptyset, \text{ } G_2 \text{ has four disjoint paths from } V(G_1 \cap G_2) - \{v\} \text{ to } A.
\end{align*}
\]

Proof. Suppose (i) fails. Then \(G_2\) has a separation \((K, L)\) such that \(v \in V(K \cap L), |V(K \cap L)| \leq 4, V(G_1 \cap G_2) \subseteq V(K), \text{ and } A \subseteq V(L)\). Hence, the separation \((G_1 \cup K, L)\) shows that \((G, u_1, u_2, A)\) satisfies Theorem 2.1(iii), a contradiction.
Thus $y$ (otherwise, the separation $(K, L)$ holds for $(G, u, A)$, a contradiction. So $V(L) = A$ and $E(G[A]) = \emptyset$. Since $N(v) \cap V(G) \neq \emptyset$, $v \in V(K \cap L)$ and, hence, $V(G_1 \cap G_2) \subseteq V(K)$. Therefore, $(G_1 \cup K, L)$ is a separation of order at most 3 in $G$, contradicting the assumption that Theorem 2.1(iii) fails for $(G, u_1, u_2, A)$.

We choose $(G_1, G_2)$ such that, subject to (1), (2) and (3),

4. $G_1$ is minimal.

In the rest of this section, we let $A' := V(G_1 \cap G_2) - \{x\} = \{y, a'_2, a'_3, a'_4\}$, and assume that

5. $xu_1, yu_2 \in E(G), N(x) \cap (V(G_1) - V(G_2)) \subseteq N[u_1]$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2]$.

**Lemma 6.2.** If $(G, u_1, u_2, A)$ is a counterexample to Theorem 2.1 with $|V(G)|$ minimum and $G$ has a separation $(G_1, G_2)$ satisfying (1)–(5) above, then (i), (ii) and (iii) of Theorem 2.1 do not hold for $(G_1, u_1, u_2, A')$ and, moreover, $(G_1, u_1, u_2, A')$ is an obstruction of type I, or II, or IV, with $\{y\}$ as a middle part.

**Proof.** By the minimality of $|V(G)|$, Theorem 2.1 holds for $(G_1, u_1, u_2, A')$. If Theorem 2.1(i) holds then a topological $H$ in $G_1$ rooted at $u_1, u_2, A'$ and four disjoint paths in $G_2 - x$ from $A'$ to $A$ (by Lemma 6.1(i)) would form a topological $H$ in $G$ rooted at $u_1, u_2, A$.

Assume that Theorem 2.1(ii) holds and that $G_1$ has a separation $(U_1, U_2)$ such that $|V(U_1 \cap U_2)| \leq 2$, $u_1 \in V(U_1) - V(U_2)$, and $A' \cup \{u_2\} \subseteq V(U_2)$. Then $|V(U_1 \cap U_2)| = 2$ and $x \in V(U_1) - V(U_2)$; as otherwise the separation $(U_1, U_2 \cup G_2)$ shows that Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. Thus $y \in V(U_1 \cap U_2)$ as $xy \in E(G_1)$ and $y \in A'$. If $|V(U_1)| = 4$ then, since $\{x, y\} \nsubseteq N(u_1)$ (by (1)), Theorem 2.1(ii) holds. So $|V(U_1)| \geq 5$. Thus, $(U_1, U_2 \cup G_2)$ is a separation in $G$ contradicting Lemma 4.2.

Now assume that Theorem 2.1(iii) holds; so $G_1$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 4, u_1, u_2 \in V(K) - V(L)$, and $A' \subseteq V(L)$. Then $x \in V(K) - V(L)$ and $|V(K \cap L)| = 4$; otherwise, the separation $(K, L \cup G_2)$ shows that Theorem 2.1(iii) would hold for $(G, u_1, u_2, A)$. Thus $y \in V(K \cap L)$ as $xy \in E(G_1)$ and $y \in A'$; so $(K, (L + x) \cup G_2)$ contradicts the choice of $(G_1, G_2)$ in (4) (that $G_1$ is minimal).

Thus, Theorem 2.1(iv) holds; so $(G_1, u_1, u_2, A')$ is an obstruction. As $y \in A'$, $y$ belongs to some middle part. Since $y \in N(u_2)$, $y$ belongs to the side containing $u_2$. Thus, by definition
of obstructions, the middle part containing \( y \) is in fact \( \{ y \} \). As a consequence, \( (G_1, u_1, u_2, A') \) cannot be an obstruction of type III.

In the next three lemmas, we consider the obstruction types of \((G_1, u_1, u_2, A')\), and show that \((G, u_1, u_2, A)\) cannot be a minimum counterexample to Theorem 2.1.

**Lemma 6.3.** If \((G, u_1, u_2, A)\) is a counterexample to Theorem 2.1 with \(|V(G)|\) minimum and \( G \) has a separation \((G_1, G_2)\) satisfying (1)–(5) above, then \((G_1, u_1, u_2, A')\) is not an obstruction of type IV.

**Proof.** Suppose \((G_1, u_1, u_2, A')\) is an obstruction of type IV with sides \(U_1, U_2\) and middle parts \(A_1, A_2, A_3, A_4\). For \(1 \leq i \leq 4\), let \( V(U_1 \cap A_i) = \{v_i\} \) and \( V(U_2 \cap A_i) = \{w_i\} \). Let \( u_1 \in V(U_1) - \{v_1, v_2, v_3, v_4\} \) and \( u_2 \in V(U_2) - \{w_1, w_2, w_3, w_4\} \). We choose such \( U_i, A_j \) so that \( U_1 \cup U_2 \) is maximal. By Lemma 6.2, let \( V(A_i) = \{y\} \) and let \( a'_i \in V(A_i) \) for \(2 \leq i \leq 4\). By (5), \( x \in V(U_1)\). If \( x = v_i \) for some \( i \in \{2, 3, 4\} \) then \((G_1 - V(A_i - \{v_i, w_i\}), G_2 \cup A_i)\) contradicts the choice of \( (G_1, G_2)\) (see (4)). So \( x \notin \{v_2, v_3, v_4\} \). By (5), \( N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2] \). Hence, we have symmetry between \(U_1 - y, u_1, x, v_2, v_3, v_4\) and \(U_2, u_2, y, w_2, w_3, w_4\). See Figure 11.

![Figure 11: \((G_1, u_1, u_2, A')\) is of type IV.](image)

(a) For each \(i \in \{2, 3, 4\}\) with \(|V(A_i)| \geq 2\), \(A_i - v_i\) has a path \(W'_i\) from \(w_i\) to \(a'_i\), \(A_i - w_i\) has a path \(V'_i\) from \(v_i\) to \(a'_i\), and \(A_i - a'_i\) has a path \(R_i\) from \(v_i\) to \(w_i\). (When \(|V(A_i)| = 1\) let \(W'_i = V'_i = R_i\) consist of only \(a'_i\).) First, suppose \(W'_i\) does not exist and, without loss of generality, let \(i = 2\). Then \(A_2\) has a separation \((A_2, A_2')\) such that \(V(A_2) \cap A_2 = \{v_2\}, w_2 \in V(A_2)\) and \(a'_2 \in V(A_2')\), and hence \((U_1 \cup U_2 \cup A_1 \cup A_3 \cup A_4, A_{22} + A')\) shows that Theorem 2.1(iii) holds for \((G_1, u_1, u_2, A')\), contradicting Lemma 6.2. So \(W'_i\) does exist. Similarly, \(V'_i\) exists. Now suppose \(R_i\) does not exist. Then \(A_i\) has a separation \((A_{1i}, A_{2i})\) such that \(V(A_{1i}) \cap A_{2i} = \{v'_i\}, v'_i \in V(A_{1i})\) and \(w_i \in V(A_{2i})\). Replacing the sides \(U_1, U_2\) with \(U_1 \cup A_{1i}, U_2 \cup A_{2i}\), and replacing the middle part \(A_i\) with \(\{a'_i\}\), we get a contradiction to the maximality of \(U_1 \cup U_2\).

(b) There exists a permutation \(ijk\) of \(\{2, 3, 4\}\) such that \((U_1 - y) - v_k\) has three independent paths \(P_1, P_2, P_3\) from \(u_1\) to \(x, v_i, v_j\), respectively, and there is a permutation \(rst\) of \(\{2, 3, 4\}\) such that \(U_2 - w_1\) has three independent paths \(Q_1, Q_2, Q_3\) from \(u_2\) to \(y, w_r, w_s\), respectively. For, otherwise, suppose by symmetry that \(P_1, P_2, P_3\) do not exist. By Lemma 5.1(i), \(N(u_1) = \{v_2, v_3, v_4\}\), contradicting (5) (that \(x \in N(u_1)\)).

(c) \(|N(u_1)| = 3 = |N(u_2)|\) and, for any choice of \(P_1, P_2, P_3\) in (b) and any choice of \(Q_1, Q_2, Q_3\) in (b), we have \(N(u_1) \cap \{v_r, v_s\} \neq \emptyset\) and \(N(u_2) \cap \{w_i, w_j\} \neq \emptyset\), where \(ijk\) and \(rst\) are permutations of \(\{1, 2, 3\}\) in (b). By symmetry we only prove the claim for \(u_2\). If \(t \neq j\) for
every choice of $Q_1, Q_2, Q_3$ above, then $U_2 - w_j$ does not have three independent paths from $u_2$ to $y, w_1, w_3$, respectively; so by Lemma 5.1(i), $|N(u_2)| = 3$ and $w_j \in N(u_2)$. Similarly, if $t \neq i$ for every choice of $Q_1, Q_2, Q_3$ above, then we have $|N(u_2)| = 3$ and $w_i \in N(u_2)$. In either case, (c) holds for $u_2$. Thus assume that we may choose $Q_1, Q_2, Q_3$ so that $t = i$ and we may choose $Q_1, Q_2, Q_3$ so that $t = j$. Suppose $a_i', a_j' \in A$, $|V(A_i)| = 1$ or $N(a_i') \cap V(G_2) = \emptyset$, and $|V(A_j)| = 1$ or $N(a_j') \cap V(G_2) = \emptyset$. If $w_k \notin A$ then $(G/yw_k, u_1, u_2, A)$ is an obstruction of type I, with sides $U_1 - y, U_2/yw_k$ and middle parts $A_1, A_2, (A_k \cup G_2) + xy - \{a_i', a_j'\}/yw_k$, contradicting Lemma 5.3. So $w_k \in A$. Then $V(A_k) = \{w_k\}$ by the maximality of $U_1 \cup U_2$; so $(G, u_1, u_2, A)$ is an obstruction of type IV with sides $U_1 - y, U_2$ and middle parts $A_1, A_2, A_3, A_4, G_2 - \{a_1', a_3', a_4'\} + xy$, a contradiction. Hence, by symmetry we may assume that $a_i' \notin A$, or $a_i' \in A, |V(A_i)| \geq 2$ and $N(a_i') \cap V(G_2) = \emptyset$. Choose $Q_1, Q_2, Q_3$ so that $t = j$. Then by Lemma 6.1, $G_2 - a_i'$ (when $a_i' \notin A$) and $G_2$ (when $a_i' \in A$) has four disjoint paths from $\{x, y, a_i', a_2'\}$ to $A$. In either case these four paths and $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_i, V_1', W_k$ (see (a)) form a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction.

(d) There exists some $\ell \in \{2, 3, 4\}$ such that $v_\ell \in N(u_1)$ and $w_\ell \in N(u_2)$. By (c), assume $w_j \notin N(u_1)$, as otherwise we may let $\ell = j$. Then by Lemma 5.1(i), $(U_1 - y) - v_j$ has three independent paths from $u_1$ to $x, v_1, v_k$, respectively; so by (c) again, $N(u_2) \cap \{w_i, w_k\} = \emptyset$. Hence $|N(u_2) \cap \{w_2, w_3, w_4\}| \geq 2$. By symmetry we could also prove $|N(u_1) \cap \{v_2, v_3, v_4\}| \geq 2$. Hence, $\ell$ exists.

Without loss of generality, let $v_3 \in N(u_1)$ and $w_3 \in N(u_2)$. By (c) and Lemma 4.1, $N(u_i) \cap A = \emptyset$ for $i = 1, 2$. So $|V(A_3)| \geq 2$ as $N(u_i) \cap N(u_2) \subseteq A$.

(e) There exists $b \in \{2, 4\}$ such that $v_b \in N(u_1)$ and $w_b \in N(u_2)$. Otherwise, by symmetry and by (c), since $|N(u_1)| = |N(u_2)| = 3$ for $\ell = 1, 2$, we may assume $v_2 \notin N(u_1)$ and $w_2 \notin N(u_2)$. Then by Lemma 5.1(i), $(U_1 - y) - v_2$ has three independent paths $P_1', P_3', P_3'$ from $u_1$ to $x, v_3, v_4$, respectively, and $U_2 - w_4$ has three independent paths $Q_1', Q_2', Q_2'$ from $u_2$ to $y, w_3, w_2$, respectively. If $a_3' \notin A$ then by Lemma 6.1(i), $G_3 - a_3'$ has four disjoint paths from $\{x, y, a_2', a_4'\}$ to $A$; if $a_3' \in A$ and $N(a_3') \cap V(G_2) = \emptyset$ then by Lemma 6.1(ii), $G_2$ has four disjoint paths from $\{x, y, a_3', a_1'\}$ to $A$. In either case the four paths in $G$ and $P_1, P_1, P_3', Q_1', Q_2', Q_3, R_3, V_1', W_2'$ (see (a)) form a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction. So $a_3' \notin A$ and $N(a_3') \cap V(G_2) = \emptyset$. Similarly, if $U_1 - y$ has three independent paths from $u_1$ to $x, v_2, v_4$ then $a_2' \in A$ and $N(a_2') \cap V(G_2) = \emptyset$. In this case, if $w_4 \notin A$ then $(G/yw_4, u_1, u_2, A)$ is an obstruction of type I with sides $U_1 - y, U_2/yw_4$ and middle parts $A_2, A_3, (G_2 + A_2 - \{a_2', a_4'\} + xy)/yw_4$, contradicting Lemma 5.3; and if $w_4 \in A$ then $V(A_2) = \{w_4\}$ by the maximality of $U_1 \cup U_2$, which implies that $(G, u_1, u_2, A)$ is an obstruction of type IV with sides $U_1 - y, U_2$ and middle parts $A_2, A_3, A_4, G_2 - \{a_2', a_3', a_4'\} + xy$, a contradiction. Thus $U_1 - y$ has no three independent paths from $u_1$ to $x, v_2, v_4$. Since $v_3 \in N(u_1)$, so by Lemma 5.1(ii), $N(v_3) \cap V(U_1 - y) \subseteq N(u_1)$. Similarly, we conclude that $N(w_3) \cap V(U_2) \subseteq N(u_2)$. Hence, $G[N[u_1]], G[N[u_2]], A_3, G - (A_3 + \{u_1, u_2\})$ (removing from last subgraph with both ends in $N(u_1)$ or in $N(u_2)$) show that $(G, u_1, u_2, A)$ is an obstruction of type II, a contradiction.

Thus, we may assume that $N(u_1) = \{x, v_2, v_3\}$ and $N(u_2) = \{y, w_2, w_3\}$. Since $N(u_i) \cap A = \emptyset$ for $i = 1, 2$ (by Lemma 4.1), $|V(A_2)| \geq 2$ and $|V(A_3)| \geq 2$ (as $N(u_1) \cap N(u_2) \subseteq A$). Suppose for $i = 2, 3$, $a_i' \in A$ and $N(a_i') \cap V(G_2) = \emptyset$. If $w_4 \notin A$ then $(G/yw_4, u_1, u_2, A)$ is an obstruction of type I with sides $U_1 - y, U_2/yw_4$ and middle parts $A_2, A_3, (G_2 + A_2 + xy)/yw_4$, contradicting Lemma 5.3. So $w_4 \in A$. Then $V(A_2) = \{w_4\}$ by the maximality of...
$U_1 \cup U_2$; so $(G, u_1, u_2, A)$ is an obstruction of type IV with sides $U_1 - y, U_2$ and middle parts $A_2, A_3, A_4, G_2 - \{a'_2, a'_3, a'_4\} + xy$, a contradiction. Hence, by symmetry we may assume $a'_3 \notin A$, or $a'_3 \in A$ and $N(a'_3) \cap V(G_2) \neq \emptyset$.

Suppose $(U_1 - y) - \{u_1, x, v_3\}$ has a path $S_2$ from $v_3$ to $v_2$ or $U_2 - \{u_2, y, w_3\}$ has a path $T_2$ from $w_4$ to $w_2$. By symmetry, assume we have $S_2$. By Lemma 6.1, $G_2 - a'_3$ (when $a'_3 \notin A$) or $G_2$ (when $a'_3 \in A$ and $N(a'_3) \cap V(G_2) \neq \emptyset$) has four disjoint paths from $V(G_1 \cap G_2) - \{a'_3\}$ to $A$. These paths and $u_1x, u_1v_3, S_2 + \{u_1, u_1v_2\}, u_2y, u_2w_2, u_2w_3, W_2, R_3, V'_4$ (see (a)) form a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction.

So neither $S_2$ nor $T_2$ exists. Then $\{x, v_3\}$ is a cut in $U_1 - y$ separating $\{u_1, v_2\}$ from $(U_1 - y) - \{u_1, v_2, x, v_3\}$, and $\{y, w_3\}$ is a cut in $U_2$ separating $\{u_2, w_2\}$ from $U_2 - \{u_2, w_2, y, w_3\}$. Hence, $a'_2 \notin A$, or $a'_2 \in A$ and $N(a'_2) \cap V(G_2) \neq \emptyset$; as otherwise, $G[N[u_1]], G[N[u_2]], A_2, (G_2 - a'_2) \cup (U_1 - u_1, v_1) - xv_3 \cup (U_2 - \{u_2, w_2\}) - yw_3 \cup A_3 \cup A_4 + xy$ show that $(G, u_1, u_2, A)$ is an obstruction of type II, a contradiction. Moreover, $(U_1 - y) - \{u_1, x, v_2\}$ has a path $S_3$ from $v_4$ to $v_3$ or $U_2 - \{u_2, y, w_2\}$ has a path $T_3$ from $w_4$ to $w_3$; otherwise by (5), we see that $(G[N[u_1]] \cup G[N[u_2]] \cup A_2 \cup A_3, G_2, G_2 \cup G[A_3 + xy] \cup (U_1 - \{u_1, v_2, v_3\}) \cup (U_2 - \{u_2, w_2, w_3\}))$ is a separation in $G$ showing that Theorem 2.1(iii) would hold for $(G, u_1, u_2, A)$. By symmetry, assume we have $S_3$. Hence $U_1 - y$ has three independent paths $S_1, S_2, S_3$ from $u_1$ to $x, v_2, v_4$, respectively. By Lemma 6.1, $G_2 - a'_2$ (when $a'_2 \notin A$) or $G_2$ (when $a'_2 \in A$ and $N(a'_2) \cap V(G_2) \neq \emptyset$) has four disjoint paths from $V(G_1 \cap G_2) - \{a'_2\}$ to $A$. These paths and $S_1, S_2, S_3, u_2y, u_2w_2, u_2w_3, W_3, R_2, V'_4$ (see (a)) form a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction.

**Lemma 6.4.** If $(G, u_1, u_2, A)$ is a counterexample to Theorem 2.1 with $|V(G)|$ minimum and $G$ has a separation $(G_1, G_2)$ satisfying (1)–(5) above, then $(G_1, u_1, u_2, A')$ is not an obstruction of type I.

**Proof.** Suppose $(G_1, u_1, u_2, A')$ is an obstruction of type I with sides $U_1, U_2$ and middle parts $A_1, A_2, A_3$. Let $V(A_1) = \{y\}$ (by Lemma 6.2), $V(U_1 \cap A_2) = \{v_2\}$, $V(U_1 \cap A_3) = \{v_3, v_4\}$, $V(U_2 \cap A_1) = \{v_1\}$ for $i = 2, 3$, $a'_2 \in V(A_2)$, $a'_3, a'_4 \in V(A_3) - \{v_3, v_4, w_3\}$, $u_1 \in V(U_1) - \{y, v_2, v_3, v_4\}$, and $u_2 \in V(U_2) - \{y, w_2, w_3\}$. We choose $U_i$ and $A_j$ so that $U_1 \cup U_2$ is maximized.

By (5), $x \in V(U_1 - y)$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2] \subseteq V(U_2)$. We claim that $x \notin \{v_2, v_3, v_4\}$; for, if $x = v_2$ then $(G_1 - V(A_2 - \{v_2, w_2\}), G_2 \cup A_2)$ contradicts the choice of $(G_1, G_2)$ (see (4)), and if $x \in \{v_3, v_4\}$ then $(G_1 - V(A_3 - \{v_3, v_4, w_3\}), G_2 \cup A_3)$ contradicts the choice of $(G_1, G_2)$ (see (4)).

![Fig. 12: $(G_1, u_1, u_2, A')$ is of type I.](image)

By Lemma 4.2, $N(u_2) = \{y, w_2, w_3\}$. As in the proof of Lemma 6.3, if $|A_2| \geq 2$ then $A_2 - v_2$
has a path $W'_2$ from $w_2$ to $a'_3$, and $A_2 - a'_3$ has a path $R_2$ from $w_2$ to $v_2$ (by the maximality of $U_1 \cup U_2$). When $|A_2| = 1$, we let $W'_2 = R_2 = A_2$.

For any $i \in \{3, 4\}$, $A_3 - v_{7-i}$ has two disjoint paths $R_i, Q_i$ from $\{v_i, w_3\}$ to $\{a'_3, a'_i\}$. Otherwise, $A_3$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 2$, $v_{7-i} \in V(A_{31} \cap A_{32})$, $\{v_i, w_3\} \subseteq V(A_{31})$, and $\{a'_3, a'_i\} \subseteq V(A_{32})$. Then $(U_1 \cup U_2 \cup A_2 \cup A_{31}, A_{32} + \{v, a'_2\})$ shows that Theorem 2.1(iii) holds for $(G, u_1, u_2, A')$, contradicting Lemma 6.2.

Moreover, for any $i \in \{3, 4\}$, $A_3 - a'_2 - a'_i$ has two disjoint paths $R'_i, Q'_i$ from $\{v_3, v_4\}$ to $\{w_3, a'_3\}$. For otherwise $A_3$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 2$, $a'_2 - a'_i \in V(A_{31} \cap A_{32})$, $\{v_3, v_4\} \subseteq V(A_{31})$, and $\{w_3, a'_i\} \subseteq V(A_{32})$. Then $U_1 \cup A_{31} + a'_i, U_2 \cup A_{32}, \{y\}, A_2, \{a'_3\}, \{a'_i\}$ (when $a'_i \in V(A_{31} \cap A_{32}) \cup \{a'_{7-i}\}$) or $U_1 \cup A_{31}, U_2 + a'_{7-i}, \{y\}, A_2, \{a'_{7-i}\}, A_3 - a'_{7-i}$ (when $a'_i \notin V(A_{31} \cap A_{32}) \cup \{a'_{7-i}\}$) shows that $(G, u_1, u_2, A')$ is an obstruction of type IV, contradicting Lemma 6.3.

Clearly, $v_3, v_4 \notin A$. We note that $v_2 \notin A$. For otherwise, by the maximality of $U_1 \cup U_2$, $v_2 = w_2 \in N(u_2)$. So $N(u_2) \cap A \neq \emptyset$. But $|N(u_2)| = 3$, contradicting Lemma 4.1.

If for all $i \in \{3, 4\}$, $a'_i \notin A$ and $N(a'_i) \cap V(G) = \emptyset$, then $(G/vx_2, u_1, u_2, A)$ is an obstruction of type III with sides $(U_i - y)/vx_2, U_2$ and middle parts $(A_2 \cup (G - \{a'_3, a'_i\}) + xy)/vx_2, A_3$, contradicting Lemma 5.5. Hence by symmetry, let $a'_3 \notin A$, or $a'_i \notin A$ and $N(a'_i) \cap V(G) \neq \emptyset$. Then by Lemma 6.1, $G_2 - a'_4$ (when $a'_4 \notin A$) or $G_2$ (when $a'_4 \in A$) has four disjoint paths $S_1, S_2, S_3, S_4$ from $V(G_1 \cap G_2) - \{a'_4\}$ to $A$.

If $a'_2 \in A$ and $N(a'_2) \cap V(G) = \emptyset$ then $(G/v_3v_4, u_1, u_2, A)$ is an obstruction of type II with sides $(U_1 - y)/v_3v_4, U_2$ and middle parts $(A_2, A_3/v_3v_4) \cup (G_2 - a'_2) + xy$, contradicting Lemma 5.4. Thus $a'_2 \notin A$, or $a'_2 \in A$ and $N(a'_2) \cap V(G) \neq \emptyset$. So by Lemma 6.1, $G_2 - a'_2$ (when $a'_2 \notin A$) or $G_2$ (when $a'_2 \in A$) has four disjoint paths $T_1, T_2, T_3, T_4$ from $V(G_1 \cap G_2) - \{a'_2\}$ to $A$.

By Lemma 5.1(i) and the fact $u_1 x \in E(G)$, there exists a permutation $ijk$ of $\{2, 3, 4\}$ such that $(U_i - y) - v_k$ has three independent paths $P_i, P_2, P_3$ from $u_1$ to $x, v_i, v_j$, respectively. If $\{i, j\} = \{3, 4\}$ then $P_1, P_2, P_3, u_2y, u_2w_2, u_2w_3, R'_3, Q'_3, W'_2, S_1, S_2, S_3, S_4$ form a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction. Thus by symmetry between $v_3$ and $v_4$, assume $\{i, j\} = \{2, 3\}$. Then $P_1, P_2, P_3, u_2y, u_2w_2, u_2w_3, R_3, Q_3, R_2, T_1, T_2, T_3, T_4$ form a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction.

Lemma 6.5. If $(G, u_1, u_2, A)$ is a counterexample to Theorem 2.1 with $|V(G)|$ minimum and $G$ has a separation $(G_1, G_2)$ satisfying (1)–(5) above, then $(G_1, u_1, u_2, A')$ is not an obstruction of type II.

Proof. Suppose $(G, u_1, u_2, A)$ is a counterexample to Theorem 2.1 with $|V(G)|$ minimum, and $(G_1, u_1, u_2, A')$ is an obstruction of type II with sides $U_1, U_2$ and middle parts $A_1, A_2$. Let $V(A_1) = \{y\}$ (by Lemma 6.2), $V(U_1 \cap A_2) = \{v_2, v_3\}$, $V(U_2 \cap A_2) = \{w_2, w_3\}$, $a'_2, a'_3, a'_4 \in V(A_2) - \{v_3, w_3, w_3, v_3\}$, $u_1 \in V(U_1) - \{y, v_2, v_3\}$, and $u_2 \in V(U_2) - \{y, v_3, w_3\}$. By (5), $x \in V(U_1 - y)$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N(u_2) \subseteq V(U_2)$. Note that $x \notin \{v_2, v_3\}$; otherwise the separation $(U_1 - y, U_2 \cup A_2 \cup G_2)$ shows that Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. By Lemma 4.2, $V(U_1 - y) = \{u_1, v_2, v_2, v_3\}$ and $V(U_2) = \{w_2, w_2, w_3, y\}$. Moreover, $N(u_1) = \{v_2, v_3, x\}$ and $N(u_2) = \{w_2, w_3, y\}$, as otherwise Theorem 2.1(ii) would hold for $(G, u_1, u_2, A)$. See Figure 13(a).
Let $s$ would form a topological $M$ and another from $\{a\}$. Lemma 3.3).

By symmetry, assume $a'_4 \notin A$ or $N(a'_4) \cap V(G_2) \neq \emptyset$. Then by Lemma 6.1, $G_2 - a'_4$ (when $a'_4 \notin A$) or $G_2$ (when $a'_4 \in A$) has four disjoint paths $S_1, S_2, S_3, S_4$ from $V(G_1 \cap G_2) - \{a'_1\}$ to $A$.

Let $A_2 - a'_4 = L \cup M \cup R$ such that $|V(L \cap M)| \leq 2$, $|V(R \cap M)| \leq 2$, $V(L \cap R) \subseteq V(M)$, $\{v_2, v_3\} \subseteq V(L)$, $\{w_2, w_3\} \subseteq V(R)$, and $\{a'_{2_2}, a'_{3_2}\} \subseteq V(M) - V(L \cup R)$. (Note that $L = \{v_2, v_3\}$, $M = A_2 - a'_4$ and $R = \{w_2, w_3\}$ satisfy this.) Choose $L, M, R$ to minimize $M$.

Then $|V(L \cap M)| = 2$ and $L$ has two disjoint paths from $\{v_2, v_3\}$ to $V(L \cap M)$, and $|V(R \cap M)| = 2$ and $R$ has two disjoint paths from $\{w_2, w_3\}$ to $V(R \cap M)$. For, suppose this is not true, and assume by symmetry that $|V(L \cap M)| \leq 1$ or $L$ has no disjoint paths from $\{v_2, v_3\}$ to $V(L \cap M)$. If $|V(L \cap M)| \leq 1$ let $L_1 = L$ and $L_2 = L \cap M$, and if $|V(L \cap M)| = 2$ then $G[L + a'_4]$ has a separation $(L_1, L_2)$ such that $|V(L_1 \cap L_2)| \leq 2$, $a'_4 \in V(L_1 \cap L_2)$, $\{v_2, v_3\} \subseteq V(L_1)$, and $V(L \cap M) \subseteq V(L_2)$. Now $V(L_1 \cap L_2) \cup \{x\}$ is a cut in $G$ separating $u_1$ from $A \cup \{w_2\}$, contradicting Lemma 4.2.

Let $V(L \cap M) = \{s_1, s_2\}$ and $V(R \cap M) = \{t_1, t_2\}$. Note that $\{s_1, s_2\} \neq \{t_1, t_2\}$: as otherwise, the separation $(G_1 - (M - \{s_1, s_2\}), G_2 \cup G[M + a'_4])$ contradicts (4). Clearly, $G[L + \{u_1, x\}]$ has three independent paths $P_1, P_2, P_3$ from $u_1$ to $x, s_1, s_2$, respectively, and $G[R + \{u_2, y\}]$ has three independent paths $Q_1, Q_2, Q_3$ from $u_2$ to $y, t_1, t_2$, respectively. If $M$ has three disjoint paths, with one from $\{s_1, s_2\}$ to $\{t_1, t_2\}$, one from $\{s_1, s_2\}$ to $\{a'_{2_2}, a'_{3_2}\}$, and another from $\{t_1, t_2\}$ to $\{a'_{2_2}, a'_{3_2}\}$, then these paths and $P_1, P_2, P_3, Q_1, Q_2, Q_3, S_1, S_2, S_3, S_4$ would form a topological $H$ in $G$ rooted at $u_1, u_2, A$. So such paths in $M$ do not exist.

We claim that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$. Suppose otherwise and, without loss of generality, let $s_1 = t_1$. Then $s_2 \neq t_2$, and $M - s_1$ does not contain disjoint paths from $\{s_2, t_2\}$ to $\{a'_{2_2}, a'_{3_2}\}$. Hence $M$ has a separation $(M_1, M_2)$ such that $V(M_1 \cap M_2)$ is of type II.

By Lemma 3.3 (with $M, s_1, s_2, t_1, t_2, a'_{2_2}, a'_{3_2}$ as $G, v_1, v_2, w_1, w_2, a_1, a_2$, respectively), $M$ has a separation $(M_1, M_2)$ such that one of $(i) - (v)$ of Lemma 3.3 holds (with $M_i, i = 1, 2, 3$, as $G_i$ in Lemma 3.3).

If Lemma 3.3(ii) holds, then the separation $(G_1[M_2 \cup L \cup R + \{a'_4, u_1, u_2, x, y\}], M_1 +
We now show that \((a'_4, y)\) shows that Theorem 2.1(iii) holds for \((G_1, u_1, u_2, A')\), contradicting Lemma 6.2. If Lemma 3.3(iii) holds, then \(G_1[L \cup M_i + (A' \cup \{u_1, x, y\})], G_1[R \cup M_{3-i} + (A' \cup \{u_2, y\})], \{y\}, \{a'_2\}, \{a'_3\}, \{a'_4\}\) show that \((G_1, u_1, u_2, A')\) is an obstruction of type IV, contradicting Lemma 6.3. If Lemma 3.3(iv) holds, then \(G_1[L + \{a'_4, u_1, x, y\}], G_1[R + \{a'_4, u_2, y\}], \{y\}, \{a'_4\}, M_1, M_2\) show that \((G_1, u_1, u_2, A')\) is an obstruction of type IV, contradicting Lemma 6.3. If Lemma 3.3(v) holds, then by symmetry assume that \(\{s_1, s_2, t_1, a'_2, a'_3\} \subseteq V(M_1)\); now \(G_1[L + \{a'_4, u_1, x, y\}], G[R \cup M_2 + \{a'_4, u_2, y\}], \{y\}, \{a'_4\}, M_1\) show that \((G_1, u_1, u_2, A')\) is an obstruction of type I, contradicting Lemma 6.4.

So Lemma 3.3(i) holds, and assume by symmetry that \(\{s_1, s_2, a'_2, a'_3\} \subseteq V(M_1)\) and \(\{t_1, t_2\} \subseteq V(M_2)\). Note that \(|V(M_1 \cap M_2)| = 2\) and \(M_2\) has disjoint paths \(T_1, T_2\) from \(\{t_1, t_2\}\) to \(V(M_1 \cap M_2)\); otherwise, \(|V(M_1 \cap M_2)| \leq 1\) (in this case let \(S := V(M_1 \cap M_2)\)), or \(M_2\) has a cut \(S\), \(|S| \leq 1\), separating \(V(M_1 \cap M_2)\) from \(\{t_1, t_2\}\), and hence, \(S \cup \{a'_4, y\}\) is a cut in \(G\) separating \(u_2\) from \(A \cup \{u_1\}\), contradicting Lemma 4.2.

Hence by the minimality of \(M\), we may assume by symmetry that \(V(M_1 \cap M_2) = \{a'_2, z\}\). Then \(z \neq a'_3\), as otherwise \(G_1[L \cup M_1 + \{a'_4, u_1, x, y\}], G_1[R \cup M_2 + \{a'_4, u_2, y\}], \{y\}, \{a'_2\}, \{a'_3\}, \{a'_4\}\) show that \((G_1, u_1, u_2, A')\) is an obstruction of type IV, contradicting Lemma 6.3.

If \(M_1 - a'_2\) contains disjoint paths from \(\{s_1, s_2\}\) to \(\{a'_3, z\}\) then these paths and \(P_1, P_2, P_3, Q_1, Q_2, Q_3, T_1, T_2, S_1, S_2, S_3, S_4\) form a topological \(H\) in \(G\) rooted at \(u_1, u_2, A\), a contradiction. So such paths do not exist in \(M_1 - a'_2\). Then \(M_1\) has a separation \((M_{11}, M_{12})\) such that \(a'_2 \in V(M_1 \cap M_2), |V(M_11 | M_12)| \leq 2, \{s_1, s_2\} \subseteq V(M_{11}), \) and \(\{a'_3, z\} \subseteq V(M_{12}).\) If \(a'_2 \notin V(M_{11} \cap M_{12})\) then \(G_1[L \cup M_{11} + \{a'_4, u_1, x, y\}], G_1[R \cup M_{2} + \{a'_4, u_2, y\}], \{y\}, \{a'_2\}, \{a'_3\}, M_{12} - a'_2\) show that \((G_1, u_1, u_2, A')\) is an obstruction of type IV, contradicting Lemma 6.3. So \(a'_2 \in V(M_{11} \cap M_{12})\). Then \(G_1[L \cup M_{11} + \{a'_4, u_1, x, y\}], G_1[R \cup M_{2} \cup M_{12} + \{a'_4, u_2, y\}], \{y\}, \{a'_2\}, \{a'_3\}, \{a'_4\}\) show that \((G_1, u_1, u_2, A')\) is an obstruction of type IV, contradicting Lemma 6.3.

7 Conclusion

We complete the proof of Theorem 2.1. Suppose that the assertion of Theorem 2.1 fails, and let \((G, u_1, u_2, A)\) be a counterexample with \(|V(G)|\) minimum.

Then \(|N(u_i)| \geq 3\) (otherwise (ii) would hold for \((G, u_1, u_2, A)\)). Also \(G\) has no separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 4, \{u_1, u_2\} \subseteq V(G_1) - V(G_2), \) and \(A \subseteq V(G_2);\) for otherwise (iii) would hold for \((G, u_1, u_2, A)\). Thus \(A\) is independent in \(G\). Moreover, for any vertex \(u \notin A \cup \{u_1, u_2\},\) the graph \(G'\), obtained from \(G - u\) by duplicating \(u_i\) with \(u'_i\) \((i = 1, 2,\) contains four disjoint paths from \(\{u_1, u'_1, u_2, u'_2\}\) to \(A\). Now these paths give rise to four independent paths \(P_1, P_2, P_3, P_4\) in \(G - u\) from \(\{u_1, u_2\}\) to \(A\), with two from each \(u_i\). We now prove properties (a) – (e) and use them to prove that \(G\) has a separation \((G_1, G_2)\) satisfies (1) – (5) in Section 6.

(a) \(u_1u_2 \notin E(G), \) and \(N(u_1) \cap N(u_2) \subseteq A.\)

For, if \(u_1u_2 \in E(G)\) then \(P_1, P_2, P_3, P_4\) and \(u_1u_2\) would form a topological \(H\) in \(G\) rooted at \(u_1, u_2, A;\) and if there exists \(u \in (N(u_1) \cap N(u_2)) - A\) then \(P_1, P_2, P_3, P_4\) and \(u_1u_2\) would form a topological \(H\) in \(G\) rooted at \(u_1, u_2, A.\)
If $G - A - \{u_1, u_2\} = \emptyset$ then we see that $N(u_i) \subseteq A$. So by Lemma 4.1, $N(u_i) = A$ for $i = 1, 2$. Hence $G[A + u_1], G[A + u_2], \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction. Thus $G - A - \{u_1, u_2\} \neq \emptyset$. In fact,

(b) $E(G - A - \{u_1, u_2\}) \neq \emptyset$.

Otherwise, by (a), for any $x \in V(G) - A - \{u_1, u_2\}, N(x) \subseteq A \cup \{u_i\}$ for some $i \in \{1, 2\}$. Thus $G$ has a separation $(U_1, U_2)$ such that $V(U_1 \cap U_2) = A$, $u_1 \in V(U_1) - V(U_2)$, and $u_2 \in V(U_2) - V(U_1)$. Now $U_1, U_2, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction.

(c) There exists $xy \in E(G - A - \{u_1, u_2\})$ such that $\{x, y\} \not\subseteq N(u_i)$ for any $i \in \{1, 2\}$.

Suppose for any $xy \in E(G - A - \{u_1, u_2\})$ we have $\{x, y\} \subseteq N(u_i)$ for some $i \in \{1, 2\}$. Then by (a), for any $v \in N(u_i) - A$, $N(v) \subseteq N[u_i] \cup A$. Thus, $G$ has a separation $(U_1, U_2)$ such that $V(U_1 \cap U_2) = A$, $U_1 = G[N[u_1] \cup A]$, and $U_2 = G - V(G_1)$. Now $U_1, U_2, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that $(G, u_1, u_2, A)$ is an obstruction of type IV, a contradiction.

Since $|V(G/xy)| < |V(G)|$, one of $(i) - (iv)$ of Theorem 2.1 holds for $(G/xy, u_1, u_2, A)$. Let $v$ denote the vertex resulting from the contraction of $xy$.

(d) For any $xy$ satisfying (c), (iii) holds for $(G/xy, u_1, u_2, A)$.

By Lemmas 5.2, 5.3, 5.4 and 5.5, (iv) does not hold for $(G/xy, u_1, u_2, A)$. If (i) holds for $(G/xy, u_1, u_2, A)$ then let $K$ be a topological $H$ in $G/xy$ rooted at $u_1, u_2, A$; now $K$ (when $v \not\in K$) or the graph obtained from $K$ by uncontracting $v$ back to $xy$ (when $v \in K$) gives a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction. Now suppose that (ii) holds for $(G/xy, u_1, u_2, A)$, and let $(G_1', G_2')$ denote a separation in $G/xy$ such that $|V(G_1' \cap G_2')| \leq 2$, $u_1 \in V(G_1') - V(G_2')$ and $A \cup \{u_2\} \subseteq V(G_2')$. Then $|V(G_1' \cap G_2')| = 2$ and $v \in V(G_1' \cap G_2')$; for otherwise (ii) would hold for $(G, u_1, u_2, A)$. Hence $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| = 3, x, y \in V(G_1 \cap G_2), u_1 \in V(G_1) - V(G_2)$, and $A \cup \{u_2\} \subseteq G_2$. Since $|N(u_1)| \geq 3$ and $\{x, y\} \not\subseteq N(u_i), |V(G)| \geq 5$, contradicting Lemma 4.2. Thus (iii) holds for $(G/xy, u_1, u_2, A)$.

By (d), for any $xy$ satisfying (c), $G/xy$ has a separation $(G_1', G_2')$ such that $|V(G_1' \cap G_2')| \leq 4$, $\{u_1, u_2\} \subseteq V(G_1') - V(G_2')$, and $A \subseteq V(G_2')$. Then $v \in V(G_1' \cap G_2')$ and $|V(G_1' \cap G_2')| = 4$; or else (iii) would hold for $(G, u_1, u_2, A)$. Hence, $G$ has a separation $(G_1, G_2)$ such that $x, y \in V(G_1 \cap G_2), |V(G_1 \cap G_2)| = 5, \{u_1, u_2\} \subseteq V(G_1) - V(G_2)$, and $A \subseteq V(G_2)$. Moreover, $N(x) \cap (V(G_1) - V(G_2)) \neq \emptyset$, and $N(y) \cap (V(G_1) - V(G_2)) \neq \emptyset$; for otherwise, (iii) would hold for $(G, u_1, u_2, A)$. We choose $xy$ satisfying (c) and $(G_1, G_2)$ so that $G_1$ is minimal (subject to $xy \in E(G_1)$). Now $(G, u_1, u_2, A)$ satisfies (1) - (4) in Section 6. We now show that $(G, u_1, u_2, A), xy$ and $(G_1, G_2)$ also satisfies (5) in Section 6. First, we claim that

(e) $x, y \in N(\{u_1, u_2\})$ and $(N(x) \cup N(y)) \cap (V(G_1) - V(G_2)) \subseteq N[\{u_1, u_2\}]$.

Suppose (e) fails, and assume by symmetry that it fails for $x$. If $x \not\in N(\{u_1, u_2\})$ let $z \in N(x) \cap (V(G_1) - V(G_2))$; and if $x \in N(\{u_1, u_2\})$ then $N(x) \cap (V(G_1) - V(G_2)) \not\subseteq N[\{u_1, u_2\}]$. 32
and let $z \in N(x) \cap (V(G_1) - V(G_2)) - N\{u_1, u_2\}$. Then $xz$ satisfies (c). By the argument following (d), $G$ has a separation $(H_1, H_2)$ such that $\{x, z\} \subseteq V(H_1 \cap H_2)$, $|V(H_1 \cap H_2)| = 5$, $\{u_1, u_2\} \subseteq V(H_1) - V(H_2)$ and $A \subseteq V(H_2)$. Thus $u_1, u_2 \in (V(G_1) - V(G_2)) \cap (V(H_1) - V(H_2))$ and $A \subseteq V(G_2 \cap H_2)$. In particular, $|V(G_2 \cap H_2)| \geq |A \cup \{x\}| \geq 5$. Thus $|V(G_1 \cap G_2 \cap H_2) \cup V(H_1 \cap H_2 \cap G_2)| \geq 5$; as otherwise the separation $(G_1 \cup H_1, G_2 \cap H_2)$ shows that (iii) would hold for $(G, u_1, u_2, A)$.

Therefore, since $|V(G_1 \cap G_2)| + |V(H_1 \cap H_2)| = 10$, we see that $|V(G_1 \cap G_2 \cap H_1) \cup V(H_1 \cap H_2 \cap G_1)| \leq 5$. In fact, $|V(G_1 \cap G_2 \cap H_1) \cup V(H_1 \cap H_2 \cap G_1)| = 5$; otherwise, the separation $(G \cap H_1, G_2 \cup H_2)$ shows that (iii) would hold for $(G, u_1, u_2, A)$. Thus $|V(G_1 \cap G_2 \cap H_2) \cup V(H_1 \cap H_2 \cap G_2)| = 5$. By the choice of $(G_1, G_2)$ (i.e., the minimality of $G_1$), the separation $(G_1 \cap H_1, G_2 \cup H_2)$ implies that $V(G_1 \cap H_2) - V(H_1) = \emptyset$ (so $V(G_1 \cap G_2 \cap H_2) = V(G_1 \cap G_2 \cap H_2)$). Now since $z \notin V(G_2)$, $|V(G_1 \cap G_2 \cap H_2) \cup V(H_1 \cap H_2 \cap G_2)| = |V(H_1 \cap H_2 \cap G_2)| \leq |V(H_1 \cap H_2) - \{z\}| = 4$, a contradiction.

By (a), (c) and (e), there exists a permutation $ij$ of $\{1, 2\}$ such that $xu_i, yu_j \in E(G)$. We now show that $N(x) \cap (V(G_1) - V(G_2)) \subseteq N[u_i]$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_j]$. Suppose this is false and, by symmetry, assume that $N(x) \cap (V(G_1) - V(G_2)) \not\subseteq N[u_i]$. Then by (a) and (e) there exists $z \in V(G_1) - V(G_2) - \{u_1, u_2\}$ such that $xz \in E(G)$ and $zu_i \notin E(G)$. By (a) again, $xz$ satisfies (c); so by (e), $zu_j \in E(G)$. Let $G_1^*$ be obtained from $G_1$ by duplicating $u_k$ with $u_k', k = 1, 2$. If $G_1^* - \{x, z\}$ has four disjoint paths from $\{u_1, u_1', u_2, u_2'\}$ to $V(G_1 \cap G_2) - \{x\}$, then these paths, $u_kzu_j$ and four disjoint paths in $G_2 - x$ from $V(G_1 \cap G_2) - \{x\}$ to $A$ (Lemma 6.1(ii)) give a topological $H$ in $G$ rooted at $u_1, u_2, A$, a contradiction. Thus, $G_1$ has a separation $(G_11, G_{12})$ such that $|V(G_11 \cap G_{12})| \leq 5$, $\{x, z\} \subseteq V(G_11 \cap G_{12})$, $\{u_1, u_2\} \subseteq V(G_11)$, and $V(G_1 \cap G_2) - \{x\} \subseteq V(G_{12})$. Now the separation $(G_11, G_{12} \cup G_2)$ contradicts the choice of $(G_1, G_2)$ (the minimality of $G_1$).

Thus, $(G, u_1, u_2, A)$ also satisfies (5) in Section 6. Hence, we get a final contradiction by invoking Lemmas 6.2 – 6.5, completing the proof of Theorem 2.1.

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References


