Infinite Paths in Planar Graphs II, structures and ladder nets

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Abstract

A graph is *k-indivisible*, where k is a positive integer, if the deletion of any finite set of vertices results in at most $k-1$ infinite components. In 1971, Nash-Williams conjectured that a 4-connected infinite planar graph contains a spanning 2-way infinite path if, and only if, it is 3-indivisible. In this paper, we prove a structural result for 2-indivisible infinite planar graphs. This structural result is then used to prove Nash-Williams conjecture for all 4-connected 2-indivisible infinite planar graphs.

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1 Introduction

We use the notation and terminology in [9]. In 1931, Whitney [8] proved that every 4-connected finite planar triangulation contains a spanning cycle. This result was generalized by Tutte [7] and by Thomassen [6]. To extend Whitney's result to infinite graphs, Nash-Williams ([2], [3], and [5]) conjectured that every 4-connected 2-indivisible infinite planar graph contains a spanning 1-way infinite path. This conjecture is verified in [1].

Nash-Williams also conjectured that every 4-connected 3-indivisible infinite planar graph contains a spanning 2-way infinite path. This conjecture is proved in [9] for those 4-connected infinite plane graphs which admit "radial nets". It is shown in [1] that for any 4-connected 2-indivisible infinite plane graph G, either G has a radial net or a special subgraph of G has a "ladder net". In this paper, we prove the following.

(1.1) Theorem. *Every 4-connected 2-indivisible infinite planar graph contains a spanning 2-way infinite path.*

To prove (1.1), we need a detailed description of structures of 4-connected 2-indivisible plane graphs, and this is done in Section 2. This structural result is then used in Section 3 to prove a result about 1-way infinite paths in infinite plane graphs with ladder nets. (This structural result will also be used in subsequent papers.) In Section 4, we use the results in Sections 2 and 3 to complete the proof of (1.1).

Throughout the rest of the paper, by a graph we mean a finite graph unless otherwise noted. For convenience, we use the notation $A := B$ to re-name B with A.

2 Nets and structures

By the Jordan curve theorem, any cycle C in an infinite plane graph G divides the plane into two closed regions (whose intersection is C). If exactly one of these two closed regions, say $\mathcal{R}$, contains only finitely many vertices and edges of G, then we use $I_G(C)$ to denote the subgraph of G consisting of vertices and edges of G contained in $\mathcal{R}$. Note that $I_G(C)$ is finite. When there is no danger of confusion, we use $I(C)$ instead of $I_G(C)$. Also note that $C \subset I(C)$, and if $I(C) = C$ then C is a facial cycle of G. For convenience, we state the definition of a net introduced in [1].

A *net* in an infinite plane graph G is a sequence $N := (C_1, C_2, \ldots)$ of cycles in G such that $I(C_i)$ is defined for all $i \geq 1$, and the following properties are satisfied:

(1) $I(C_i) \subset I(C_{i+1})$ for all $i \geq 1$,

(2) $\bigcup_{i=1}^{\infty} I(C_i) = G$, and

(3) either $C_i \cap C_j = \emptyset$ for all $i \neq j$, or, for each $i \geq 1$, $C_i \cap C_{i+1}$ is a non-trivial path, $C_i \cap C_{i+1} \subset C_{i+1} \cap C_{i+2}$, and neither endvertex of $C_i \cap C_{i+1}$ is an endvertex of $C_{i+1} \cap C_{i+2}$.

If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then N is called a *radial net*; otherwise, N is a *ladder net.* Let $\partial N = \emptyset$ if N is a radial net; otherwise, let $\partial N := \bigcup_{i=1}^{\infty}(C_i \cap C_{i+1})$.

Note that if an infinite plane graph has a net, then it is locally finite. Also note that if N is a ladder net in an infinite plane graph, then ∂N is a 2-way infinite path.

For our purpose, we need a detailed description of structures of 2-indivisible infinite plane graphs. We say that an infinite plane graph G is *nicely embedded* or is a *nice embedding* if, for any cycle C in G for which $I(C)$ is defined, $I(C)$ is contained in the closed disc bounded by C.

(2.1) Lemma. *If G is an infinite plane graph with a net and C is a facial cycle of G, then G has a nice embedding in which C is also a facial cycle.*

Proof. With respect to the given embedding of G in the plane, any cycle D in G divides the plane into an unbounded closed region $\mathcal{U}(D)$ and a bounded closed region $\mathcal{B}(D)$. Since G is infinite and 2-indivisible, exactly one of these two regions contains a finite subgraph of G. This finite subgraph will be denoted by $I(D)$. In other words, the notation $I(D)$ is in this proof defined with reference to the given embedding of G in the plane.

Since G has a net, G has a sequence of cycles $(D_1, D_2, \ldots)$ such that $I(D_i) \subset I(D_{i+1})$ for all $i \geq 1$, and $G = \bigcup_{i \geq 1} I(D_i)$. Because C is finite, $C \subset I(D_k)$ for all sufficiently large k. Therefore we may assume that $(D_1, D_2, \ldots)$ is chosen so that it also satisfies the additional condition that $C \subset I(D_1)$. Since $I(D_1) \subset I(D_2) \subset \ldots$, it follows that either (a) $\mathcal{B}(D_i)$ contains $I(D_i)$ for all $i \geq 1$ or (b) there is a positive integer r such that $\mathcal{B}(D_i)$ contains $I(D_i)$ for all $i < r$ and $\mathcal{U}(D_i)$ contains $I(D_i)$ for all $i \geq r$. In case (a), the given embedding of G in the plane is nice, and by hypothesis, has C as a facial cycle. Now suppose that (b) occurs. For each i, $I(D_i)$ can be embedded in the plane so that D_i is its outer cycle and so that, in the case $i = 1$, C remains a facial cycle. Hence, for each $i \geq 1$, $I(D_{i+1}) - (I(D_i) - D_i)$ can be embedded in the plane so that D_{i+1} is its outer cycle and D_i is a facial cycle. Since G is the union of $I(D_1)$ and all $I(D_{i+1}) - (I(D_i) - D_i)$ with $i \geq 1$, it follows that there is an embedding of G in the plane such that, for each $i \geq 1$, $I(D_i)$ is contained in the closed disc in the plane bounded by D_i. Since $G = \bigcup_{i \geq 1} I(D_i)$, for any cycle D in G, $I(D) \subset I(D_i)$ for all sufficiently large i. Hence in the new embedding of G, for any cycle D in G, $I(D)$ is contained in the closed disc in the plane bounded by D. Thus, the new embedding of G is nice. $\square$

Before we prove a structural result for 4-connected 2-indivisible infinite plane graphs, we prove two lemmas for a larger class of graphs. An infinite graph G is *cohesive* if, for any finite $X \subset V(G)$, $G - X$ has only finitely many components exactly one of which is infinite. It is easy to verify that if G is a 3-connected infinite planar graph then G is cohesive.

In order to describe our lemmas, we need the concept of bridge. For a subgraph H (finite or infinite) of a graph G (finite or infinite), an *H-bridge* of G is a subgraph (finite or infinite) of G which either (1) is induced by an edge of $E(G) - E(H)$ whose incident vertices are in H or (2) is induced by the edges contained in a component D of $G - V(H)$ and the edges of G from D to H. If B is an H-bridge of G, then the vertices of $H \cap B$ are called the *attachments* of B (on H).

(2.2) Lemma. *Let G be a cohesive 2-connected infinite plane graph. For every cycle D in G, there is a cycle $D' \neq D$ in G such that*

(1) $I(D) \subset I(D')$, and $D \cap D'$ is minimal among all subgraphs $D \cap D^*$ arising from cycles D^* in G such that $I(D) \subset I(D^*)$, and

(2) G has no finite $I(D')$-bridge.

Proof. Since G is infinite, G has a vertex $u \notin I(D)$. Since G is 2-connected, G has paths P, Q from u to $p, q \in V(D)$, respectively, such that $(P - u) \cap (Q - u) = \emptyset$ and $((P \cup Q) - \{p, q\}) \cap I(D) = \emptyset$. Then either $I(D) \subset I(P \cup Q \cup pDq)$ or $I(D) \subset I(P \cup Q \cup qDp)$. Therefore, there is a cycle D' in G such that (1) holds. Since G is cohesive, we may further select D' such that the number of edges of G contained in finite $I(D')$-bridges of G is minimum.

Next, we prove (2). Suppose for a contradiction that G has a finite $I(D')$-bridge, say B. Since G is 2-connected, B has at least two attachments on D'. So let x, y be distinct vertices of $B \cap D'$. Then B contains a path R from x to y such that $(R - \{x, y\}) \cap I(D') = \emptyset$. Now $I(D') \subset I(D'')$, where either $D'' := R \cup xD'y$ or $D'' := R \cup yD'x$. Note that $D \cap D'' \subset D \cap D'$ (since $I(D) \subset I(D') \subset I(D'')$), and every infinite $I(D')$-bridge of G is an infinite $I(D'')$-bridge of G (because $R \subset B$ and B is a finite $I(D')$-bridge of G). Also note that the number of edges in finite $I(D'')$-bridges of G is less than the number of edges in finite $I(D')$-bridges of G. Hence, D'' contradicts the choice of D', and so, (2) holds. □

Let G be a graph and C a subgraph of G. We say that G is *$(4, C)$-connected* if for any $T \subset V(G)$ with $|T| \leq 3$, every component of $G - T$ contains a vertex of C.

(2.3) Lemma. *Let G be a cohesive 2-connected infinite plane graph and let C be a facial cycle of G such that G is (4,C)-connected and, for any cycle D in G, $I(D)$ is defined. Then there is an infinite sequence $(D_1, D_2, \ldots)$ of cycles in G such that $C \subset I(D_1)$ and the following properties hold:*

(1) for each $i \geq 1$, $I(D_i) \subset I(D_{i+1})$, and $D_i \cap D_{i+1}$ is minimal among all subgraphs $D_i \cap D^*$ arising from cycles D^* in G such that $I(D_i) \subset I(D^*)$,

(2) for each $i \geq 1$, G has no finite $I(D_i)$-bridge,

(3) for each $i \geq 1$, $D_i \cap D_{i+1} \subset D_{i+1} \cap D_{i+2}$, and

(4) $\bigcup_{i\geq 1} I(D_i) = G$.

Proof. Let $D := C$ and let $D_1 := D'$ as in (2.2). Then $C \subset I(D_1)$, and G has no finite $I(D_1)$-bridges. Suppose we have constructed cycles $D_1, \ldots, D_l$, where $l \geq 1$, such that (1) holds for $1 \leq i \leq l-1$ and (2) holds for $1 \leq i \leq l$. By (2.2) (with D_l, D_{l+1} as D, D', respectively), there is a cycle D_{l+1} in G such that (a) $I(D_l) \subset I(D_{l+1})$, and $D_l \cap D_{l+1}$ is minimal among all subgraphs $D_l \cap D^*$ arising from cycles D^* in G such that $I(D_l) \subset I(D^*)$, and (b) G has no finite $I(D_{l+1})$-bridge. Therefore, continuing this process, we obtain an infinite sequence of cycles $(D_1, D_2, \ldots)$ such that (1) and (2) hold. We need to show that (3) and (4) also hold.

Since $I(D_i) \subset I(D_{i+1})$ for all $i \geq 1$, we have $D_{i+2} \cap D_i \subset D_{i+2} \cap D_{i+1}$ and $D_{i+2} \cap D_i \subset D_{i+1} \cap D_i$ for all $i \geq 1$. Moreover, $D_{i+2} \cap D_i = D_{i+1} \cap D_i$; otherwise, D_{i+2} would contradict the choice of D_{i+1} (by (a) above). Thus, $D_i \cap D_{i+1} = D_i \cap D_{i+2} \subset D_{i+1} \cap D_{i+2}$ for all $i \geq 1$. Hence, (3) holds for $(D_1, D_2, \ldots)$.

Next we show that $(D_1, D_2, \ldots)$ also satisfies (4). Let $H := \bigcup_{i\geq 1} I(D_i)$ and let $\partial H := \bigcup_{i\geq 1}(D_i \cap D_{i+1})$.

First, we claim that $V(G) = V(H)$. Otherwise, let $v \in V(G) - V(H)$. Since G is (4,C)-connected, G contains four paths P_i from v to $v_i \in V(H)$, $i = 1, 2, 3, 4$, such that $V(P_i \cap H) = \{v_i\}$, and $V(P_i \cap P_j) = \{v\}$ for $i \neq j$. Then $v_i \in \partial H$. Since $D_i \cap D_{i+1} \subset D_{i+1} \cap D_{i+2}$ for all $i \geq 1$, there is some integer k such that $\{v_1, v_2, v_3, v_4\} \subset V(D_j \cap D_{j+1})$ for all $j \geq k$. Hence, let $l > k$ be an integer, and assume that v_1, v_2, v_3, v_4 occur on D_l in that clockwise order. Let $D_l' := P_1 \cup P_3 \cup v_3 D_l v_1$ and $D_l'' := P_1 \cup P_3 \cup v_1 D_l v_3$. Note that either $I(D_l) \subset I(D_l')$ or $I(D_l) \subset I(D_l'')$. First, assume $I(D_l) \subset I(D_l')$. Then $I(D_k) \subset I(D_l')$ (because $I(D_k) \subset I(D_l)$). Note that $D_k \cap D_l' = D_k \cap v_3 D_l v_1 \subset D_k \cap D_l \subset D_k \cap D_{k+1}$. However, $v_2 \notin V(D_l' \cap D_k)$ and $v_2 \in V(D_k \cap D_{k+1})$. This means that D_l' contradicts the choice of D_{k+1} (because $(D_1, D_2, \ldots)$ satisfies (1)). The case when $I(D_l) \subset I(D_l'')$ gives a contradiction in a similar way because $v_4 \notin V(D_l'' \cap D_k)$ and $v_4 \in V(D_k \cap D_{k+1})$.

Now let $e = uv \in E(G)$. Since $V(G) = V(H)$, $u, v \in V(H)$. Hence, $u, v \in I(D_j)$ for some sufficiently large j. Then $e \in E(H)$, for otherwise e would induce a finite $I(D_j)$-bridge in G, contradicting (2). Hence, $G = H$, and (4) holds for $(D_1, D_2, \ldots)$. □

We are now ready to state and prove the main result of this section.

(2.4) Theorem. *Let G be a 4-connected 2-indivisible infinite plane graph with a facial cycle C, and let S denote the set of vertices of G of infinite degree. Then $|S| \leq 2$, and there is a set F of edges of G such that*

(1) for any $f \in F$, f is incident with a vertex in S,

(2) $G - F$ has a net $N = (C_1, C_2, \ldots)$, $C \subset I(C_1)$, $S \subset \partial N$, and for any $f \in F$ both incident vertices of f are contained in a common infinite S-bridge of ∂N,

(3) if $|S| = 1$, then either one S-bridge of ∂N contains all vertices incident with edges in F or each S-bridge of ∂N contains infinitely many vertices incident with edges in F, and

(4) if $|S| = 2$, then for any $T \subset V(G) - S$ with $|T| \leq 3$, S is contained in a component of $(G - F) - T$.

Proof. Since G is 2-indivisible, $I(D)$ is defined for every cycle D in G. By (2.1), we may assume that G is nicely embedded in the plane. Since G is 4-connected and G is 2-indivisible, G is $(4, C)$-connected and G is cohesive. By (2.3), G has a sequence $(D_1, D_2, \ldots)$ of cycles with $C \subset I(D_1)$ and satisfying (1) - (4) of (2.3).

If $D_i \cap D_{i+1} = \emptyset$ for all $i \geq 1$, then let $C_i := D_i$ for all $i \geq 1$. In this case, $|S| = 0$ and G has a radial net $N = (C_1, C_2, \ldots)$, and (1) - (4) are satisfied with $F = \emptyset$.

Hence, we may assume that $D_n \cap D_{n+1} \neq \emptyset$ for some positive integer n. By (3) of (2.3), $D_n \cap D_{n+1} \subset D_i \cap D_{i+1}$ for all $i \geq n$. Therefore, $D_i \cap D_{i+1} \neq \emptyset$ for all $i \geq n$.

Claim 1. For each $i \geq n$, $D_i \cap D_{i+1}$ is a path.

Otherwise, suppose that $D_i \cap D_{i+1}$ is not a path for some $i \geq n$. Then there are at least two D_i-bridges in $D_i \cap D_{i+1}$, say T_1 and T_2. Note that $T_1 - D_i \neq \emptyset \neq T_2 - D_i$; for otherwise, T_1 or T_2 would be a finite $I(D_i)$-bridge in G, contradicting (2) of (2.3). By (2) of (2.3) and since G is 2-indivisible, G has a unique $I(D_i)$-bridge which is infinite and contains both $T_1 - D_i$ and $T_2 - D_i$. So there is a path R in G with endvertices $w_j \in V(T_j - D_i)$, $j = 1, 2$, such that $(R - \{w_1, w_2\}) \cap I(D_{i+1}) = \emptyset$. Hence $I(D_{i+1}) \subset I(D)$, where either $D := R \cup w_1 D_{i+1} w_2$ or $D := R \cup w_2 D_{i+1} w_1$. Clearly, $I(D_i) \subset I(D)$ and $D_i \cap D$ is properly contained in $D_{i+1} \cap D_i$. Hence, D contradicts the choice of D_{i+1} (because of (1) of (2.3)). This proves Claim 1.

The following claim is straightforward to verify.

Claim 2. If $(D'_1, D'_2, \ldots)$ is a subsequence of $(D_1, D_2, \ldots)$, then $(D'_1, D'_2, \ldots)$ also satisfies Claim 1 and (1) - (4) of (2.3).

Claim 3. There is a subsequence $(D'_1, D'_2, \ldots)$ of $(D_1, D_2, \ldots)$ such that exactly one of the following holds for all $k \geq 1$, where u'_k and v'_k are the endvertices of $D'_k \cap D'_{k+1}$ and $u'_k D'_k v'_k = D'_k \cap D'_{k+1}$:

(a) $u'_k \neq u'_{k+1}$ and $v'_k \neq v'_{k+1}$;
(b) $u'_k = u'_{k+1}$ and $v'_k \neq v'_{k+1}$;
(c) $u'_k \neq u'_{k+1}$ and $v'_k = v'_{k+1}$; or
(d) $u'_k = u'_{k+1}$ and $v'_k = v'_{k+1}$.

For $i \geq n$, let u_i, v_i be the endvertices of $D_i \cap D_{i+1}$ such that $u_i D_i v_i = D_i \cap D_{i+1}$.

If $(D_1, D_2, \ldots)$ has a subsequence $(D_{i_1}, D_{i_2}, \ldots)$ with $u_{i_k} \neq u_{i_{k+1}}$ for all $k \geq 1$, and has a subsequence $(D_{j_1}, D_{j_2}, \ldots)$ with $v_{j_k} \neq v_{j_{k+1}}$ for all $k \geq 1$, then $(D_1, D_2, \ldots)$ has a subsequence $(D_{l_1}, D_{l_2}, \ldots)$ with $u_{l_k} \neq u_{l_{k+1}}$ and $v_{l_k} \neq v_{l_{k+1}}$ for all $k \geq 1$. We re-name

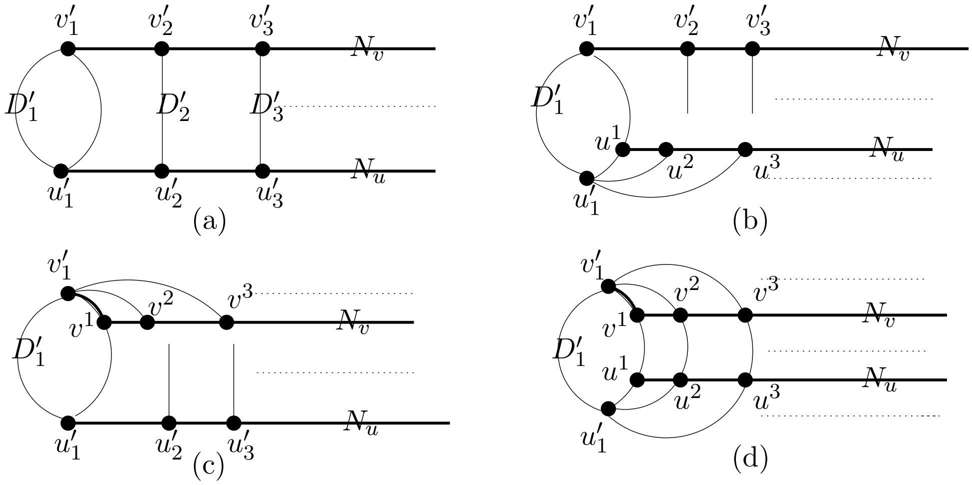


Figure 1: The sequence $(D_1', D_2', D_3, \ldots)$, and N_u and N_v

$D_{l_k}, u_{l_k}, v_{l_k}$ as D_k', u_k', v_k', respectively. Then $u_k' \neq u_{k+1}'$ and $v_k' \neq v_{k+1}'$ for all $k \geq 1$. In this case, (a) holds. See Figure 1(a).

If $(D_1, D_2, \ldots)$ has no subsequence $(D_{i_1}, D_{i_2}, \ldots)$ with $u_{i_k} \neq u_{i_{k+1}}$ for all $k \geq 1$, but has a subsequence $(D_{j_1}, D_{j_2}, \ldots)$ with $v_{j_k} \neq v_{j_{k+1}}$ for all $k \geq 1$, then there is some integer m such that $u_i = u_{i+1}$ for all $i \geq m$. Note that we can choose $(D_{j_1}, D_{j_2}, \ldots)$ so that $j_1 \geq m$. We re-name $D_{j_k}, u_{j_k}, v_{j_k}$ as D_k', u_k', v_k', respectively. Then $u_k' = u_{k+1}'$ and $v_k' \neq v_{k+1}'$ for all $k \geq 1$. In this case, (b) holds. See Figure 1(b).

If $(D_1, D_2, \ldots)$ has a subsequence $(D_{i_1}, D_{i_2}, \ldots)$ with $u_{i_k} \neq u_{i_{k+1}}$ for all $k \geq 1$, but has no subsequence $(D_{j_1}, D_{j_2}, \ldots)$ with $v_{j_k} \neq v_{j_{k+1}}$ for all $k \geq 1$, then there is some integer m with $v_i = v_{i+1}$ for all $i \geq m$. We can choose $(D_{i_1}, D_{i_2}, \ldots)$ so that $i_1 \geq m$. We re-name $D_{i_k}, u_{i_k}, v_{i_k}$ as D_k', u_k', v_k', respectively. Then $u_k' \neq u_{k+1}'$ and $v_k' = v_{k+1}'$ for all $k \geq 1$. In this case, (c) holds. See Figure 1(c).

Finally, if $(D_1, D_2, \ldots)$ has no subsequence $(D_{i_1}, D_{i_2}, \ldots)$ with $u_{i_k} \neq u_{i_{k+1}}$ for all $k \geq 1$, and has no subsequence $(D_{j_1}, D_{j_2}, \ldots)$ with $v_{j_k} \neq v_{j_{k+1}}$ for all $k \geq 1$, then there is a positive integer m such that $u_i = u_{i+1}$ and $v_i = v_{i+1}$ for $i \geq m$. We re-name D_i, u_i, v_i ($i \geq m$) as $D_{i-m+1}', u_{i-m+1}', v_{i-m+1}'$, respectively. Then $u_k' = u_{k+1}'$ and $v_k' = v_{k+1}'$ for all $k \geq 1$. In this case, (d) holds. See Figure 1(d). This completes the proof of Claim 3.

It is clear that $S = \emptyset$ if (a) of Claim 3 occurs; $S = \{u_1'\}$ if (b) of Claim 3 occurs; $S = \{v_1'\}$ if (c) of Claim 3 occurs; and $S = \{u_1', v_1'\}$ (possibly $u_1' = v_1'$) if (d) of Claim 3 occurs. Hence, $|S| \leq 2$.

Since G is 4-connected, if $|S| = 2$ then G contains four paths between the vertices in S which are pairwise vertex disjoint except for vertices in S. Since $\bigcup_{i\geq 1} I(D_i') = G$, these four paths are contained in $I(D'_i)$ for all sufficiently large i. Therefore, we may assume that $(D'_1, D'_2, \ldots)$ is chosen to satisfy the following.

Claim 4. If $|S| = 2$ then $I(D'_1)$ contains four paths between the vertices in S which are pairwise vertex disjoint except for vertices in S.

Let F be the set of edges in G with one incident vertex in S and the other in $G - I(D'_1)$. Clearly (1) holds. Note that $F = \emptyset$ if (a) of Claim 3 occurs, and otherwise, F is an infinite set. Our next objective is to describe a ladder net N in $G - F$ so that (2), (3), and (4) hold. To do this, we first describe 1-way infinite paths N_u and N_v in $G - F$.

Let v^1 denote the neighbor of v'_1 in $v'_1 D'_1 u'_1$, and let u^1 denote the neighbor of u'_1 in $v'_1 D'_1 u'_1$. Since G is 4-connected, $u^1 \neq v^1$ (otherwise, $G - \{u'_1, v'_1, u^1 = v^1\}$ would not be connected). See Figure 1.

If $v'_1 \notin S$, then let N_v denote the 1-way infinite path in $\bigcup_{i \geq 1}(D'_i \cap D'_{i+1})$ from v'_1 and containing $v'_2, v'_3, \ldots$. See Figure 1(a) and Figure 1(b).

If $v'_1 \in S$, then, for each $k \geq 1$, we can label the edges in $F \cap (E(I(D'_k)) - E(I(D'_1)))$ incident with v'_1 as $v'_1 v^2, v'_1 v^3, \ldots, v'_1 v^{m_k}$ such that the vertices $v^1, v^2, \ldots, v^{m_k}$ occur around v'_1 in this counter clockwise order. Since $I(D'_k) \subset I(D'_{k+1})$ for $k \geq 1$ and since G is nicely embedded, this labeling is consistent for all $k \geq 1$. For $j \geq 1$, let F_v^j denote the facial cycle of G containing $v'_1 v^j$ and $v'_1 v^{j+1}$. Since G is 4-connected, all $F_v^j - v'_1$ are vertex disjoint except for their endvertices (because if $x \in V(F_v^j - v'_1) \cap V(F_v^l - \{v'_1\})$ and x is not adjacent to v'_1, then $G - \{v'_1, x\}$ would not be connected). Hence, $N_v := ((\bigcup_{j \geq 1}(F_v^j - v'_1)) \cup \{v'_1\}) + v'_1 v^1$ is a 1-way infinite path in $G - F$ from v'_1. See Figure 1(c) and Figure 1(d).

Similarly, if $u'_1 \notin S$, then let N_u denote the 1-way infinite path in $\bigcup_{i \geq 1}(D'_i \cap D'_{i+1})$ from u'_1 and containing $u'_2, u'_3, \ldots$. See Figure 1(a) and Figure 1(c). If $u'_1 \in S$, then for each $k \geq 1$, we can label the edges in $F \cap (E(I(D'_k)) - E(I(D'_1)))$ incident with u'_1 as $u'_1 u^2, u'_1 u^3, \ldots, u'_1 u^{n_k}$ such that the vertices $u^1, u^2, \ldots, u^{n_k}$ occur around u'_1 in this clockwise order, and for each $j \geq 1$ let F_u^j denote the facial cycle of G containing $u'_1 u^j$ and $u'_1 u^{j+1}$. Then $N_u := ((\bigcup_{j \geq 1}(F_u^j - u'_1)) \cup \{u'_1\}) + u'_1 u^1$ is a 1-way infinite path in $G - F$ from u'_1. See Figure 1(b) and Figure 1(d). This completes the description of N_u and N_v.

Since G is 4-connected, $(N_u - u'_1) \cap (N_v - v'_1) = \emptyset$ (because if $z \in V(N_u - u'_1) \cap V(N_v - v'_1)$, then $G - \{u'_1, v'_1, z\}$ would not be connected). Hence, $Q := u'_1 D'_1 v'_1 \cup N_u \cup N_v$ is a 2-way infinite path. Next, we describe a ladder net $N = (C_1, C_2, \ldots)$ with $\partial N = Q$.

Since G is 2-indivisible, $G - I(D'_1)$ contains a path Q_1 from $x_1 \in V(N_u)$ to $y_1 \in V(N_v)$ such that $V(Q_1 \cap N_u) = \{x_1\}$ and $V(Q_1 \cap N_v) = \{y_1\}$. See Figure 2. Let $C_1 := x_1 Q y_1 \cup Q_1$. Then C_1 is a cycle in $G - F$, and $I(D'_1) - F = I(D'_1) \subset I(C_1)$. Since $G = \bigcup_{i \geq 1} I(D'_i)$ and $I(D'_i) \subset I(D'_{i+1})$ for all $i \geq 1$, $I(C_1) \subset I(D'_{i_1})$ for some $i_1 > 1$.

Suppose that we have constructed paths Q_j, $j = 1, \ldots, k$, from $x_j \in V(N_u)$ to $y_j \in V(N_v)$ and there are $1 = i_0 < i_1 < \ldots < i_k$ such that $V(Q_j \cap N_u) = \{x_j\}$,

$V(Q_j \cap N_v) = \{y_j\}$, and $I(D'_{i_{j-1}}) - F \subset I(C_j) \subset I(D'_{i_j})$, where $C_j := x_jQy_j \cup Q_j$. See Figure 2. Since G is 2-indivisible, $G - I(D'_{i_k})$ contains a path Q_{k+1} from $x_{k+1} \in V(N_u)$ to $y_{k+1} \in V(N_v)$ such that $V(Q_{k+1} \cap N_u) = \{x_{k+1}\}$ and $V(Q_{k+1} \cap N_v) = \{y_{k+1}\}$. Let $C_{k+1} := x_{k+1}Qy_{k+1} \cup Q_{k+1}$. Then C_{k+1} is a cycle in $G - F$, and $I(D'_{i_k}) - F \subset I(C_{k+1})$. Since $G = \bigcup_{i\geq 1} I(D'_i)$ and $I(D'_i) \subset I(D'_{i+1})$ for all $i \geq 1$, $I(C_{k+1}) \subset I(D'_{i_{k+1}})$ for some $i_{k+1} > i_k$.

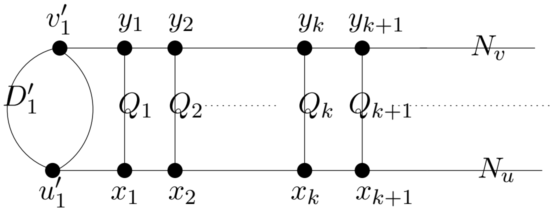


Figure 2: The ladder net $N = (C_1, C_2, \ldots)$.

Note that $u'_1, x_1, x_2, \ldots$ occur on N_u in that order, and $v'_1, y_1, y_2, \ldots$ occur on N_v in that order. Let $N = (C_1, C_2, \ldots)$. Then $I(C_i) \subset I(C_{i+1})$ for all $i \geq 1$. Since $C_i \cap C_{i+1} = x_iQy_i$, $C_i \cap C_{i+1}$ is a non-trivial path, $C_i \cap C_{i+1} \subset C_{i+1} \cap C_{i+2}$, and no endvertex of $C_{i+1} \cap C_{i+2}$ is an endvertex of $C_i \cap C_{i+1}$. Since $I(D'_{i_{j-1}}) - F \subset I(C_j)$ for all $j \geq 1$, $G - F = \bigcup_{i\geq 1}(I(D'_i) - F) \subset \bigcup_{j\geq 1} I(C_j) \subset G - F$. Hence, $G - F = \bigcup_{j\geq 1} I(C_j)$, and so, N is a ladder net in $G - F$ with $\partial N = Q$. Note that $C \subset I(D) \subset I(D'_1) \subset I(C_1)$ and $S \subset \{u'_1, v'_1\} \subset Q = \partial N$. Also note that the edges in F incident with v'_1 (respectively, u'_1) have their other neighbors in $N_v - v'_1$ (respectively, $N_u - u'_1$). Therefore, (2) holds for F and N.

Now assume that $|S| = 1$. The first alternative of (3) occurs when (b) or (c) of Claim 3 occurs, and the second alternative of (3) occurs when $u'_1 = v'_1$ and (d) of Claim 3 occurs. Hence (3) holds for F and N.

Finally, assume that $|S| = 2$. By Claim 4, $I(D'_1) \subset G - F$ has four paths between the vertices in S which are pairwise vertex disjoint except for the vertices in S. So (4) holds for F and N. □

3 Tutte paths

In this section, we prove the existence of a certain type of 1-way infinite paths in a 2-indivisible infinite plane graph. This result will be used in Section 4 to prove (1.1).

Let G be a graph (finite or infinite) and H, C be subgraphs (finite or infinite) of G. We say that H is a *Tutte subgraph* of G if every H-bridge of G is finite and has at most three attachments. We say that H is a *C-Tutte subgraph* of G if H is a Tutte subgraph of G and every H-bridge of G containing an edge of C has at most two attachments. A *Tutte path* (finite or infinite) is a path (finite or infinite) which is a Tutte subgraph.

A standard approach to proving the existence of spanning subgraphs in 4-connected graphs is to prove the existence of Tutte subgraphs in 2-connected graphs. The concept "C-Tutte subgraph" is for the sake of induction. The following result is the main theorem in [6], where a P-bridge is called a "P-component".

(3.1) Lemma. *Let G be a 2-connected plane graph with a facial cycle C. Assume that $x \in V(C)$, $e \in E(C)$, and $y \in V(G-x)$. Then G contains a C-Tutte path P from x to y and through e.*

The next result is due to Thomas and Yu ([4], (2.6)), where a vCu-Tutte path is called an $E(vCu)$-snake.

(3.2) Lemma. *Let G be a 2-connected plane graph with a facial cycle C. Let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in that clockwise order. Then G contains a vCu-Tutte path P from u to v and through e and f.*

The rest of this section is devoted to proving the existence of certain 1-way infinite Tutte paths in 2-indivisible infinite plane graphs. We need the following fact which allows us to "construct" a 1-way infinite path from a sequence of paths. (This fact is a variation of König's lemma).

(3.3) Lemma. *Let G be an infinite locally finite graph and let $x \in V(G)$. Suppose $\{P_n\}$ is an infinite sequence of finite paths from x such that the length of P_n increases. Then $\{P_n\}$ has a subsequence $\{P_{n_k}\}$ converging to a 1-way infinite path P from x, that is, for any $v \in V(P)$, $xPv = xP_{n_k}v$ for all sufficiently large n_k.*

In later proofs, we need to find a sequence of Tutte paths converging to a 1-way infinite Tutte path. For this reason, we need such Tutte paths to be "forward".

(3.4) Definition. Let $N = (H_1, H_2, \ldots)$ be a sequence of finite subgraphs of a graph G (finite or infinite). A path P in G is said to be *N-forward* or *$(H_1, H_2, \ldots)$-forward* if, for any $i \geq 1$ and for any $a, b, c \in V(P)$ with $a \in V(bPc)$, $\{b, c\} \subset V(H_i)$ implies that $a \notin V(H_j)$ for all $j \geq i+2$.

Note that if, for each $i \geq 2$, $\bigcup_{j=1}^{i-1} H_j$ and $\bigcup_{j \geq i+1} H_j$ are contained in different components of $G - V(H_i)$, then "P is $(H_1, H_2, \ldots)$-forward" means that if P starts from H_1, then, after reaching H_{i+2}, P never visits H_i again.

(3.5) Lemma. *Let G be a 2-connected infinite plane graph with a net $N = (C_1, C_2, \ldots)$. If N is a radial net then let C be a facial cycle of G with $C \subset I(C_1)$, and if N is a ladder* *net then let $C := \partial N$. Let $x \in V(C)$, and let H_i denote the path obtained from C_i by deleting $C_i \cap C_{i+1}$ except its endvertices. Suppose that, for each $n \geq 1$, $I(C_n)$ contains a Tutte path P_n between x and a vertex of H_n such that P_n is $(H_1, H_2, \ldots)$-forward in G. Then $\{P_n\}$ has a subsequence $\{P_{n_k}\}$ converging to a 1-way infinite Tutte path P from x in G and, for any given P-bridge B of G, B is a P_{n_k}-bridge of $I(C_{n_k})$ for all sufficiently large n_k.*

Proof. Note that $I(C_1) - H_1$ and $\bigcup_{j\geq 2} H_j$ are contained in different components of $G - V(H_1)$, and for each $i \geq 2$, $\bigcup_{j=1}^{i-1} H_j$ and $\bigcup_{j\geq i+1} H_j$ are contained in different components of $G - V(H_i)$.

Since G has a net, G is locally finite. Since $I(C_i) \subset I(C_{i+1})$ and $I(C_i) \neq I(C_{i+1})$ and since P_n is between x and a vertex of H_n, $\{P_n\}$ contains a subsequence $\{P_{n_i}\}$ such that the length of P_{n_i} increases. By (3.3), $\{P_{n_i}\}$ contains a subsequence converging to a 1-way infinite path P from x. So let $\{P_{n_k}\}$ be a subseqence of $\{P_n\}$ converging to P.

Claim 1. For any given positive integer l, $P_{n_k} \cap I(C_l) = P \cap I(C_l)$ for all sufficiently large n_k.

Let $y \in V(P \cap I(C_l))$ with xPy maximal. Then $P \cap I(C_l) = xPy \cap I(C_l)$. Since $\{P_{n_k}\}$ converges to P, $xP_{n_k}y = xPy$ for all sufficiently large n_k. Hence, $P \cap I(C_l) = xPy \cap I(C_l) = xP_{n_k}y \cap I(C_l) \subset P_{n_k} \cap I(C_l)$ for all sufficiently large n_k.

It remains to show that $P_{n_k} \cap I(C_l) \subset P \cap I(C_l)$ for all sufficiently large n_k. Let $a \in V(P \cap H_{l+2})$. Since $\{P_{n_k}\}$ converges to P, $xPa = xP_{n_k}a$ for all sufficiently large n_k.

We claim that, for any n_k such that $a \in V(P_{n_k})$ and for any $z \in V(P_{n_k} - xP_{n_k}a)$, $z \notin I(C_l)$. For otherwise, there exists some $c \in V(zP_{n_k}a) \cap V(H_l)$. Since $x \in I(C_1)$, there is a vertex $b \in V(xP_{n_k}a) \cap V(H_l)$. Since $a \in V(bP_{n_k}c)$ and P_{n_k} is $(H_1, H_2, \ldots)$-forward in G, $a \notin V(H_j)$ for all $j \geq l+2$, a contradiction.

Thus, for all sufficiently large n_k, $P_{n_k} \cap I(C_l) = xP_{n_k}a \cap I(C_l) = xPa \cap I(C_l) \subset P \cap I(C_l)$. This completes the proof of Claim 1.

Let B be a P-bridge of G. We need to show that B is a P_{n_k}-bridge of $I(C_{n_k})$ for all sufficiently large n_k.

Claim 2. B is finite.

Suppose that B is infinite. Since G (and hence $B - P$) is locally finite and $B - P$ is connected, $B - P$ contains an infinite path. Hence, $B - P$ contains a path R from H_i to H_j for some i and j with $j - i \geq 4$. Since R is finite, $R \subset I(C_l)$ for some l. By Claim 1, $R \cap P_{n_k} = \emptyset$ for all sufficiently large n_k. Hence, R is contained in a P_{n_k}-bridge B' of $I(C_{n_k})$ for all sufficiently large n_k. Since $R \cap H_s \neq \emptyset$ and $P_{n_k} \cap H_s \neq \emptyset$ for all s with $i \leq s \leq j$, B' has at least four attachments on P_{n_k}, contradicting the fact that P_{n_k} is a Tutte path in $I(C_{n_k})$. Hence B is finite.

By Claim 2, $B \subset I(C_l)$ for some l. By Claim 1, B is a P_{n_k}-bridge of $I(C_{n_k})$ for all sufficiently large n_k. Since P_{n_k} is a Tutte path in $I(C_{n_k})$, B has at most 3 attachments. So P is a 1-way infinite Tutte path from x in G. $\square$

We can now prove the main result in this section.

(3.6) Theorem. *Let G be a 2-connected infinite plane graph with a ladder net N. Let $x, y \in V(\partial N)$ be distinct. Then G contains a 1-way infinite ∂N-Tutte path P from x and through y.*

Proof. Since G has a net, for any cycle D in G, $I(D)$ is defined. By (2.1), we may assume that G is nicely embedded in the plane. Let N_x and N_y denote the infinite $x\partial Ny$-bridges of ∂N such that $x \in N_x$ and $y \in N_y$. Then N_x and N_y are 1-way infinite paths from x and y, respectively. See Figure 3. Let $G_1 := G$, $x_1 := x$, $y_1 := y$, and $H_1 := x_1\partial Ny_1$. Let $C_1 := x_1\partial Ny_1 \cup H_1$, and $I(C_1) := H_1$.

Claim 1. There are distinct vertices $x_i \in V(N_x - x_1)$ and $y_i \in V(N_y - y_1)$, $i = 2, 3, \ldots$, and there are disjoint paths H_i in G from x_i to y_i, $i = 2, 3, \ldots$, such that

(i) $x_1, x_2, x_3, \ldots$ occur on N_x in that order and $y_1, y_2, y_3, \ldots$ occur on N_y in that order,

(ii) $V(H_i \cap \partial N) = \{x_i, y_i\}$ for all $i \geq 2$, and

(iii) for each $i \geq 2$, $C_i := x_i\partial Ny_i \cup H_i$ is a cycle, and for each $i \geq 1$, $I(C_i) \subset I(C_{i+1})$ and every $I(C_i)$-bridge of $I(C_{i+1})$ has at most one attachment on H_{i+1}.

Suppose that we have defined C_i, H_i, x_i, y_i for some $i \geq 1$. Since G is 2-indivisible, $G - I(C_i)$ contains a path H_{i+1} from some $x_{i+1} \in V(N_x)$ to some $y_{i+1} \in V(N_y)$ such that $V(H_{i+1} \cap \partial N) = \{x_{i+1}, y_{i+1}\}$. See Figure 3. Then $C_{i+1} := x_{i+1}\partial Ny_{i+1} \cup H_{i+1}$ is a cycle in G. Choose such H_{i+1} that $I(C_{i+1})$ is minimal. Then every $(I(C_i) \cup H_{i+1})$-bridge of $I(C_{i+1})$ has at most one attachment on H_{i+1}. This proves Claim 1.

It is easy to check that $N' = (C_2, C_3, \ldots)$ is a ladder net in G with $\partial N' = \partial N$. For $n \geq i \geq 1$, let $G_{n,i} = I(C_n) - (I(C_i) - C_i)$ and $D_{n,i} = x_n\partial Nx_i \cup H_i \cup y_i\partial Ny_n$. See Figure 3.

Claim 2. $G_{n,i}$ contains a $D_{n,i}$-Tutte path $P_{n,i}$ between x_i and a vertex of H_n such that $\{x_i, \ldots, x_n, y_i, \ldots, y_n\} \subset V(P_{n,i})$ and $P_{n,i}$ is $(H_1, H_2, \ldots)$-forward in G, and $G_{n,i}$ contains a $D_{n,i}$-Tutte path $R_{n,i}$ between y_i and a vertex of H_n such that $\{x_i, \ldots, x_n, y_i, \ldots, y_n\} \subset V(R_{n,i})$ and $R_{n,i}$ is $(H_1, H_2, \ldots)$-forward in G.

We use induction on $n - i$. If $n - i = 0$, then $G_{n,i} = H_i = H_n$, and in this case, H_n gives the desired $P_{n,i}$ and $R_{n,i}$. Now assume that $n - i \geq 1$, $G_{n,i+1}$ contains an $D_{n,i+1}$-Tutte path $P_{n,i+1}$ between x_{i+1} and a vertex of H_n such that $\{x_{i+1}, \ldots, x_n, y_{i+1}, \ldots, y_n\} \subset V(P_{n,i+1})$ and $P_{n,i+1}$ is $(H_1, H_2, \ldots)$-forward in G, and $G_{n,i+1}$ contains an $D_{n,i+1}$-Tutte

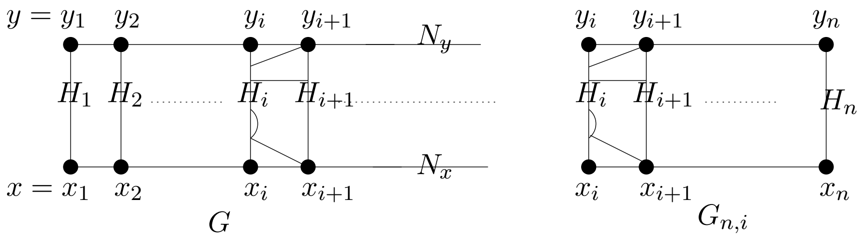


Figure 3: The graphs G and $G_{n,i}$

path $R_{n,i+1}$ between y_{i+1} and a vertex of H_n such that $\{x_{i+1}, \ldots, x_n, y_{i+1}, \ldots, y_n\} \subset V(R_{n,i+1})$ and $R_{n,i+1}$ is $(H_1, H_2, \ldots)$-forward in G.

Next we extend $R_{n,i+1}$ (respectively, $P_{n,i+1}$) to the desired $P_{n,i}$ (respectively, $R_{n,i}$). We will only show how to obtain $P_{n,i}$ from $R_{n,i+1}$, because the other case is symmetric.

Let W be the set of attachments on H_{i+1} of $(H_i \cup G_{n,i+1})$-bridges of $G_{n,i}$. For $w, w' \in W$, we say $w \sim w'$ if $w = w'$ or $\{w, w'\} \subset V(B - R_{n,i+1})$ for some $R_{n,i+1}$-bridge B of $G_{n,i+1}$ (such B contains an edge of $D_{n,i+1}$, and hence, has just two attachments). Then $\sim$ is an equivalence relation. Let $W_1, W_2, \ldots, W_m$ be the equivalence classes of W with respect to $\sim$. Let $B_i := W_i$ if $W_i \subset R_{n,i+1}$ (in this case, $|W_i| = 1$), and otherwise, let B_i denote the $R_{n,i+1}$-bridge of $G_{n,i+1}$ containing W_i. Without loss of generality, we may assume that $W_1, \ldots, W_m$ occur on H_{i+1} in that order, and $W_1 = \{x_{i+1}\}$ and $W_m = \{y_{i+1}\}$.

Let $s_j, t_j \in V(H_i)$ such that there are $w_s, w_t \in W_i$ such that $\{s_j, w_s\}$ is contained in an $(H_i \cup G_{n,i+1})$-bridge of $G_{n,i}$ and $\{t_j, w_t\}$ is contained in an $(H_i \cup G_{n,i+1})$-bridge of $G_{n,i}$, and subject to this, $s_j H_i t_j$ is maximal. Without loss of generality, assume that $s_1, t_1, s_2, t_2, \ldots, s_m, t_m$ occur on H_i in that order. Then $s_1 = x_i$ and $t_m = y_i$.

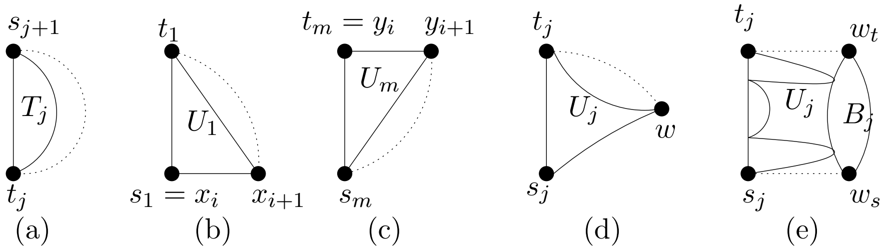


Figure 4: The graphs $U_1, \ldots, U_m$ and $T_1, \ldots, T_{m-1}$.

For $j = 1, \ldots, m-1$, let T_j be the union of $t_j H_i s_{j+1}$ and those $(H_i \cup G_{n,i+1})$-bridges of $G_{n,i}$ whose attachments are all contained in $V(t_j H_i s_{j+1})$.

(1) We claim that T_j contains a $(T_j \cap D_{n,i})$-Tutte path R_j from t_j to s_{j+1}.

If $|V(t_j H_i s_{j+1})| \leq 2$, then let $R_j := t_j H_i s_{j+1}$. If $|V(t_j H_i s_{j+1})| \geq 3$, then apply (3.1) to $T_j + t_j s_{j+1}$ to find a $(T_j \cap D_{n,i})$-Tutte path R_j from t_j to s_{j+1} through an edge of $t_j H_i s_{j+1}$. See Figure 4(a). It is easy to verify that R_j gives the desired path.

For $j = 1, \ldots, m$, let U_j denote the union of $s_j H_i t_j$, B_j, and those $(H_i \cup G_{n,i+1})$-bridges of $G_{n,i}$ whose attachments are all contained in $V(s_j H_i t_j) \cup W_j$. Then $U_j \cap H_i = s_j H_i t_j$ and $|V(U_j \cap R_{n,i+1})| = |V(B_j \cap R_{n,i+1})| \leq 2$.

(2) We claim that $U_1 - x_{i+1}$ contains a path Q_1 from $x_i = s_1$ to t_1 such that $Q_1 \cup \{x_{i+1}\}$ is a $(U_1 \cap D_{n,i})$-Tutte subgraph of U_1.

If $t_1 = x_i$, then let $Q_1 := s_1 H_i t_1$. If $t_1 \neq x_i$, then apply (3.1) to $U_1 + t_1 x_{i+1}$ to find a $(U_1 \cap D_{n,i})$-Tutte path Q_1' from $x_i = s_1$ to x_{i+1} and through $t_1 x_{i+1}$; and let $Q_1 := Q_1' - x_{i+1}$. See Figure 4(b). It is easy to see that Q_1 gives the desired path.

(3) We claim that U_m contains $(U_m \cap D_{n,i})$-Tutte path Q_m from s_m to y_{i+1} and through y_i.

If $|V(U_m)| = 2$, then let $Q_m := U_m$. If $|V(U_m)| \geq 3$, then apply (3.1) to $U_m + s_m y_{i+1}$ to find a $(U_m \cap D_{n,i})$-Tutte path Q_m from s_m to y_{i+1} through an edge of $U_m \cap D_{n,i}$ incident with y_i. See Figure 4(c). It is easy to see that Q_m gives the desired path.

(4) For each $j = 2, \ldots, m-1$, we claim that $U_j - V(U_j \cap R_{n,i+1})$ contains a path Q_j from s_j to t_j such that $Q_j \cup (U_j \cap R_{n,i+1})$ is a $(U_j \cap D_{n,i})$-Tutte subgraph of U_j.

If $s_j = t_j$ then $Q_j := s_j H_i t_j$ gives the desired path. So we may assume that $s_j \neq t_j$. First, assume $B_j = W_j$. Then $|W_j| = 1$, and let w be the only vertex in W_j. See Figure 4(d). In $U_j + t_j w$, we apply (3.1) to find a $(U_j \cap D_{n,i})$-Tutte path Q_j' from s_j to w and through $t_j w$; and let $Q_j := Q_j' - w$. It is easy to see that Q_j gives the desired path. Now assume that $B_j \neq W_j$. Let w_s, w_t denote the vertices of $U_j \cap R_{n,i+1}$ such that $x_{i+1}, w_s, w_t, y_{i+1}$ occur on H_{i+1} in that order. See Figure 4(e). In $U_j + \{s_j w_s, t_j w_t\}$, we apply (3.2) to find a $(U_j \cap D_{n,i})$-Tutte path Q_j' from w_s to w_t and through $s_j w_s$ and $t_j w_t$; and let $Q_j := Q_j' - \{w_s, w_t\}$. It is easy to verify that Q_j gives the desired path.

Let $P_{n,i} := R_{n,i+1} \cup (\bigcup_{j=1}^{m} Q_j) \cup (\bigcup_{j=1}^{m-1} R_j)$. Then $P_{n,i}$ is a path in $G_{n,i}$ between x_i and a vertex of H_n. Since $\{x_{i+1}, \ldots, x_n, y_{i+1}, \ldots, y_n\} \subset V(P_{n,i})$, $\{x_i, \ldots, x_n, y_i, \ldots, y_n\} \subset V(P_{n,i})$. Note that every $P_{n,i}$-bridge of $G_{n,i}$ is one of the following: an $R_{n,i+1}$-bridge of $G_{n,i+1}$ not contained in any U_j, or an R_j-bridge of T_j, or a $(Q_1 \cup \{x_{i+1}\})$-bridge of U_1, or a Q_m-bridge of U_m, or a $(Q_j \cup (U_j \cap R_{n,i+1}))$-bridge of U_j. So by (1)-(4), $P_{n,i}$ is a $D_{n,i}$-Tutte path in $G_{n,i}$.

It remains to show that $P_{n,i}$ is $(H_1, H_2, \ldots)$-forward in G. Let $a, b, c \in V(P_{n,i})$, $b, c \in V(H_k)$, and $a \in V(b P_{n,i} c)$. We need to show that $a \notin V(H_j)$ for all $j \geq k+2$. First, assume that $b, c \in V(P_{n,i}) - V(R_{n,i+1} - y_{i+1})$. Then $b P_{n,i} c \subset P_{n,i} - (R_{n,i+1} - y_{i+1}) \subset H_i \cup H_{i+1}$. Since $\{b, c\} \subset V(H_k)$, $H_k = H_i$ or $H_k = H_{i+1}$. Thus $a \notin V(H_j)$ for any

$j \geq k+2 \geq i+2$. Now assume $b, c \in V(R_{n,i+1})$. Then $a \notin V(H_j)$ for all $j \geq k+2$ because $R_{n,i+1}$ is $(H_1, H_2, \ldots)$-forward in G. Finally, assume by symmetry that $b \in V(P_{n,i}) - V(R_{n,i+1})$ and $c \in V(R_{n,i+1} - y_{i+1})$. Then $b \in V(H_i \cup H_{i+1})$ and $c \notin V(H_i)$. Since $b, c \in V(H_k)$, $H_k = H_{i+1}$, and so, $y_{i+1} \in V(H_k)$ (because $y_{i+1} \in V(H_{i+1})$). If $a \notin V(y_{i+1}R_{n,i+1}c)$ then $a \in V(H_i \cup H_{i+1})$, and so, $a \notin V(H_j)$ for all $j \geq k+2$. So $a \in V(y_{i+1}R_{n,i+1}c)$. Since $\{y_{i+1}, c\} \subset V(H_k)$, $a \notin V(H_j)$ for all $j \geq k+2$ (because $R_{n,i+1}$ is $(H_1, H_2, \ldots)$-forward in G). This completes the proof of Claim 2.

Let $P_n := P_{n,1}$. Note that $D_{n,1} = x_n \partial N y_n$. Hence, P_n is an $x_n \partial N y_n$-Tutte path of $I(C_n)$ between x and a vertex of H_n and through y such that P_n is $(H_1, H_2, \ldots)$-forward in G. By (3.5) (with $N' = (C_2, C_3, \ldots)$ as $N = (C_1, C_2, \ldots)$), $\{P_n\}$ has a subsequence $\{P_{n_k}\}$ converging to a 1-way infinite Tutte path P from x and, for any given P-bridge B of G, B is a P_{n_k}-bridge of $I(C_{n_k})$ for all sufficiently large n_k. Since $y \in P_{n_k}$ for all n_k and since each P_{n_k} is an $x_n \partial N y_n$-Tutte path of $I(C_{n_k})$, P is a 1-way infinite ∂N-Tutte path in G from x and through y. □

The following consequence will be useful in a later paper.

(3.7) Corollary. *Let G be a 2-connected infinite plane graph with a ladder net N, and let $x \in V(\partial N)$ and $e = uv \in E(\partial N)$ such that $u \in V(x \partial N v)$. Then G contains a 1-way infinite ∂N-Tutte path P from x through e such that $u \in V(xPv)$.*

Proof. Let G' be the graph obtained from G by subdividing the edge e with a vertex y. It is easy to see that G' is a 2-connected infinite plane graph with a ladder net N' such that $\partial N'$ is obtained from ∂N by subdividing the edge e with y. Now apply (3.6) to G', we see that G' has a 1-way infinite $\partial N'$-Tutte path P' from x and through y. Let P be the 1-way infinite path obtained from P' by deleting y and by adding the edge $e = uv$. It is easy to see that P is a 1-way infinite ∂N-Tutte path in G from x and through e. By planarity, $u \in V(x \partial N v)$. □

4 2-Way infinite paths

In this section, we complete the proof of (1.1). First, we prove a result about 2-way infinite Tutte paths.

(4.1) Theorem. *Let G be a 2-connected graph with a ladder net N, and let $e \in E(\partial N)$. Then G contains a 2-way infinite ∂N-Tutte path P through e.*

Proof. Since G has a ladder net, for any cycle D in G, $I(D)$ is defined. So by (2.1), we may assume that G is nicely embedded in the plane. Let x, y be the vertices of G incident with e, and let X, Y be the components of $\partial N - e$ such that $x \in V(X)$ and

$y \in V(Y)$. Then X and Y are 1-way infinite paths from x and y, respectively. Let M be the minimal subgraph of $G - Y$ such that $X \subset M$ and M is a union of blocks of $G - Y$. Then the blocks of M can be labeled as $M_1, \dots, M_{n+1}$ or $M_1, M_2, \dots$, where $x \in V(M_1)$, $M_i \cap M_j = \emptyset$ if $j \geq i + 2$, and $M_i \cap M_j$ has just one vertex x_i if $j = i + 1$. Let $x_0 = x$. Then clearly $x_0, x_1, \dots, x_n$ or $x_0, x_1, x_2, \dots$ lie on X in that order. See Figures 5(a) and 5(b).

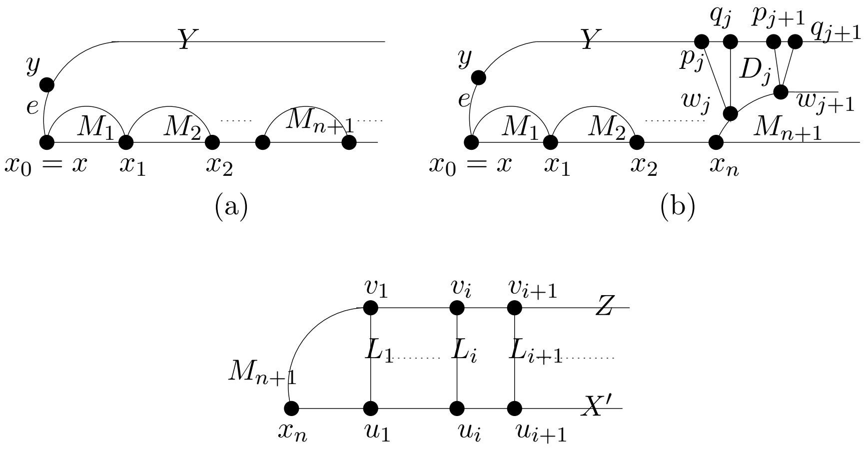
Since G is locally finite, for every integer $i \geq 1$ there is some $y_i \in V(Y)$ such that $(\bigcup_{j=1}^{i} M_j) - \{x_i, y_i\}$ and $\bigcup_{j>i} M_j - \{x_i, y_i\}$ are contained in different components of $G - \{x_i, y_i\}$. Therefore, if M has infinitely many blocks $M_1, M_2, \dots$, then M_i is finite for all $i \geq 1$. The following claim describes the situation when M has only finitely many blocks.

Claim 1. Suppose M has finitely many blocks $M_1, \dots, M_{n+1}$. Then M_{n+1} is the only infinite block in M. Moreover, M_{n+1} has a ladder net N' such that $X' := X \cap M_{n+1} \subset \partial N'$, X' is a 1-way infinite path, and the attachments on M_{n+1} of $(Y \cup M)$-bridges of G are contained in $\partial N' - (X' - x_n)$.

Since G is 2-indivisible and since X is a 1-way infinite path, M_{n+1} must be the only infinite block. Since G is nicely embedded, so is M_{n+1}.

Let W' denote the set of those attachments on $M_{n+1} - x_n$ of $(Y \cup M)$-bridges of G. For each $w_j \in W'$, let $p_j, q_j \in V(Y)$ such that (1) $\{p_j, w_j\}$ is contained in a $(Y \cup M)$-bridge of G and $\{q_j, w_j\}$ is contained in a $(Y \cup M)$-bridge of G, and subject to (1), (2) $p_j Y q_j$ is maximal. By planarity, we may assume that $y, p_1, q_1, p_2, q_2, \dots$ occur on Y in that order. See Figure 5(b). Let D_j denote the facial cycle of G containing $\{q_j, p_{j+1}, w_{j+1}, w_j\}$, and let D_0 denote the facial cycle of G containing $\{x_n, w_1, p_1\}$. For $j \geq 0$, let $Z_j = w_{j+1} D_j w_j$, where $w_0 = x_n$. Since M_{n+1} is 2-connected, $Z_i \cap Z_j = \emptyset$ if $j \geq i + 2$, and $V(Z_i \cap Z_j) = \{w_{i+1}\}$ if $j = i + 1$. Let $Z := \bigcup_{j \geq 0} Z_j$. Clearly, Z is a 1-way infinite path from x_n. Since M_{n+1} is a block of $G - Y$ and by planarity, $V(Z \cap X') = \{x_n\}$. So $Z \cup X'$ is a 2-way infinite path.

Next, we show that M_{n+1} contains a ladder net N' such that $\partial N' = X' \cup Z$. Note that for any cycle D in M_{n+1}, $I_G(D) = I_{M_{n+1}}(D)$, and hence, we will simply use $I(D)$. Since M_{n+1} is 2-connected, $M_{n+1} - x_n$ contains a path L_1 from some $u_1 \in V(X')$ to some $v_1 \in V(Z)$ such that $V(L_1 \cap X') = \{u_1\}$ and $V(L_1 \cap Z) = \{v_1\}$. See Figure 5(c). Let $C_1 := x_n Z v_1 \cup L_1 \cup x_n X' u_1$. Then C_1 is a cycle. Suppose that we have constructed a path L_i and a cycle C_i, where L_i is from some $u_i \in V(X')$ to some $v_i \in V(Z)$ such that $V(L_i \cap X') = \{u_i\}$ and $V(L_i \cap Z) = \{v_i\}$, and $C_i := x_n Z v_i \cup L_i \cup x_n X' u_i$. Then $M_{n+1} - I(C_i)$ contains a path L_{i+1} from some $u_{i+1} \in V(X')$ to some $v_{i+1} \in V(Z)$. (For otherwise, $M_{n+1} - I(C_i)$ has two infinite components Z^* and X^* containing $Z - I(C_i)$ and $X' - I(C_i)$, respectively. Since $W' \subset Z$, $G - I(C_i)$ has two infinite components: X^*, and the other containing $Z^* \cup Y$. This contradicts the 2-indivisibility of G.) We can choose L_{i+1} such that $V(L_{i+1} \cap X') = \{u_{i+1}\}$ and $V(L_{i+1} \cap Z) = \{v_{i+1}\}$. Let



(a) (b)

(c)

Figure 5: The graph M and its blocks.

$C_{i+1} := x_n Z v_{i+1} \cup L_{i+1} \cup x_n X' u_{i+1}$. It is straightforward to verify that $N' = (C_1, C_2, \ldots)$ is a ladder net in M_{n+1} with $\partial N' = Z \cup X'$. This completes the proof of Claim 1.

If M_i is finite, let ∂M_i denote the subgraph of M_i consisting of all vertices and edges of M_i incident with its infinite face. If M_i is infinite, then $M_i = M_{n+1}$ and by Claim 1, we let $\partial M_{n+1} := \partial N'$. Let $\partial M = \bigcup_{i \geq 1} \partial M_i$.

Claim 2. M contains a 1-way infinite ∂M-Tutte path P_M from x.

First, assume that all blocks of M are finite. Then M has infinitely many blocks. If $V(M_i) = \{x_{i-1}, x_i\}$, then let $P_i := M_i$. If $V(M_i) \neq \{x_{i-1}, x_i\}$, then we apply (3.1) to find a ∂M_i-Tutte path P_i in M_i from x_{i-1} to x_i. Clearly, $P_M := \bigcup_{i=1}^{\infty} P_i$ is a 1-way infinite ∂M-Tutte path from x in M.

Now assume that M has exactly one infinite block. Then M has finitely many blocks $M_1, M_2, \ldots, M_n, M_{n+1}$, where M_{n+1} is the infinite block. By Claim 1 and by (3.6), M_{n+1} contains a 1-way infinite $\partial N'$-Tutte path P_{n+1} from x_n. For each $i \leq n$, let $P_i := M_i$ if $V(M_i) = \{x_{i-1}, x_i\}$, and otherwise, we apply (3.1) to find a ∂M_i-Tutte path P_i in M_i from x_{i-1} to x_i. Then $P_M := \bigcup_{i=1}^{n+1} P_i$ is a 1-way infinite ∂M-Tutte path from x in M. This completes the proof of Claim 2.

We complete our proof by proving the following.

Claim 3. There is a 1-way infinite path Q in $G - V(P_M - x)$ from x and through e such that $P := P_M \cup Q$ is a 2-way infinite ∂N-Tutte path through e in G.

Let W be the set of attachments on M of $(Y \cup M)$-bridges of G. For $w, w' \in W$, we say $w \sim w'$ if $w = w'$ or $\{w, w'\} \subset V(B) - V(P_M)$ for some P_M-bridge B of M

(such B contains an edge of ∂M, and hence, has just two attachments). Then $\sim$ is an equivalence relation. Let $W_1, W_2, \ldots$ be the equivalence classes of W with respect to $\sim$. Let $B_i := W_i$ if $W_i \subset V(P_M)$ (in this case, $|W_i| = 1$), and otherwise, let B_i denote the P_M-bridge of M containing W_i.

Let $s_i, t_i \in V(Y)$ such that there are $w_s, w_t \in W_i$ such that $\{s_i, w_s\}$ is contained in a $(Y \cup M)$-bridge of G and $\{t_i, w_i\}$ is contained in a $(Y \cup M)$-bridge of G, and subject to this, s_iYt_i is maximal. Without loss of generality, assume that $s_1, t_1, s_2, t_2, \ldots$ occur on Y in that order, where $s_1 = y$. See Figure 6(a).

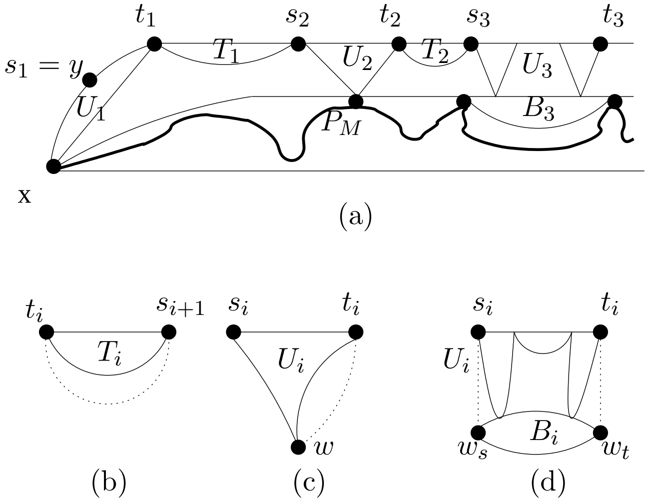


Figure 6: Graphs G, T_i, U_i.

For $i \geq 1$, let T_i be the union of t_iYs_{i+1} and those $(Y \cup M)$-bridges of G whose attachments are all contained in $V(t_iYs_{i+1})$. Note that $T_i \cap \partial N = t_iYs_{i+1}$.

(1) We claim that T_i contains a t_iYs_{i+1}-Tutte path R_i from t_i to s_{i+1}.

If $|V(t_iYs_{i+1})| \leq 2$, then let $R_i := t_iYs_{i+1}$. If $|V(t_iYs_{i+1})| \geq 3$, then apply (3.1) to $T_i + t_is_{i+1}$ to find a t_iYs_{i+1}-Tutte path R_i from t_i to s_{i+1} and through an edge of t_iYs_{i+1}. See Figure 6(b). It is clear that R_i gives the desired path.

For $i \geq 1$, let U_i denote the union of s_iYt_i, B_i, and those $(Y \cup M)$-bridges of G whose attachments are all contained in $V(s_iYt_i) \cup W_i$. Then $U_i \cap \partial N = U_i \cap Y = s_iYt_i$ and $|V(U_i \cap P_M)| = |V(B_i \cap P_M)| \leq 2$.

(2) We claim that U_1 contains a $(U_1 \cap \partial N)$-Tutte path Q_1 from x to t_1 through e.

If $s_1 = t_1$, then let $Q_1 := x\partial Ny$. If $s_1 \neq t_1$, then in $U_1 + t_1x$ we apply (3.1) to find a $(U_1 \cap \partial N)$-Tutte path Q_1 from x to t_1 and through e. See Figure 6(c) (with $w = x$ and $i = 1$).

(3) We claim that for each $i \geq 2$, $U_i - V(P_M)$ contains a path Q_i from s_i to t_i such that $Q_i \cup (U_i \cap P_M)$ is a $(U_i \cap \partial N)$-Tutte subgraph of U_i.

If $s_i = t_i$ then let $Q_i := s_i Y t_i$. Now assume $s_i \neq t_i$. First assume that $W_i \subset V(P_M)$. Then $|W_i| = 1$, and let w be the only vertex in W_i. See Figure 6(c). We apply (3.1) to $U_i + t_i w$ to find a $(U_i \cap \partial N)$-Tutte path Q_i' from s_i to w and through $t_i w$; and let $Q_i := Q_i' - w$. It is easy to check that Q_i is the desired path. Now assume that $W_i \not\subset V(P_M)$. Then $|V(B_i \cap P_M)| = 2$, and let w_s, w_t be the vertices of $B_i \cap P_M$ such that w_s and s_i are incident with a common face of G and w_t and t_i are incident with a common face of G. See Figure 6(d). In $U_i + \{s_i w_s, t_i w_t\}$, we apply (3.2) to find a $(U_i \cap Y)$-Tutte path Q_i' from w_s to w_t and through $s_i w_s$ and $t_i w_t$; and let $Q_i := Q_i' - \{w_s, w_t\}$. It is easy to check that Q_i is as desired.

Let $Q := \bigcup_{i \geq 1}(Q_i \cup R_i)$. Then Q is a 1-way infinite path from x and through e, and $V(P_M \cap Q) = \{x\}$. Let $P := P_M \cup Q$. Then every P-bridge of G is one of the following: a P_M-bridge of M not in any U_i, or a Q_1-bridge of U_1, or a $(Q_i \cup (U_i \cap P_M))$-bridge of U_i for some $i \geq 2$, or an R_i-bridge of T_i for some $i \geq 1$. Note that the P_M-bridges of M containing an edge of X are P-bridges of G with no attachment on Q. Hence, it is easy to see that P is a 2-way infinite ∂N-Tutte path in G through e. □

Proof of (1.1). Let G be a 4-connected 2-indivisible infinite plane graph. Hence, G is cohesive and by (2.1), we may assume that G is nicely embedded. To show that G has a spanning 2-way infinite path, it suffices to show that G contains a 2-way infinite Tutte path. Let C be a facial cycle of G. Let S denote the set of vertices of G with infinite degree. By (2.4), $|S| \leq 2$, and there exists a set F of edges of G such that

(1) for any $f \in F$, f is incident with a vertex in S,

(2) $G - F$ has a net $N = (C_1, C_2, \ldots)$, $C \subset I(C_1)$, $S \subset \partial N$, and for any $f \in F$ both incident vertices of f are contained in a common infinite S-bridge of ∂N,

(3) if $|S| = 1$, then either one S-bridge of ∂N contains all vertices incident with edges in F or each S-bridge of ∂N contains infinitely many vertices incident with edges in F, and

(4) if $|S| = 2$, then, for any $T \subset V(G) - S$ with $|T| \leq 3$, S is contained in a component of $(G - F) - T$.

If N is a radial net, then let $e \in E(C)$. By the main result in [9], G contains a 2-way infinite C-Tutte path P through e. Since G is 4-connected, P is a spanning 2-way infinite path in G.

So we may assume that N is a ladder net. If $S = \emptyset$, then let $e \in E(\partial N)$. If $|S| = 1$, then let $e \in E(\partial N)$ be incident with the vertex in S. If $|S| = 2$, then let e be an edge on the subpath of ∂N between the vertices in S such that e is incident with a vertex in S. By (4.1), $G - F$ contains a 2-way infinite ∂N-Tutte path P through e.

We claim that $S \subset V(P)$. Otherwise, let $s \in S - V(P)$. Then $|S| = 2$, and s is contained in a P-bridge B of $G - F$ with $|V(B \cap P)| = 2$. Since e is on the subpath of

∂N between the vertices in S, $S-\{s\} \not\subset V(B\cap P)$. Hence the vertices in S are contained in different components of $(G-F)-V(B\cap P)$, contradicting (4).

Next we show that P is a spanning 2-way infinite path in G. Suppose for a contradiction that P is not spanning. Then there is a vertex $x \in V(G)-V(P)$. Let B be the P-bridge of G containing x. Then B is one of the following: (i) a P-bridge of $G-F$, or (ii) a subgraph of G induced by an edge in F, or (iii) a subgraph of G obtained from a P-bridge B' of $G-F$ by adding edges in F from $S\cap V(B')$ to $V(B'-P)$, or (iv) a subgraph of G obtained from a P-bridge B' of $G-F$ by adding a vertex $s \in S-V(B')$ and edges from $(S\cap V(B'))\cup\{s\}$ to $V(B'-P)$. If any of (i) - (iii) occurs, then clearly, B has at most three attachments, a contradiction (since G is 4-connected). So assume that (iv) occurs. Then $B'-P$ contains an edge of ∂N, and hence, B' has just two attachments on P. Since $B'-P$ contains neighbors of at most one vertex in $S-V(B)$, B has three attachments, again, a contradiction. $\square$

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