Independent paths and $K_5$-subdivisions

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Abstract

A well known theorem of Kuratowski states that a graph is planar iff it contains no subdivision of $K_5$ or $K_{3,3}$. Seymour conjectured in 1977 that every 5-connected nonplanar graph contains a subdivision of $K_5$. In this paper, we prove several results about independent paths (no vertex of a path is internal to another), which are then used to prove Seymour’s conjecture for two classes of graphs. These results will be used in a subsequent paper to prove Seymour’s conjecture for graphs containing $K_{4}^-$, which is a step in a program to approach Seymour’s conjecture.

AMS Subject Classification: 05C38, 05C40, 05C75
Keywords: Subdivision of graph, independent paths, nonseparating path, planar graph

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1 Introduction

Only finite simple graphs are considered. We follow Diestel [5] for notation and terminology not explicitly defined. In particular, for a graph $K$ we use $TK$ to denote a subdivision of $K$. Thus, the well known Kuratowski’s theorem can be stated as follows: A graph is planar iff it contains no $TK_5$ or $TK_{3,3}$. It is known that any 3-connected nonplanar graph other than $K_5$ contains a $TK_{3,3}$. Seymour [15] conjectured in 1977 that every 5-connected nonplanar graph contains a $TK_5$, which was also posed by Kelmans [10] in 1979.

For convenience, the vertices with degree 4 in a $TK_5$ are called branch vertices. Suppose $G$ is a 5-connected graph and an edge $xy$ of $G$ is contained in three triangles, say $xyv_1x$, $xyv_2x$ and $xyv_3x$. Then $G - \{x, y\}$ is 3-connected, and hence contains a cycle $C$ such that $\{v_1, v_2, v_3\} \subseteq C$. Clearly, $C$ and these three triangles form a $TK_5$ in $G$ with branch vertices $x, y, v_1, v_2, v_3$.

A graph has an edge in two triangles iff it contains $K_4^-$, the graph obtained from $K_4$ by deleting an edge. As a first step in a program to approach Seymour’s conjecture, we wish to avoid $K_4^-$, i.e., to prove it for graphs containing a $K_4^-$. Note that $K_4^-$-free graphs have nice structural properties; for example, it is shown in [7] that if $G$ is 5-connected and $K_4^-$-free then $G$ contains a contractible edge (see [8] for more results).

It turns out to be quite difficult to find a $TK_5$ in a 5-connected nonplanar graph containing $K_4^-$. We will see in a subsequent paper that given a $K_4^-$ in a 5-connected nonplanar graph, we may be forced to find a $TK_5$ in which no vertex of this $K_4^-$ is a branch vertex.

The paths $P_1, \ldots, P_k$ are said to be independent if for any $1 \leq i \neq j \leq k$ no vertex of $P_i$ is an internal vertex of $P_j$. In this paper we prove several results on independent paths, which will be used to prove Seymour’s conjecture for two classes of graphs. All these results will be used in a subsequent paper to prove Seymour’s conjecture for graphs containing $K_4^-$. We use $\emptyset$ to denote both the empty set and the empty graph. Let $G$ be a graph; then $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. By $H \subseteq G$, we mean that $H$ is a subgraph of $G$. For $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$. For $X \subseteq V(G) \cup E(G)$ or $X \subseteq G$, $G - X$ denotes the graph obtained from $G$ by deleting the vertices in $X$ and those edges in $G$ incident with vertices in $X$. If $x \in V(G) \cup E(G)$, we write $G - x$ instead of $G - \{x\}$.

We can now state our first result.

Theorem 1.1 Let $G$ be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2$ be distinct vertices of $G$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Suppose there is an induced path $P$ in $G - x_1x_2$ from $x_1$ to $x_2$ such that $G - V(P)$ is 2-connected and $\{y_1, y_2\} \cap V(P) = \emptyset$. Then $G$ contains a $TK_5$ in which $x_1, x_2, y_1, y_2$ are branch vertices.

For subgraphs $G$ and $H$ of a graph, $G \cup H$ and $G \cap H$ denote the union and intersection of $G$ and $H$, respectively. We say that $G$ and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. We use $G - H$ instead of $G - V(G \cap H)$. A separation of a graph $G$ is a pair $(G_1, G_2)$ of subgraphs of $G$ such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$, and $E(G_i) \cup V(G_i - G_{3-i}) \neq \emptyset$ for $i \in \{1, 2\}$. If $|V(G_1 \cap G_2)| = k$, then $(G_1, G_2)$ is a $k$-separation.

The following result says that whenever a 5-connected nonplanar graph has a 5-separation and one side of the 5-separation is planar and nontrivial then it contains a $TK_5$. By an edge crossing we mean an intersection of two edges in a drawing of a graph in the plane (vertices
are represented by points and edges by polygonal arcs). A drawing of a graph in the plane without edge crossings is also said to be a planar representation of that graph.

**Theorem 1.2** Let $G$ be a 5-connected nonplanar graph and let $(G_1, G_2)$ be a 5-separation in $G$. Suppose $|G_2| \geq 7$ and $G_2$ has a planar representation in which the vertices in $V(G_1 \cap G_2)$ are incident with a common face. Then $G$ contains a $TK_5$.

Another step in our program is to prove that if $G$ is a 5-connected nonplanar graph with a 5-separation $(G_1, G_2)$ such that $|G_i| \geq 2$ for $i = 1, 2$ then $G$ admits a $TK_5$. This was also suggested by Kawarabayashi.

One of the key ideas in our proof is to find, in a 5-connected graph, an induced path with given ends whose removal results in a graph that is at least 2-connected. This is related to the conjecture of Lovász [13] that there is a minimum integer $c(k) > 0$ such that for any integer $k \geq 1$ and any two vertices $u$ and $v$ in a $(k)$-connected graph $G$, there is a path $P$ from $u$ to $v$ in $G$ such that $G - V(P)$ is $k$-connected. A result of Tutte [19] implies $c(1) = 3$. That $c(2) = 5$ follows from results of Chen, Gould and Yu [3] and Kriesell [12], which are further extended in [4,9].

Let $x_1, x_2, y_1, y_2$ be the vertices of a $K_4^-$ in a 5-connected nonplanar graph $G$ such that $y_1y_2 \notin E(G)$. We show in Section 2 that there is an induced path $P$ in $G - \{x_1x_2, x_1y_1, x_2y_1, x_2y_2\}$ between $x_1$ and $x_2$ such that $\{y_1, y_2\} \not\subseteq V(P)$ and $G - V(P)$ is 2-connected. We then prove Theorem 1.1 in Section 3 (the case when $\{y_1, y_2\} \cap V(P) = \emptyset$), using a result of Watkins and Mesner [20] on cycles through three given vertices. (The remaining case when $|\{y_1, y_2\} \cap V(P)| = 1$ is more difficult, and will be proved in another paper with the help of Theorem 1.2.) In Section 4, we prove Theorem 1.2.

We mention several results and problems related to Seymour’s conjecture. Mader [14] proved that if $G$ is a simple graph with $n \geq 3$ vertices and at least $3n - 5$ edges then $G$ contains a $TK_5$, establishing a conjecture of Dirac [6]. Kézdy and McGuinness [11] showed that Seymour’s conjecture if true would imply Mader’s result. Seymour’s conjecture is also related to a conjecture of Hajós (see [1]) that every graph containing no $TK_{k+1}$ is $k$-colorable. Hajós’ conjecture is false for $k \geq 6$ [1] and true for $k = 1, 2, 3$, and remains open for the case $k = 4$ and $k = 5$.

We conclude this section with additional notation and terminology. Let $G$ be a graph. If there is no confusion, we may write $S \subseteq G$ instead of $S \subseteq V(G)$ or $S \subseteq E(G)$, and write $x \in G$ instead of $x \in V(G)$ or $x \in E(G)$. Let $Y \subseteq G$; then $N_G(Y)$, or $N(Y)$ if $G$ is understood, denotes the set of vertices in $V(G) - V(Y)$ adjacent to vertices in $V(Y)$. If $Y = \{y\} \subseteq V(G)$, then we use $N_G(y)$ or $N(y)$ instead of $N_G(\{y\})$ or $N(\{y\})$. Let $T$ be a set of 2-element subsets of $V(G)$; then $G + T$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup T$. If $T = \{\{x, y\}\}$, we write $G + xy$ instead of $G + \{\{x, y\}\}$.

Given a path $P$ in a graph and $x, y \in V(P)$, $xPy$ denotes the subpath of $P$ between $x$ and $y$ (inclusive). The ends of the path $P$ are the vertices of the minimum degree in $P$, and the other vertices of $P$ are its internal vertices. A path $P$ with ends $u$ and $v$ is also said to be from $u$ to $v$ or between $u$ and $v$. Let $H_1$ and $H_2$ be subgraphs of $G$; a path $P$ in $G$ is an $H_1$-$H_2$ path if $P$ has one end in $H_1$ and another in $H_2$, and is otherwise disjoint from $H_1 \cup H_2$. A path $P$ from $x$ to $y$ in a graph $G$ is said to be internally disjoint from $H \subseteq G$ if $P \cap H \subseteq \{x, y\}$.

Let $G$ be a graph. A set $S \subseteq V(G)$ is a $k$-cut or a cut of size $k$ in $G$, where $k$ is a positive integer, if $|S| = k$ and $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = S$ and
$V(G_i - S) \neq \emptyset$ for $i \in \{1, 2\}$. If $v \in V(G)$ and $\{v\}$ is a cut of $G$, then $v$ is said to be a cut vertex of $G$.

For a subgraph $H$ of a graph $G$, an $H$-bridge of $G$ is a subgraph of $G$, say $B$, for which there exists a component $D$ of $G - V(H)$ such that $B$ is induced by the edges which are either contained in $D$ or from $D$ to $H$. The vertices in $H$ that are neighbors of $D$ are called the attachments of this $H$-bridge. For $S \subseteq V(G)$, the $G[S]$-bridges of $G$ are also called $S$-bridges.

2 Nonseparating paths

In this section we prove three lemmas, two on nonseparating paths and one on independent paths. A nonseparating path in a graph $G$ is a path $P$ such that $G - V(P)$ is connected. We need the following concept of connectivity.

**Definition 2.1** Let $G$ be a graph and $S \subseteq V(G)$, and let $k$ be a positive integer. We say that $G$ is $(k, S)$-connected if, for any cut $T$ of $G$ with $|T| < k$, every component of $G - T$ contains a vertex from $S$.

We also need a result of Seymour [16]; equivalent formulations can be found in [2, 17, 18].

**Theorem 2.2** (Seymour) Let $G$ be a graph and let $s_1, s_2, t_1, t_2$ be distinct vertices of $G$. Then either $G$ contains disjoint paths from $s_1$ to $s_2$ and from $t_1$ to $t_2$, or there exist pairwise disjoint sets $A_i \subseteq V(G)$ ($k \geq 0$ and $1 \leq i \leq k$), such that

(a) for $i \neq j$, $N(A_i) \cap A_j = \emptyset$,

(b) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and

(c) the graph, obtained from $G$ by (for each $i$) deleting $A_i$ and adding new edges joining every pair of distinct vertices in $N(A_i)$, can be drawn in a closed disc with no edge crossings such that $s_1, t_1, s_2, t_2$ occur on the boundary of the disc in cyclic order.

As a consequence, if $G$ is $(4, \{s_1, s_2, t_1, t_2\})$-connected, then either $G$ has disjoint paths from $s_1$ to $s_2$ and from $t_1$ to $t_2$, or $G$ can be drawn in a closed disc in the plane with no edge crossings such that $s_1, t_1, s_2, t_2$ occur on the boundary in cyclic order.

Let $G$ be a graph; a chain of blocks in $G$ is a sequence $B_1B_2\ldots B_k$ such that each $B_i$ is a block of $G$, $B_i \cap B_j = \emptyset$ when $|i - j| \geq 2$, and $|V(B_i \cap B_{i+1})| = 1$ for $1 \leq i \leq k - 1$. If $k = 1$ and $x, y \in V(B_1)$, or if $k \geq 2$ and $x \in V(B_1 - B_2)$ and $y \in V(B_k - B_{k-1})$, then $B_1B_2\ldots B_k$ is said to be a chain of blocks from $x$ to $y$ (or from $x$, or from $y$).

The lemma below allows one to modify an existing path to a good nonseparating path.

**Lemma 2.3** Let $G$ be a graph and let $x_1, x_2, y_1, y_2$ be distinct vertices of $G$ such that $G$ is $(5, \{x_1, x_2, y_1, y_2\})$-connected. Suppose $X$ is an induced path in $G$ from $x_1$ to $x_2$, and $H$ is a chain of blocks in $G - V(X)$ from $y_1$ to $y_2$. Then precisely one of the following holds:

(i) $H = y_1y_2$ and $G - y_1y_2$ can be drawn in a closed disc in the plane without edge crossings such that $x_1, y_1, x_2, y_2$ occur on the boundary of the disc in this cyclic order.
(ii) There is an induced path $X'$ from $x_1$ to $x_2$ such that $H \subseteq G - V(X')$, and $G - V(X')$ is a chain of blocks from $y_1$ to $y_2$.

Proof. First, we may assume that if $y_1y_2 \in E(G)$ then $H \neq y_1y_2$; in particular, $|V(H)| \geq 3$. For, suppose $y_1y_2 \in E(G)$ and $H = y_1y_2$. If $G - y_1y_2$ contains disjoint paths $X', Y$ from $x_1, y_1$ to $x_2, y_2$, respectively, then we see that in $G - X'$, $\{y_1, y_2\}$ is contained in a block $H'$ which contains the cycle $H \cup Y$; so we may replace $X, H$ by $X', H'$, respectively. On the other hand, (i) follows from Lemma 2.2 and the assumption that $G$ is $\{x_1, x_2, y_1, y_2\}$-connected.

We now choose such $X$ and $H$ that

(1) $H$ is maximal (under subgraph containment), and

(2) subject to (1), the number of components of $G - V(X)$ is minimum.

Next, we show that $G - V(X)$ is connected. For, suppose there is a component of $G - V(X)$ disjoint from $H$, and let $D$ be such a component. Let $v_1, v_2$ denote the neighbors of $D$ on $X$ with $v_1Xv_2$ maximal. ($D$ has at least 5 neighbors on $X$; so $v_1v_2 \notin E(G)$.) Since $G$ is $\{x_1, x_2, y_1, y_2\}$-connected, $v_1Xv_2 - \{v_1, v_2\}$ contains a neighbor of some component of $G - V(X)$ other than $D$, say $C$. Now let $X'$ be obtained from $X$ by deleting $v_1Xv_2 - \{v_1, v_2\}$ and adding an induced path in $G[V(D) \cup \{v_1, v_2\}]$ from $v_1$ to $v_2$. Let $D'$ denote the union of those components of $D - X'$ with no neighbor in $v_1Xv_2 - \{v_1, v_2\}$. (Possible $D' = \emptyset$.) We choose $X'$ so that

(3) $D'$ is minimal.

If $D' = \emptyset$ then $(D - X') \cup C \cup (v_1Xv_2 - \{v_1, v_2\})$ is contained in a component of $G - X'$, and the number of components of $G - V(X')$ is smaller than $G - V(X)$, contradicting to (2) (since $H$ will not get smaller). So we may assume $D' \neq \emptyset$. Let $D_1, \ldots, D_k$ be the components of $D'$. Let $a_i, b_i$ ($1 \leq i \leq k$) denote the neighbors of $D_i$ in $v_1X'v_2$ with $a_iX'b_i$ maximal. Since $G$ is $\{x_1, x_2, y_1, y_2\}$-connected, $\{a_i, b_i, v_1, v_2\}$ in not a cut in $G$, so there exists $c_i \in V(a_iX'b_i) - \{a_i, b_i\}$ such that $c_i$ has a neighbor in $D - (X' \cup D_i)$ or in $v_1Xv_2 - \{v_1, v_2\}$. If $c_i$ has a neighbor that belongs to $v_1Xv_2 - \{v_1, v_2\}$, or that is not in $D'$ but is contained in a component of $D - X'$, then let $X''$ be obtained from $X'$ by deleting $a_iX'b_i - \{a_i, b_i\}$ and adding an induced path between $a_i$ and $b_i$ in $G[V(D_i) \cup \{a_i, b_i\}]$; it is easy to see that $X''$ contradicts the choice of $X'$ in (3). Thus, for any $1 \leq i \leq k$, $N(a_iX'b_i - \{a_i, b_i\}) \subseteq X' \cup D'$. Therefore, $\bigcup_{i=1}^k a_iX'b_i$ is a subpath of $v_1X'v_2$; let $a, b$ denote its ends. Now $\{a, b, v_1, v_2\}$ is not a cut in $G$, so there exists $c \in V(aX'b) - \{a, b\}$ such that $c$ has a neighbor in $v_1Xv_2 - \{v_1, v_2\}$, or in a component of $D - X'$ that is not a component of $D'$. Then there exists some $1 \leq i \leq k$ such that $c \in a_iX'b_i - \{a_i, b_i\}$, which is a contradiction since we have shown that $N(a_iX'b_i - \{a_i, b_i\}) \subseteq X' \cup D'$.

Having shown that $G - V(X)$ is connected, we may now assume that $G - V(X) \neq H$; as otherwise $X' := X$ is the desired path for (ii). Let $D$ be an arbitrary $H$-bridge of $G - V(X)$ with $V(D) \cap V(H) = \{v\}$. Let $v_1, v_2$ denote the neighbors of $D - v$ on $X$ with $v_1Xv_2$ maximal.

Suppose there are independent paths $Q, R$ in $G$ from $v_1Xv_2 - \{v_1, v_2\}$ to distinct vertices of $H$ which are also internally disjoint from $D \cup X \cup H$. Then let $X'$ be obtained from $X$ by deleting $v_1Xv_2 - \{v_1, v_2\}$ and adding an induced path in $G[V(D - v) \cup \{v_1, v_2\}]$ from $v_1$ to $v_2$. Clearly, in $G - V(X')$ the chain of blocks from $y_1$ to $y_2$ contains $H \cup Q \cup R$, contradicting (1).
So all paths from \( v_1Xv_2 - \{v_1, v_2\} \) to \( H \) internally disjoint from \( D \cup X \cup H \) must end at the same vertex, say \( u \), in \( H \). Moreover, at least one such path has length at least 2; for otherwise, because \( |V(H)| \geq 3 \), \( \{v, u, v_1, v_2\} \) would be a 4-cut in \( G \) (contradicting the assumption that \( G \) is \((5, \{x_1, x_2, y_1, y_2\}\))-connected). Hence there exists some \( H \)-bridge \( C \) of \( G - V(X) \) such that \( V(C \cap H) = \{u\} \) and \( C - u \) contains a neighbor of \( v_1Xv_2 - \{v_1, v_2\} \). Let \( u_1, u_2 \) denote the neighbors of \( C - u \) on \( X \) with \( u_1Xu_2 \) maximal.

Suppose \( v_1Xv_2 \subseteq u_1Xu_2 \). Then since \( G \) is \((5, \{x_1, x_2, y_1, y_2\}\))-connected, \( \{u, v, u_1, u_2\} \) is not a cut in \( G \). Hence, since \( |V(H)| \geq 3 \), there is a path \( R \) in \( G \) from \( u_1Xu_2 - \{u_1, u_2\} \) to \( H - \{u, v\} \) internally disjoint from \( C \cup D \cup X \cup H \). Let \( X' \) be obtained from \( X \) by deleting \( u_1Xu_2 - \{u_1, u_2\} \) and adding an induced path in \( G[V(C - u) \cup \{u_1, u_2\}] \) from \( u_1 \) to \( u_2 \). Clearly, in \( G - V(X') \), the chain of blocks from \( y_1 \) to \( y_2 \) contains \( H \cup R \) and part of \( D \cup u_1Xu_2 \), contradicting (1).

If \( u_1Xu_2 \subseteq v_1Xv_2 \), then the same argument above (by simply exchanging the roles of \( C, u, u_1, u_2 \) with \( D, v, v_1, v_2 \), respectively) gives a contradiction to (1).

So neither \( v_1Xv_2 \) nor \( u_1Xu_2 \) is contained in the other. By symmetry we may assume that \( x_1, u_1, v_1, u_2, v_2, x_2 \) occur on \( X \) in this order. Since \( G \) is \((5, \{x_1, x_2, y_1, y_2\}\))-connected, \( \{u, v, u_1, v_1\} \) is not a cut in \( G \). Hence, since \( |V(H)| \geq 3 \), there is a path \( R \) in \( G \) from \( r \in V(u_1Xv_2) - \{u_1, v_2\} \) to \( H - \{u, v\} \) internally disjoint from \( C \cup D \cup X \cup H \). Note that \( r \notin v_1Xv_2 - \{v_1, v_2\} \), and so \( r \in u_1Xu_2 - \{u_1, u_2\} \). Let \( X' \) be obtained from \( X \) by deleting \( u_1Xu_2 - \{u_1, u_2\} \) and adding an induced path in \( G[V(C - u) \cup \{u_1, u_2\}] \) from \( u_1 \) to \( u_2 \). In \( G - V(X') \), the chain of block from \( y_1 \) to \( y_2 \) contains \( H \cup R \) and part of \( D \cup u_1Xu_2 \), contradicting (1).

We now prove that in a 5-connected nonplanar graph containing \( K_4^- \), one can find a TK5 or a good nonseparating path.

**Lemma 2.4** Let \( G \) be a 5-connected nonplanar graph and \( x_1, x_2, y_1, y_2 \) distinct vertices of \( G \) such that \( G[\{x_1, x_2, y_1, y_2\}] \cong K_4^- \) and \( y_1y_2 \notin E(G) \). Then one of the following holds:

(i) \( G \) has a TK5 in which \( x_1, x_2, y_1, y_2 \) are branch vertices.

(ii) There is an induced path \( X \) in \( G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\} \) from \( x_1 \) to \( x_2 \) such that \( G - V(X) \) is 2-connected, and \( \{y_1, y_2\} \not\subseteq V(X) \).

**Proof.** For convenience, let \( H := G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\} \). Then \( H \) is \((5, \{x_1, x_2, y_1, y_2\}\))-connected. Note that \( H \) has disjoint paths from \( x_1, y_1 \) to \( x_2, y_2 \), respectively; for otherwise it would follow from Lemma 2.2 that \( G \) is planar. Hence by Lemma 2.3, there exists an induced path \( X \) from \( x_1 \) to \( x_2 \) in \( H \) such that \( H - V(X) \) is a chain of blocks. For \( i = 1, 2 \), let \( B_i \) denote the block of \( H - V(X) \) containing \( y_i \).

We claim that for some \( i \), say \( i = 1 \), \( B_1 \) is 2-connected. For, assume that neither \( B_1 \) nor \( B_2 \) is 2-connected, and let \( y_i' \) \((i = 1, 2)\) denote the unique neighbor of \( y_i \) in \( B \). Since \( G \) is 5-connected, each \( y_i' \) is adjacent to distinct \( u_i, v_i \in V(X) - \{x_1, x_2\} \), and each \( x_i \) must have a neighbor \( x_i' \in V(B) - \{y_1, y_2\} \). Let \( X' \) denote the path obtained from a path in \( H - \{y_1, y_2\} \) from \( x_1' \) to \( x_2' \) by adding \( x_1, x_2, x_1x_1' \) and \( x_2x_2' \). Clearly, \( \{y_1, y_2\} \) and \( X - \{x_1, x_2\} \) are contained in a chain of blocks in \( G - V(X') \) from \( y_1 \) to \( y_2 \), and both \( y_1 \) and \( y_2 \) are contained in a 2-connected block of \( G - V(X') \). So our claim follows from another application of Lemma 2.3.
We choose \( X \) so that \( H - V(X) \) is a chain of blocks from \( y_1 \) to \( y_2 \), the block of \( H \) containing \( y_1 \) is 2-connected, and subject to this, \( B_1 \) is maximal.

Let \( a \) be the cutvertex of \( H \) contained in \( B_1 \). Note that \( A := H - V(B_1 - a) \) is a chain of blocks from \( a \) to \( y_2 \), and let \( A_1, \ldots, A_m \) denote the blocks of \( B - V(B_1 - a) \) such that \( A_i \cap A_j = \emptyset \) when \( |i - j| \geq 2 \), \( V(A_i \cap A_{i+1}) = a_i \), and \( A_m = B_2 \). Let \( v_1, v_2 \) be the neighbors of \( A - a \) on \( X \) with \( v_1Xv_2 \) maximal, and we may assume that \( x_1, v_1, v_2 \) occur on \( X \) in order.

Suppose \( V(A) = \{a, y_2\} \) and \( v_1v_2 \in X \). Since \( G \) is 5-connected, \( \{v_1, v_2\} \neq \{x_1, x_2\} \). By symmetry, let \( v_1 \neq x_1 \). Then again since \( G \) is 5-connected, \( v_1 \) has a neighbor \( v \in B_1 - a \). Let \( P_a, P_v \) denote independent paths in \( B_1 \) from \( y_1 \) to \( a, v_1 \), respectively. Then \( P_a \cup \{x_2, v_1, ay_2, y_2v_1\} \cup X \cup G(\{x_1, x_2, y_1, y_2\}) \) form a TK5 with branch vertices \( x_1, x_2, y_1, y_2, v_1 \).

So \( V(A) \neq \{a, y_2\} \) or \( v_1v_2 \in E(X) \). Since \( G \) is 5-connected, \( \{a, y_2, v_1, v_2\} \) is not a cut in \( G \). Therefore, \( v_1Xv_2 - \{v_1, v_2\} \) must have neighbors in \( B_1 - a \).

Indeed, \( v_1Xv_2 - \{v_1, v_2\} \) has exactly one neighbor in \( B_1 - a \). For, assume that \( v_1Xv_2 - \{v_1, v_2\} \) has two distinct neighbors in \( B_1 - a \), say \( b_1, b_2 \). Let \( X' \) be obtained from \( X \) by deleting \( v_1Xv_2 - \{v_1, v_2\} \) and adding a path in \( G[V(A - a) \cup \{v_1, v_2\}] \) from \( v_1 \) to \( v_2 \). Then we see that the block of \( G - V(X') \) containing \( y_1 \) contains \( B_1 \) properly, a contradiction.

So let \( b \) be the unique neighbor of \( v_1Xv_2 - \{v_1, v_2\} \) in \( B_1 - a \). Then \( A' := G[V(A \cup v_1Xv_2) \cup \{b\}] \) has a plane embedding in which \( v_1, a, v_2, b \) occur on a facial boundary in cyclic order. For, otherwise, by Lemma 2.2 and \( A' \) is \( (5, \{v_1, v_2, a, b, y_2\}) \)-connected, \( A' \) contains disjoint paths \( X_1, Y_1 \) from \( v_1, a \) to \( v_2, b \), respectively. Let \( X' \) be obtained from \( X \) by deleting \( v_1Xv_2 - \{v_1, v_2\} \) and adding \( X_1 \). Now \( B_1 \cup Y_1 \) is contained in a block of \( G - V(X') \). So we derive a contradiction to the maximality of \( B_1 \) by another application of Lemma 2.3.

Since \( B_1 \) is 2-connected, \( B_1 \) has independent paths \( P_a, P_b \) from \( y_1 \) to \( a, b \), respectively. Since \( \{v_1, v_2, y_2, a\} \) is not a cut of \( G \), \( b \) must have a neighbor in \( v_1Xv_2 - \{v_1, v_2\} \), say \( z \). Since \( G \) is 5-connected and \( X \) is induced in \( H \), \( z \) has a neighbor in \( A - a \). If \( G[V(A + z)] \) has a path \( P_z \) from \( z \) to \( a \) through \( y_2 \), then \( P_a \cup (P_b + P_z + b) \cup X \cup G(\{x_1, x_2, y_1, y_2, z\}) \) form a TK5 with branch vertices \( x_1, x_2, y_1, y_2, z \). So assume \( P_z \) does not exist. Then \( G[A + z] \) has a cut vertex \( w \) separating \( v_2 \) from \( \{a, z\} \). Let \( Y \) denote the \( w \)-bridge of \( G[A + z] \) containing \( y_2 \). By the planarity of \( A' \), all neighbors of \( Y - w \) are contained in \( v_1Xv_2 + \{w, x_1, x_2\} \), for some \( i \in \{1, 2\} \). Since \( G \) is connected, both \( x_1 \) and \( x_2 \) have a neighbor in \( Y - w \). So \( G[Y - w + \{x_1, x_2\}] \) contains a path \( X' \) from \( x_1 \) to \( x_2 \) such that in \( G - X' \), \( B_1 + y_2 \) are contained in a block, a contradiction.

From Lemma 2.4 we see that in order to prove Seymour’s conjecture for graphs with \( K_4^- \), it suffices to prove Theorem 1.1 (when \( \{y_1, y_2\} \cap V(X) = \emptyset \)) and deal with the case when \( |\{y_1, y_2\} \cap V(X)| = 1 \). (The later will be done in another paper.) Before we prove Theorem 1.1, we need a lemma about independent paths.

**Lemma 2.5** Let \( G \) be a graph and \( S \subseteq V(G) \) such that \( |S| \geq 4 \) and \( G \) is \( (4, S) \)-connected. Assume that there exist \( a_1, a_2 \in S \), \( a \in V(G) - S \), and two independent paths in \( G - (S - \{a_1, a_2\}) \) from \( a \) to \( a_1, a_2 \) respectively. Then there exist four independent paths in \( G \) from \( a \) to distinct vertices in \( S \), one from \( a \) to \( a_1 \) and another from \( a \) to \( a_2 \).

**Proof.** Since \( G \) is \( (4, S) \)-connected, \( |S| \geq 4 \); and it follows from Menger’s theorem that there exist four independent paths \( P_i \), \( i = 1, 2, 3, 4 \), in \( G \) from \( a \) to \( b_1 \in S \), respectively, and internally
disjoint from $S$. For convenience, let $P := \bigcup_{i=1}^{4} P_i$. We choose $P_1, P_2, P_3, P_4$ so that $\ell := |\{a_1, a_2\} \cap \{b_1, b_2, b_3, b_4\}|$ is maximum.

Note that $0 \leq \ell \leq 2$. If $\ell = 2$ then $P_1, P_2, P_3, P_4$ are the desired paths. So we may assume $\ell = 0$ or $\ell = 1$. By assumption, let $Q_i$ ($i = 1, 2$) be independent paths in $G - (S - \{a_1, a_2\})$ from $a$ to $a_1$, and let $x_i \in V(Q_i \cap P)$ such that $V(a_iQ_ix_i \cap P) - \{a_i\} = \{x_i\}$.

Suppose $\ell = 0$. Without loss of generality, we may assume that $x_2 \in P_1$. Then the paths $aP_1x_2 \cup x_2Q_2a_2, P_2, P_3, P_4$ contradict the choice of $P_1, P_2, P_3, P_4$ (the maximality of $\ell$).

So $\ell = 1$, and we may assume, without loss of generality, that $a_1 = b_1$ and $a_2 \notin \{b_1, b_2, b_3, b_4\}$.

We may assume $x_2 \in P_1$; otherwise, assume without loss of generality that $x_2 \in P_2$, and then $aP_1x_2 \cup x_2Q_2a_2, P_3, P_4$ are the desired paths for the lemma. We may also assume $x_1 \in P_1$; for, otherwise, assume (without loss of generality) that $x_1 \in P_2$, and then $aP_1x_1 \cup x_1Q_1a_1, aP_1x_2 \cup x_2Q_2a_2, P_3, P_4$ are the desired paths for the lemma.

Now suppose there exists $i \in \{1, 2\}$ such that $Q_i \cap (P_2 \cup P_3 \cup P_4) = \{a\}$. We only deal with $i = 1$; the case when $i = 2$ is symmetric. Suppose then that $Q_i \cap (P_2 \cup P_3 \cup P_4) = \{a\}$. Then we may assume $Q_3 \cap (P_2 \cup P_3 \cup P_4) \neq \{a\}$, since otherwise, $Q_1, Q_2, P_3$ are the desired paths for the lemma. So let $y_2 \in V(Q_2) \cap V(P_2 \cup P_3 \cup P_4)$ such that $y_2 \notin a$ and $V(a_2Q_2y_2) \cap V(P_2 \cup P_3 \cup P_4) = \{y_2, a\}$, and we may assume without loss of generality that $y_2 \in P_2$. Now $Q_1, aP_2y_2 \cup y_2Q_2a_2, P_3, P_4$ are the desired paths for the lemma.

Thus, we may assume that $Q_i \cap (P_2 \cup P_3 \cup P_4) \neq \{a\}$ for $i \in \{1, 2\}$. Let $y_i \in V(Q_i) \cap V(P_2 \cup P_3 \cup P_4)$ such that $y_i \notin a$ and $V(a_iQ_iy_i) \cap V(P_2 \cup P_3 \cup P_4) = \{y_i\}$. Note that $a_iQ_ix_i \subseteq a_iQ_iy_i$ and $x_i \neq y_i$. Let $x'_i \in V(x_iQ_iy_i \cap P_i)$ such that $x'_iP_ia$ is minimum. Note that $x'_1 \neq x'_2$. Suppose $x'_2 \in a_1P_1x'_1$. Without loss of generality assume $y_1 \in P_2$. Then $aP_2y_1 \cup y_1Q_1a_1, aP_1x'_2 \cup x'_2Q_2a_2, P_3, P_4$ are the desired paths. Now assume $x'_1 \in a_1P_1x'_2$, and $y_2 \in P_2$ (without loss of generality). Then $aP_1x'_1 \cup x'_1Q_1a_1, aP_2y_2 \cup y_2Q_2a_2, P_3, P_4$ are the desired paths.

\section{Proof of Theorem 1.1}

We need a result of Watkins and Mesner [20] that characterizes those graphs in which no cycle contains a set of three specified vertices. This result is also used in [22] in the reduction of Hajós’ conjecture to 4-connected graphs. See Figure 1 for an illustration.

\textbf{Theorem 3.1} (Watkins and Mesner) Let $R$ be a 2-connected graph and let $y_1, y_2, v$ be three distinct vertices of $R$. Then there is no cycle through $y_1, y_2$ and $v$ in $R$ if, and only if, one of the following statements holds.

(i) There exists a 2-cut $S$ in $R$ and, for $u \in \{y_1, y_2, v\}$, there exist pairwise disjoint subgraphs $D_u$ of $R - S$ such that $u \in D_u$ and each $D_u$ is a union of components of $R - S$.

(ii) For $u \in \{y_1, y_2, v\}$, there exist 2-cuts $S_u$ of $R$ and pairwise disjoint subgraphs $D_u$ of $R$, such that $u \in D_u$, each $D_u$ is a union of components of $R - S_u$, $S_{y_1} \cap S_{y_2} \cap S_v = \{z\}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_v - \{z\}$ are pairwise disjoint.

(iii) For $u \in \{y_1, y_2, v\}$, there exist pairwise disjoint 2-cuts $S_u$ in $R$ and pairwise disjoint subgraphs $D_u$ of $R - S_u$ such that $u \in D_u$, $D_u$ is a union of components of $R - S_u$, and
prove the following claim.

Figure 1: The subgraphs \( D_u \) in \( R, u \in \{y_1, y_2, v\} \).

For future use we prove the following (slightly stronger) version of Theorem 1.1.

**Theorem 3.2** Let \( G \) be a 5-connected nonplanar graph and let \( x_1, x_2, y_1, y_2 \) be distinct vertices of \( G \) such that \( G[\{x_1, x_2, y_1, y_2\}] \cong K_5^+ \) and \( y_1y_2 \notin E(G) \). Suppose that there is a path \( X \) in \( G - x_1x_2 \) from \( x_1 \) to \( x_2 \) such that \( G - X \) is 2-connected, \( X - x_2 \) is induced in \( G \), and \( \{y_1, y_2\} \cap V(X) = \emptyset \). Let \( v \in V(X) \) such that \( x_2v \in E(X) \). Then \( G \) contains \( TK_5 \) in which \( x_2v \) is an edge and \( x_1, x_2, y_1, y_2 \) are branch vertices.

**Proof.** If there exists \( x \in V(X - \{x_1, x_2\}) \) such that \( \{x, y_1, y_2\} \) is contained in some cycle, say \( D \), in \( G - V(X - x) \), then \( D \cup X \cup G[\{x_1, x_2, y_1, y_2\}] \) is a \( TK_5 \) in \( G \), containing \( x_2v \) and with branch vertices \( x_1, x_2, y_1, y_2, x \). Hence we may assume that

1. for any \( x \in V(X - \{x_1, x_2\}) \), no cycle in \( G - V(X - x) \) contains \( \{x, y_1, y_2\} \).

Since \( |N(v)| \geq 5 \) and \( X - x_2 \) is an induced path in \( G \), \( |N(v) - V(X)| \geq 3 \). Let \( R = G - V(X - v) \). Clearly, \( R \) is 2-connected. By (1), \( \{y_1, y_2, v\} \) is not contained in any cycle in \( R \). Hence, (i) or (ii) or (iii) of Theorem 3.1 holds (see Figure 1). We choose \( X \) so that

2. \( D_{y_1} \cup D_{y_2} \cup D_v \) is maximal.

We shall treat all three cases, (i), (ii) and (iii), simultaneously. For this we need some notation. If (i) occurs let \( S_v := S = \{z_1, z_2\} \), and if (ii) or (iii) occurs let \( S_v = \{z_1, z_2\} \). Let \( Z_i \) denote the component of \( R - V(D_{y_1} \cup D_{y_2} \cup D_v) \) containing \( z_i \). If (i) occurs then let \( a_1 = a_2 = z_1 \) and \( b_1 = b_2 = z_2 \); and if (iii) occurs let \( S_{y_1} = \{a_1, b_1\} \) and \( S_{y_2} = \{a_2, b_2\} \) such that \( a_1, a_2 \in Z_1 \) and \( b_1, b_2 \in Z_2 \). If (ii) occurs let \( S_{y_1} = \{a_1, b_1\} \) and \( S_{y_2} = \{a_2, b_2\} \) such that either \( z = z_1 = a_1 = a_2 \) or \( z = z_2 = b_1 = b_2 \) (we do not fix this notation for the purpose of symmetry in the arguments to follow).

Note that if (i) occurs then \( Z_i := \{z_i\} \) for \( i = 1, 2 \); and if (ii) or (iii) occurs then by (2) and the fact that \( R \) is 2-connected, \( S_u \cap V(Z_i) \), for \( u \in \{y_1, y_2, v\} \), are not cuts in \( Z_i \). Also note that if (ii) occurs, then \( Z_1 = Z_2 \), or \( Z_1 \neq Z_2 \) and \( Z_1 = \{z_1\} \), or \( Z_1 \neq Z_2 \) and \( Z_2 = \{z_2\} \). So the case when (ii) occurs with \( Z_1 \neq Z_2 \) may also be viewed as that when (iii) occurs. We now prove the following claim.
For any $x \in V(Z_1 - z_2)$, $Z_1 - z_2$ has independent paths $A_1, A_2$ from $\{x, z_1\}$ to $\{a_1, a_2\}$ with $a_1 \in A_1$ and $x, z_1 \in A_1 \cup A_2$, and $Z_1 - z_2$ has a path $A$ between $a_1$ and $a_2$ and independent from $z_1$; for any $x \in V(Z_2 - z_1)$, $Z_2 - z_1$ has independent paths $B_1, B_2$ from $\{x, z_2\}$ to $\{b_1, b_2\}$ with $b_1 \in B_1$ and $x, z_2 \in B_1 \cup B_2$, and $Z_2 - z_1$ has a path $B$ between $b_1$ and $b_2$ and independent from $z_2$.

Since the two statements of (3) are symmetric, we only prove the existence of $B_1, B_2, B$. If $b_1 = b_2 = z_2$ then we simply take $B_1 = B_2 = B = \{z_2\}$. So we may assume by (2) that $b_1, b_2, z_2$ are pairwise distinct (and hence (ii) or (iii) occurs).

If $B_1, B_2$ do not exist, then $Z_2 - z_1$ has a cut vertex $z'_2$ separating $\{b_1, b_2\}$ from $\{x, z_2\}$; and we see that $S_{y_1}, S_{y_2}, S'_v := \{z_1, z'_2\}$ contradict (2).

Now suppose the path $B$ does not exist. Then $z_2$ is a cut vertex in $Z_2 - z_1$ separating $b_1$ and $b_2$. If (ii) occurs then $S = \{z_1, z_2\}$ is a cut in $R$ such that $y_1, y_2, v$ are contained in different components of $G - S$, contradicting (2). If (iii) occurs then $S'_y := \{a_1, z_2\}$, $S'_{y_2} := \{a_2, z_2\}$ and $S_v$ are cuts in $R$ contradicting (2). This completes the proof of (3).

Since $R$ is 2-connected, for each $u \in \{y_1, y_2, v\}$, $R[D_u \cup S_u]$ is a chain of blocks between the vertices of $S_u$, and there is a path $P_u$ in $R[D_u \cup S_u]$ between the vertices of $S_u$ and containing $u$. Let $P'_u$ denote the subpath of $P_u$ from $u$ to $S_u \cap Z_i$ (in the case when $Z_1 = Z_2$ let $P'_u$ be from $u$ to $z_i$).

(4) We may assume that $N(z_i) \cap (X - \{x_1, x_2, v\}) = \emptyset$ for $i = 1, 2$, and that $D_v$ is connected.

Suppose (4) fails. By symmetry, we may assume $N(z_1) \cap (X - \{x_1, x_2, v\}) \neq \emptyset$ or $D_v$ is not connected. Then we can find a path $P$ from $z_1$ to $a \in V(x_1Xv) - \{x_1, v\}$ and internally disjoint from $X \cup P_{y_1} \cup P_{y_2} \cup P_v$, as follows. If $N(z_1) \cap V(X - \{x_1, x_2, v\}) \neq \emptyset$ let $a \in N(z_1) \cap V(x_1Xv - \{x_1, v\})$ and let $P := z_1a$. If $D_v$ is not connected, then let $D$ be a component of $D_v$ such that $v \notin D$. Since $G$ is 5-connected, there exists $a \in N(D) \cap V(x_1Xv - \{x_1, v\})$. Let $P$ be a path in $R[V(D) \cup S_v \cup \{a\}] - z_2$ from $z_1$ to $a$.

Choose $A_1, A_2, B$ as in (3) with $x = z_1$. Then $(A_1 \cup P'_{y_1}) \cup (A_2 \cup P'_{y_2}) \cup (P'_{v} \cup vX_1) \cup (P''_{v} \cup B \cup P''_{y_2}) \cup G[x_1, x_2, y_1, y_2]$ is a TK5 in $G$ containing $x_2v$ and with branch vertices $x_1, x_2, y_1, y_2, z_1$. This proves (4).

Note that $N(D_v) \subseteq S_v \cup X$. Let $u \in N(D_v) \cap V(X)$ with $x_1Xu$ minimal, and let $u' \in N(u) \cap V(D_v)$. Since $\{z_1, z_2, u, x_2\}$ is not a cut in $G$, there exists an edge $cc'$ with $c \in uXx_2 - \{x_2, u\}$ and $c' \in (D_{y_1} \cup D_{y_2} \cup Z_1 \cup Z_2) - \{z_1, z_2\}$. Note that $c \neq v$, for otherwise $S_v$ is not a 2-cut in $R$ separating $D_v$ from $D_{y_1} \cup D_{y_2} \cup Z_1 \cup Z_2$. So $c \in uXv - \{u, v\}$. Since $X$ is induced, $u' \neq v$. Hence by (4), let $Q'_{u'}$ denote a path in $D_v$ from $u'$ to $w \in V(P_v)$ such that $Q'_{u'} \cap P_v = \{w\}$. By symmetry, we may assume $w \in P''_{y_2}$. Note $w \neq z_2$, since $Q'_{u'} \subset D_v$.

(5) We may assume that $N(c) \cap V(Z_1) = \emptyset$ when $Z_1 \neq Z_2$ or when $b_1 = b_2 = z_2$, and we may assume that if $x \in N(c) \cap V(D_{y_1})$ then for any path $P_x$ in $R[D_{y_1} \cup S_{y_1}]$ from $x$ to $P_{y_1}, P_x$ intersects $P''_{y_1} - y_1$ first (starting from $x$).

First, suppose $x \in N(c) \cap Z_1$ and $Z_1 \neq Z_2$ or $b_1 = b_2 = z_2$. By (4), $x \neq z_1$ and $x \neq z_2$; and so $Z_1 \neq \{z_1\}$. Let $A_1, A_2$ be the paths as in (3), and by symmetry we may assume $x \in A_1$. Let $B = \{z_2\}$ if $Z_1 = Z_2$ and $b_1 = b_2 = z_2$, and otherwise let $B$ be the path as in (3). Then
(vP^2_{v} w \cup Q_{u'} \cup u'u \cup uX) \cup v x_{2} \cup (v X c \cup c x \cup A_{1} \cup P_{y_{1}}) \cup (P_{v} \cup A_{2} \cup P_{y_{1}}^{1}) \cup (P_{y_{2}}^{2} \cup B \cup P_{y_{2}}) \cup G[\{x_{1}, x_{2}, y_{1}, y_{2}\}] is a TK_5 in G containing x_{2}v and with branch vertices x_1, x_2, y_1, y_2, v.

Now suppose x \in N(c) \cap V(D_{y_{1}}), and there is a path R_i in R[D_{y_{1}} \cup S_{y_{1}}] from x to x' \in V(P_{y_{1}}) such that R_i \cap P_{y_{1}} = \{x'\}. Without loss of generality, assume i = 1. Choose B as in (3). Also by (3), let A be a path in Z_1 = z_2 from z_1 to a_2 and independent from A = \{a_1\}. Note that if Z_1 \neq Z_2 then B \cap (A_1 \cup A_2) = \emptyset; and if Z_1 = Z_2 then either a_1 = a_2 = z_1 (with A_1 = A_2 = \{z_1\}) or b_1 = b_2 = z_2 (B = \{z_2\}), and we have B \cap (A_1 \cup A_2) = \emptyset as well.

Suppose x' = a_1 and a_1 = a_2 = z_1. Then R_1 is contained in a component D of D_{y_{1}} such that y_1 \notin D. Hence (R_1 \cup xC \cup CX_{2}) \cup P_{y_{1}}^{1} \cup P_{y_{2}}^{1} \cup (P_{v} \cup v x_{2}) \cup (P_{y_{1}}^{2} \cup B \cup P_{y_{2}}^{2}) \cup G[\{x_{1}, x_{2}, y_{1}, y_{2}\}] is a TK_5 in G containing x_{2}v and with branch vertices x_1, x_2, y_1, y_2, z_1. So assume x' \neq a_1 or a_1 \notin \{a_2, z_1\}. Then (vP^2_{v} w \cup Q_{u'} \cup u'u \cup uX) \cup v x_{2} \cup (v X c' \cup c x \cup R_{1} \cup x'P_{y_{1}}^{1} \cup (P_{v} \cup A_{2} \cup P_{y_{2}}^{1}) \cup (P_{y_{2}}^{2} \cup B \cup P_{y_{2}}^{2}) \cup G[\{x_{1}, x_{2}, y_{1}, y_{2}\}] is a TK_5 in G containing x_{2}v with branch vertices x_1, x_2, y_1, y_2, v.

(6) We may assume that N(c) \cap V(D_{y_{1}} \cup D_{y_{2}} \cup Z_{1} \cup Z_{2}) = \{c'\}.

Otherwise, we may assume by (4) and (5) that there exists a \in N(c) \cap V(D_{y_{1}} \cup D_{y_{2}} \cup (Z_{2} - \{z_1, z_2\})) such that a \neq c'.

Suppose \{a, c'\} \subseteq Z_{2} - \{z_1, z_2\}. Then only (ii) or (iii) can occur. First, if \exists Z_2 - z_1 has disjoint paths B'_1, B'_2 from \{a, c'\} to b_1, b_2, respectively. Then w_1 \neq w_2, and hence, either Z_1 \neq Z_2 or Z_1 = Z_2 and a_1 = a_2 = z_1. So let A := \{z_1\} if Z_1 = Z_2 and a_1 = a_2 = z_1; otherwise let A be the path in (3). Now \{y_1, y_2, c\} is contained in the cycle B'_1 \cup P_{y_{1}} \cup B'_2 \cup A \cup P_{y_{2}} \cup \{c, c', ca\} in G - V(X - c), contradicting (1). Therefore, we may assume that such paths B'_1, B'_2 do not exist for any choice of \{a, c', w_1, w_2\} from \{a, c'\} \subseteq Z_{2} - \{z_1, z_2\}. Then by (2), there is a cut vertex z in Z_2 - z_1 separating N(c) \cap Z_{2} from \{b_1, b_2\}. Suppose \exists Z_1 \neq Z_{2}. Since R is 2-connected, z must separate (in Z_{2}) \{b_1, b_2\} from (N(c) \cap Z_{2}) \subseteq \{z_{1}, z_{2}\}. But then \exists u := \{z, z_1, S_{y_{1}}, S_{y_{2}}\} contradict (2). So Z_1 = Z_{2}, and hence by (5), a_1 = a_2 = z_1. If z_2 is in the z-bridge of Z_{2} that also contains N(c) \cap Z_{2}, then \exists v := \{z, z_1, S_{y_{1}}, S_{y_{2}}\} contradict (2). So in Z_{2} - z_1, z separates \{b_1, b_2\}, z from N(c) \cap Z_{2}. Note that c' \neq z or a \neq z. Without loss of generality, assume c' \neq z. Then since R is 2-connected, Z_{2} contains disjoint paths R_{1}, R_{3} from z_1, b_1 to c', b_2, respectively. Now (R_{1} \cup c'c' \cup X_{1}c') \cup (P_{v} \cup B_{1}x_{2}) \cup (P_{y_{1}}^{1} \cup P_{y_{2}}^{1}) \cup (P_{y_{1}}^{2} \cup B_{2}P_{y_{2}}^{2}) \cup G[\{x_{1}, x_{2}, y_{1}, y_{2}\}] is a TK_5 in G containing x_{2}v and with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume (a, c') \subseteq Z_{2} - \{z_1, z_2\}. Then N(c) \cap V(D_{y_{1}} \cup D_{y_{2}}) \neq \emptyset (by (4) when Z_1 = Z_{2}, and by (4) and (5) when Z_1 \neq Z_{2}). We may thus assume by symmetry that c' \in D_{y_{2}}. Let P_{v} be a path in D_{y_{1}} from a to c' \in V(P_{y_{1}}) such that P_{v} \cap P_{y_{1}} = \{c'\}. By (5), c' \in P_{y_{2}} - y_{2}. Recall the path A from (3). Now \{c, c', ca\} \cup P_{v} \cup c''P_{y_{1}}a_{1} \cup A \cup a_{2}P_{y_{2}}a' \cup P_{a} is a cycle in G - V(X - c) containing \{y_{1}, y_{2}, c\}, contradicting (1). If a \in Z_2 - \{z_1, z_2\}, then there is a path P_{a} in Z_2 - z_1 from a to b_2. Again, \{c, c', ca\} \cup P_{v} \cup c'P_{y_{1}}a_{1} \cup A \cup P_{y_{2}} \cup P_{a} is a cycle in G - V(X - c) containing \{y_{1}, y_{2}, c\}, contradicting (1).

So we may assume a \in D_{y_{1}}. Since R[S_{y_{1}} \cup D_{y_{1}}] is a chain of blocks, it has disjoint paths P_{a}, P_{v} from a, c' to a', c'' \in V(P_{y_{1}}) such that P_{a} \cap P_{y_{1}} = \{a'\} and P_{v} \cap P_{y_{1}} = \{c''\}. By (5), we have \{a', c''\} \subseteq P_{y_{2}}^{2}. Without loss of generality, we may assume that a' \in b_{1}P_{y_{2}}^{2}. If Z_1 = Z_{2} and z_1 = a_1 = a_2 let A = \{z_{1}\} and B be as in (3); if Z_1 = Z_{2} and b_1 = b_2 = z_2
then let \( B = \{ z_2 \} \) and \( A \) be as in (3); and if \( Z_1 \neq Z_2 \) let \( A \) and \( B \) be as in (3). Then \( \{ c, cc', ca \} \cup P_v \cup c''P_{y_1}a_1 \cup A \cup P_{y_2} \cup B \cup a'P_{y_1}b_1 \cup P_a \) is a cycle in \( G - V(X - c) \) containing \( \{ y_1, y_2, c \} \), contradicting (1) and completing the proof of (6).

Since \( X - x_2 \) is induced and \( G \) is 5-connected, \( c \) has at least two neighbors in \( G - V(X) \). So by (6), \( N(c) \cap V(D_v - v) \neq \emptyset \). Without loss of generality and by (5) and (6), we may assume that \( c' \in D_{y_1} \cup (Z_2 - \{ z_1, z_2 \}) \). Moreover, if \( c' \in D_{y_1} \), let \( P_{c'} \) be a path in \( D_{y_1} \) from \( c' \) to \( c'' \in V(P_{y_1}) \) such that \( P_{c'} \cap P_{y_1} = \{ c'' \} \) and \( c'' \in P_{y_1}^2 - y_1 \) (by (5)).

(7) We may assume that \( v \) is a cut-vertex of \( R[S_v \cup D_v] - z_1z_2 \) separating \( z_2 \) from \( (N(c) \cap V(D_v)) \cup \{ z_1 \} \).

Otherwise, since \( R[S_v \cup D_v] - z_1z_2 \) is a chain of blocks from \( z_1 \) to \( z_2 \), there is a path \( P_a \) in \( R[S_v \cup V(D_v)] - \{ v, z_1 \} \) from some \( a \in N(c) \cap V(D_v) \) to \( z_2 \).

Suppose \( c' \in D_{y_1} \). If \( Z_1 = Z_2 \) and \( a_1 = a_2 = z_1 \) let \( A = \{ z_1 \} \) and \( P \) be a path in \( Z_2 - z_1 \) from \( z_2 \) to \( b_2 \); if \( Z_1 = Z_2 \) and \( b_1 = b_2 = z_2 \) let \( P = \{ z_2 \} \) and \( A \) be as in (3); and if \( Z_1 \neq Z_2 \), let \( A \) be as in (3) and \( P \) be a path in \( Z_2 - z_1 \) from \( z_2 \) to \( b_2 \). It is easy to see that \( \{ c, cc', ca \} \cup P_a \cup P \cup P_{y_2} \cup A \cup a_1P_{y_1}c'' \cup P_{c'} \) is a cycle in \( G - V(X - c) \) containing \( \{ y_1, y_2, c \} \), contradicting (1).

So \( c' \in Z_2 - \{ z_1, z_2 \} \). Then by (5), \( Z_1 \neq Z_2 \), or \( Z_1 = Z_2 \) and \( a_1 = a_2 = z_1 \). If \( Z_1 \neq Z_2 \) let \( A, B_1, B_2 \) be as in (3); and otherwise let \( A = \{ z_1 \} \), and let \( B_1, B_2 \) be as in (3) with \( c' \in B_1 \). Now \( \{ c, cc', ca \} \cup P_a \cup B_2 \cup P_{y_2} \cup A \cup P_{y_1} \cup B_1 \) is a cycle in \( G - V(X - c) \) containing \( \{ y_1, y_2, c \} \), contradicting (1) and completing the proof of (7).

Let \( T \) denote the \( v \)-bridge of \( R[S_v \cup D_v] - z_1z_2 \) containing \( z_2 \). Recall \( u, u', w \) and \( u' \neq v \) (the paragraph above (5)). Since \( w \in P_v^2 \subseteq T \), \( u' \in T - v \). Since \( G \) is 5-connected and by the choice of \( u \), \( G[V(T) \cup V(uXx_2)] \) is \( (5, V(uXx_2) \cup \{ z_2, v \}) \)-connected, and so \( G' := G[V(T) \cup V(Ux_2 - u)] \) is \( (4, V(uXx_2 - u) \cup \{ z_2, v \}) \)-connected. So by Lemma 2.5, there exist four independent paths \( P_1, P_2, P_3, P_4 \) in \( G' \) from \( u' \) to \( uXx_2 - u \) \( \{ z_2, v \} \) such that \( P_1 \) ends at \( z_2 \), \( P_2 \) ends at \( v \), and \( P_3, P_4 \) both end in \( uXx_2 - \{ u, v \} \). Since \( vXx_2 \in E(X) \), we may assume that \( P_3 \) ends at \( x' \in V(uXx_2 - \{ u, v \}) \).

Suppose \( c' \in D_{y_1} \). If \( Z_1 = Z_2 \) and \( a_1 = a_2 = z_1 \) let \( A = \{ z_1 \} \) and let \( B'_2 \) be a path in \( Z_2 - z_1 \) from \( z_2 \) to \( b_2 \); if \( Z_1 = Z_2 \) and \( b_1 = b_2 = z_2 \) let \( B'_2 = \{ z_2 \} \) and \( A \) be as in (3); and if \( Z_1 \neq Z_2 \) let \( A \) be as in (3) and \( B'_2 \) be a path in \( Z_2 \) from \( z_2 \) to \( b_2 \). Now \( (u' \cup Ux_1) \cup (P_4 \cup B'_2 \cup P_{y_2}^2) \cup (P_2 \cup vXx_2) \cup (P_3 \cup x'Xc \cup cc' \cup B_1 \cup b_1P_{y_1}y_1) \cup (P_{y_1}^1 \cup A \cup P_{y_2}^1) \cup G[\{ x_1, x_2, y_1, y_2 \}] \) is a TK_5 in \( G \) containing \( x_2v \) and with branch vertices \( x_1, x_2, y_1, y_2, u' \).

So we may assume \( c' \in Z_2 - \{ z_1, z_2 \} \). Then by (5), \( Z_1 \neq Z_2 \), or \( Z_1 = Z_2 \) and \( a_1 = a_2 = z_1 \). If \( Z_1 = Z_2 \) and \( a_1 = a_2 = z_1 \) let \( A = \{ z_1 \} \); if \( Z_1 \neq Z_2 \) let \( A \) be defined as in (3). Let \( B_1, B_2 \) be defined as in (3) (with \( c' \) as \( x \)). If \( z_2 \in B_2 \) and \( c' \in B_1 \), then \( (u' \cup Ux_1) \cup (P_1 \cup B_2 \cup P_{y_2}^2) \cup (P_2 \cup vXx_2) \cup (P_3 \cup x'Xc \cup cc' \cup B_1 \cup b_1P_{y_1}y_1) \cup (P_{y_1}^1 \cup A \cup P_{y_2}^1) \cup G[\{ x_1, x_2, y_1, y_2 \}] \) is a TK_5 in \( G \) containing \( x_2v \) and with branch vertices \( x_1, x_2, y_1, y_2, u' \). So assume \( z_2 \in B_1 \) and \( c' \in B_2 \). Then \( (u' \cup Ux_1) \cup (P_1 \cup B_1 \cup P_{y_1}^2) \cup (P_2 \cup vXx_2) \cup (P_3 \cup x'Xc \cup cc' \cup B_2 \cup b_2P_{y_2}^2y_2) \cup (P_{y_1}^1 \cup A \cup P_{y_2}^1) \cup G[\{ x_1, x_2, y_1, y_2 \}] \) is a TK_5 in \( G \) containing \( x_2v \) and with branch vertices \( x_1, x_2, y_1, y_2, u' \).
4 Planar graphs

In this section we prove Theorem 1.2, using an approach similar to that in [21] where rooted $K_4$-subdivisions are considered. This result will be useful in situations where we force a 5-separation in a 5-connected nonplanar graph such that one side of the separation is planar.

It is well known that every face of a 2-connected plane graph is bounded by a cycle. The outer cycle of a 2-connected plane graph is the boundary of its infinite face. In a plane graph, two vertices are said to be cofacial if they are incident with a common face. Let $C$ be a cycle in a plane graph and $x, y \in V(C)$; if $x \neq y$ we use $xCy$ to denote the path in $C$ clockwise from $x$ to $y$, and if $x = y$ then $xCy$ represents the path consisting of the vertex $x = y$. For a vertex $x$ in a graph, we use $d(x)$ to denote the degree of $x$.

**Lemma 4.1** Let $G$ be a graph drawn in a closed disc in the plane without edge crossings, and let $a_1, a_2, a_3, a_4, a_5$ be distinct vertices of $G$ on the boundary of the disc, and let $A := \{a_1, a_2, a_3, a_4, a_5\}$. Suppose $G$ is $(5, A)$-connected and $|V(G)| \geq 7$. Then $G - A$ is 2-connected, and $G - A$ is not spanned by its outer cycle. Moreover, for each $w \in V(G) - A$ which is not on the outer cycle of $G - A$, all vertices of $G$ that are cofacial with $w$ induce a cycle in $G - A$.

**Proof.** Without loss of generality we may assume that $a_1, a_2, a_3, a_4, a_5$ lie on the boundary of the disc in the clockwise order listed. Since $G$ is $(5, A)$-connected, $d(v) \geq 5$ for all $v \in G - A$.

First, we claim that $G - A$ is connected and has no cut vertex. Otherwise, there is a separation $(G_1, G_2)$ in $G - A$ of order at most 1 such that $G_1 - G_2 \neq \emptyset$ and $G_2 - G_1 \neq \emptyset$. Note that $|N(G_1 - G_2) \cap A| \geq 4$ since otherwise $V(G_1 \cap G_2) \cup (N(G_1 - G_2) \cap A)$ is a cut in $G$ separating $A$ from $G_1$, contradicting the assumption that $G$ is $(5, A)$-connected. Therefore, by planarity, we may assume (with appropriate notation change) that $a_1, a_2, a_3, a_4$ all have neighbors in $G_1 - G_2$. Then by planarity we see that $\{a_4, a_5, a_1\} \cup V(G_1 \cap G_2)$ is a cut in $G$ separating $G_2$ from $A$, a contradiction (since $G$ is $(5, A)$-connected).

Therefore, $G - A \equiv K_2$ or $G - A$ is 2-connected. Indeed, $G - A$ must be 2-connected. For, suppose $G - A \equiv K_2$. Let $V(G - A) = \{a, b\}$. Then $|N(a) \cap A| \geq 4$, or else $(N(a) \cap A) \cup \{b\}$ is a cut of size at most 4 separating $A$ from $b$, a contradiction. Similarly, $|N(b) \cap A| \geq 4$. However, this contradicts planarity.

Let $C$ denote the outer cycle of $G - A$. We now show that $V(G - A) \neq V(C)$. For, suppose $V(G - A) = V(C)$; we will derive a contradiction. If $|V(C)| = 3$, then each vertex in $V(C)$ has at least 3 neighbors in $A$, which is not possible due to planarity. So $|V(C)| \geq 4$. Since all edges of $G - A$ are on $C$ or inside $C$ (with both ends on $C$), it follows from planarity that there are two vertices on $C$ with degree 2 in $G - A$, say $u$ and $v$, such that $uv \notin E(G)$. Since $G$ is $(5, A)$-connected and by planarity, we may assume $a_1, a_2, a_3 \in N(u)$ and $a_3, a_4, a_5 \in N(v)$; and hence no other vertex of $G - A$ has degree 2, and each edge of $G - A$ not on $C$ joins $uCv - \{u, v\}$ to $vCu - \{u, v\}$. Since $G$ is $(5, A)$-connected and $ua_3, va_3 \in E(G)$, $|N(z) \cap V(C)| \geq 4$ for all $z \in uCu - \{u, v\}$. Let $w$ be the neighbor of $u$ in $uCv - \{u, v\}$, and let $w_1, w_2$ denote the neighbors of $w$ on $vCu - \{v, u\}$ with $v, w_1, w_2, u$ on $vCu$ in order and $w_1w_2$ maximal. Let $w'_2, w''_2$ be the neighbors of $w_2$ in $w_1Cu$. Then by planarity and the fact that $d(w_2) \geq 5$, $N(w_2) = \{w'_2, w''_2, w, a_1, a_5\}$. Because $d(w_1) \geq 5$ and $a_1 \notin N(w_1)$ (by planarity), there exists $x \in wCu - \{w, v\}$ such that $x \in N(w_1)$. Then we may pick $y \in V(xCu - v)$ such that $yCu$ minimal and $y$ has a neighbor in $vCuw_1 - v$. By planarity and the fact $d(y) \geq 5$, $|N(y) \cap V(C)| \geq 4$. Let $y_1, y_2$ denote neighbors of $y$ on $vCu - v$ with $v, y_1, y_2, w_1$ on $vCu$ in
order. Let \( y_1', y_1'' \) be the neighbors of \( y_1 \) in \( vCw_1 \). Then by planarity, \( N(y_1) \subseteq \{ y, y_1', y_1'', a_5 \} \), contradicting \( d(y_1) \geq 5 \).

Let \( w \in V(G - A) \) such that \( w \not\in C \). Then, since \( G \) is \((5, A)\)-connected and by planarity, the vertices of \( G \) that are cofacial with \( w \) induce a cycle in \( G - A \).

The lemma below is essential to the proof of Theorem 1.2.

**Lemma 4.2** Let \( G \) be a connected graph drawn in a closed disc in the plane without edge crossings, let \( a_1, a_2, a_3, a_4, a_5 \) be distinct vertices of \( G \) on the boundary of the disc, and let \( A = \{ a_1, a_2, a_3, a_4, a_5 \} \). Suppose \( G \) is \((5, A)\)-connected and \(|V(G)| \geq 7\), and assume \( G \) has no 5-separation \((G_1, G_2)\) such that \( A \subseteq G_1 \) and \(|V(G)| > |V(G_2)| \geq 7\). Let \( w \in V(G - A) \) such that the vertices of \( G \) cofacial with \( w \) induce a cycle \( C_w \) in \( G - A \). Then there exist four paths \( P_1, \ldots, P_4 \) from \( w \) to \( A \) such that

(i) for \( 1 \leq i < j \leq 4 \), \( V(P_i \cap P_j) = \{ w \} \), and

(ii) for \( 1 \leq i \leq 4 \), \(|V(P_i \cap C_w)| = 1\).

**Proof.** By assumption, we have

(1) \( G \) has no 5-separation \((G_1, G_2)\) such that \( A \subseteq G_1 \) and \(|V(G)| > |V(G_2)| \geq 7\).

By Lemma 4.1, \( G - A \) is 2-connected. So \(|V(G) - A| \geq 3\). Hence by (1), each \( a_i \) has at least two neighbors in \( G - A \), and so \( G \) is 2-connected. Let \( C \) denote the outer cycle of \( G \), and let \( C' \) denote the outer cycle of \( G - A \). By Lemma 4.1 again, there exists \( w \in V(G - A) \) such that the vertices of \( G \) which are cofacial with \( w \) induce a cycle \( C_w \) and \( C_w \subseteq G - A \). By planarity, \( w \notin C' \).

By Menger’s theorem, there exist four paths \( Q_1, \ldots, Q_4 \) from \( w \) to \( A \) such that \( V(Q_i \cap Q_j) = \{ w \} \) for \( 1 \leq i \neq j \leq 4 \), and for each \( i \) (by planarity, we may assume that) \( Q_i \cap C_w \) is a path. Let \( \alpha(Q_1, Q_2, Q_3, Q_4) \) denote the number of \( Q_i \) such that \(|V(Q_i \cap C_w)| \geq 2\). We choose such \( Q_1, Q_2, Q_3, Q_4 \) that

(2) \( \alpha(Q_1, Q_2, Q_3, Q_4) \) is minimum.

We may assume that the notation is such that \( a_i \in Q_i \) for \( i = 1, \ldots, 4 \), and that \( a_1, a_2, a_3, a_4 \) occur on \( C \) in clockwise order \((a_5) \) could be anywhere on \( C \). Let \( w_i, v_i \in V(Q_i) \) such that \( ww_i \in Q_i \) and \( V(v_i Q_i a_i \cap C_w) = \{ v_i \} \).

If \( \alpha(Q_1, Q_2, Q_3, Q_4) = 0 \), then \( P_i := Q_i, 1 \leq i \leq 4 \), are the desired paths. So we may assume without loss of generality that \(|V(Q_1 \cap C_w)| \geq 2\). By symmetry, we may further assume that \( v_1 \in w_1 C_w w_2 \). See Figure 2 for an illustration. We may also assume that \( w \) has no neighbor in \( w_1 C_w v_1 - w_1 \); for otherwise let \( w' \) be a neighbor of \( w \) in \( w_1 C_w v_1 - w_1 \) with \( w' C_v_1 \) minimal, and we may replace \( Q_1 \) with \( w'Q_1 a_1 + \{ w, w' \} \).

For \( 1 \leq i \leq 4 \), let \( H_i \) denote the maximal subgraph of \( G \) contained in the closed region in the plane with boundary \( Q_i \cup Q_{i+1} \cup a_i C_{a_{i+1}} \cup w_i C_w w_{i+1} \) for \( i = 1, 2, 3 \) and \( Q_4 \cup Q_1 \cup a_4 C_a_1 \cup w_4 C_w w_1 \) for \( i = 4 \). Let \( S_1 \) denote the vertices of \( G \), each of which is cofacial with some vertex of \( w_1 C_w v_1 - w_1 \). Then

(3) \( S_1 \cap V(v_4 C_w w_1 - w_1) = \emptyset \), and \( S_1 \cap V(v_4 Q_4 a_4 - v_4) \neq \emptyset \).
If $S_1 \cap V(v_4C_wv_1 - w_1) \neq \emptyset$, then there exist $x \in V(v_4C_wv_1 - w_1)$ and $y \in V(w_1C_wv_1 - w_1)$ such that $\{x, y, w\}$ is a cut in $G$ separating $w_1$ from $\{a_1, a_2, a_3, a_4\}$. Since $a_5 \notin C_w$, $w_1 \neq a_5$. So $\{x, y, w, a_5\}$ is a cut in $G$ separating $w_1$ from $A$, contradicting the assumption that $G$ is $(5, A)$-connected. So $S_1 \cap V(v_4C_wv_1 - w_1) = \emptyset$.

Now suppose $S_1 \cap V(v_4Q_3a_4 - v_4) = \emptyset$. Then by planarity $H_4$ has a path $Q'_1$ from $w_1$ to $a_1$ disjoint from $Q_4 \cup (C_w - w_1)$. Let $Q'^*_1 = Q'_1 + \{w, ww_1\}$ if $a_5 \notin Q'_1$, and let $Q'^*_1 = w_1Q'_1a_5 + \{w, ww_1\}$ otherwise. Now $\alpha(Q'^*_1, Q_2, Q_3, Q_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2). So $S_1 \cap V(v_4Q_4a_4 - v_4) \neq \emptyset$, completing the proof of (3).

Let $S_4$ denote the vertices of $G$ each of which is cofacial with a vertex in $S_1 \cap V(v_4Q_4a_4 - v_4)$. Then

(4) $S_4 \cap V(v_3C_wv_4) = \emptyset$, and $S_4 \cap V(v_3Q_3a_3 - v_3) \neq \emptyset$.

Suppose there exists $u \in S_4 \cap V(v_3C_wv_4)$. Then there exist $u_4 \in S_1 \cap V(v_4Q_4a_4 - v_4)$ and $v \in V(w_1C_wv_1 - w_1)$ such that $u$ and $u_4$ are cofacial, and $u_4$ and $v$ are cofacial. Note that $\{u, u_4, v, w\}$ is a cut in $G$; so, since $G$ is $(5, A)$-connected, $\{u, u_4\} \subseteq C$ and $a_5 \in uC_{u_4}$ or $\{u_4, v\} \subseteq C$ and $a_5 \in u_4Cv$. If $w_1a_5 \notin E(G)$, then the cut $\{u, u_4, v, w, a_5\}$ contradicts (1) (as $w_1$ has at least 5 neighbors); if $w_1a_5 \in E(G)$ then $\alpha(w_1a_5, Q_2, Q_3, Q_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2). Hence, $S_4 \cap V(v_3C_wv_4) = \emptyset$.

Now assume $S_4 \cap V(v_3Q_3a_3 - v_3) = \emptyset$. Then by planarity and by the fact that $S_4 \cap V(v_3C_wv_4) = \emptyset$, there is a path $Q'_4$ in $H_3 - (S_1 \cap V(Q_4))$ from $v_4$ to $a_4$ disjoint from $Q_3$ and $C_w - v_4$. Moreover, by (3) and planarity, $H_4 - V(Q'_4)$ has a path $Q'_1$ from $w_1$ to $a_1$ disjoint from $C_w - w_1$ (which necessarily contains $S_1 \cap V(Q_4)$). Let $Q'^*_1 = Q'_1 + \{w, ww_1\}$ if $a_5 \notin Q'_1$, and let $Q'^*_1 = w_1Q'_1a_5 + \{w, ww_1\}$ otherwise. Similarly, define $Q'^*_4 = Q'_4 + wQ_4v_4$ if $a_5 \notin Q'_4$, and let $Q'^*_4 = v_4Q'_4a_5 \cup wQ_4v_4$ otherwise. Then $\alpha(Q'^*_1, Q_2, Q_3, Q'^*_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2) and completing the proof of (4).

Let $S_3$ denote the vertices of $G$ each of which is cofacial with a vertex in $S_4 \cap V(v_3Q_3a_3 - v_3)$. Then

(5) $S_3 \cap V(v_2C_wv_3) = \emptyset$, and $S_3 \cap V(v_2Q_2a_2 - v_2) \neq \emptyset$.  

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First, suppose there exists \( u \in S_3 \cap V(v_2C_wv_3) \). Then there exist \( u_3 \in S_1 \cap V(v_3Q_3a_3 - v_3) \), \( u_4 \in S_1 \cap V(v_4Q_4a_4 - v_4) \), and \( v \in V(w_1C_wv_1 - w_1) \) such that \( u \) and \( u_3 \) are cofacial, \( u_3 \) and \( u_4 \) are cofacial, and \( u_4 \) and \( v \) are cofacial. Choose \( u, u_3, u_4, v \) so that \( uC_wv_3 = v_3Q_3u_3, v_4Q_4u_4, \) and \( w_1C_wv \) are minimal (in the order listed).

Let \( H'_2 \) denote the \( \{u, u_3\} \)-bridge of \( H_2 \) containing \( uC_wv_3 \cup v_3Q_3u_3 \); let \( H'_3 \) denote the \( \{u_3, u_4\} \)-bridge of \( H_3 \) containing \( v_3C_wv_4 \); and let \( H'_4 \) denote the \( \{u_4, v\} \)-bridge of \( H_4 \) containing \( v_4C_wv \). Note that \( \{u, u_3, u_4, v, w\} \) is a cut in \( G \); so by (1), \( a_i \in H'_i \) for some \( 2 \leq i \leq 4 \).

Suppose \( a_5 \in H'_2 \). Then in \( H_2 - Q_2 \) there is a path \( Q'_3 \) from \( v_3 \) to \( a_5 \) disjoint from \( S_4 \cap V(v_3Q_3u_3 - v_3) \) and \( (C_w - v_3) \cup v_3Q_3a_3 \) (by minimality of \( uC_wv_3 \) and \( v_3Q_3u_3 \)). In \( H_3 - Q_3 \) there is a path \( Q'_4 \) from \( v_4 \) to \( a_5 \) disjoint from \( S_1 \cap V(v_4Q_4u_4 - v_4) \) and \( (C_w - v_4) \cup u_4Q_4a_4 \) (by (4) and minimality of \( v_4Q_4u_4 \)). In \( H_4 - V(Q'_4) \) there is a path \( Q'_1 \) from \( w_1 \) to \( a_4 \) disjoint from \( C_w - w_1 \) (by (3) and minimality of \( w_1C_wv \)). Then \( \alpha(Q'_1 + \{w, w_1\}, Q_2, Q_3 \cup wQ_3v_3, Q_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4) \), contradicting (2).

Now assume \( a_5 \in H'_3 \). In \( H_2 - Q_2 \) there is a path \( Q'_3 \) from \( v_3 \) to \( a_5 \) disjoint from \( S_4 \cap V(v_3Q_3u_3 - \{v_3, v_3\}) \) and \( C_w - v_3 \) (by minimality of \( uC_wv_3 \) and \( v_3Q_3u_3 \)). In \( H_3 - Q_3 \) there is a path \( Q'_4 \) from \( v_4 \) to \( a_5 \) disjoint from \( S_1 \cap V(v_4Q_4u_4 - \{v_4, u_4\}) \) and \( C_w - v_4 \) (by (4) and minimality of \( v_4Q_4u_4 \)). In \( H_4 - V(Q'_4) \) there is a path \( Q'_1 \) from \( w_1 \) to \( a_4 \) disjoint from \( C_w - w_1 \) (by (3) and minimality of \( w_1C_wv \)). Then \( \alpha(Q'_1 + \{w, w_1\}, Q_2, Q_3 \cup wQ_3v_3, Q_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4) \), contradicting (2).

Finally, assume \( a_5 \in H'_4 \). In \( H_3 - Q_2 \) there is a path \( Q'_3 \) from \( v_3 \) to \( a_5 \) disjoint from \( S_4 \cap V(v_3Q_3u_3 - \{v_3, v_3\}) \) and \( C_w - v_3 \) (by minimality of \( uC_wv_3 \) and \( v_3Q_3u_3 \)). In \( H_3 - Q_3 \) there is a path \( Q'_4 \) from \( v_4 \) to \( a_5 \) disjoint from \( S_1 \cap V(v_4Q_4u_4 - \{v_4, u_4\}) \) and \( C_w - v_4 \) (by (4) and minimality of \( v_4Q_4u_4 \)). In \( H_4 - V(Q'_4) \) there is a path \( Q'_1 \) from \( w_1 \) to \( a_4 \) disjoint from \( C_w - w_1 \) (by (3)). Let \( Q'_i = Q'_1 + \{w, w_1\} \) if \( a_5 \in Q'_1 \), and let \( Q'_i = w_1Q'_1a_5 + \{w, w_1\} \) otherwise. Similarly, for \( i = 2, 3, 4 \), define \( Q'_i = Q'_i \cup wQ_iu_i \) if \( a_5 \notin Q'_i \), and let \( Q'_i = v_3Q'_3a_5 + wQ_iu_i \). Now \( \alpha(Q'_1, Q_2, Q'_3, Q'_4) < \alpha(Q_1, Q_2, Q_3, Q_4) \), contradicting (2) and completing the proof of (5).

Let \( S_2 \) denote the vertices of \( G \) each of which is cofacial with a vertex in \( S_3 \cap V(v_2C_wv_3) \). Then

(6) \( S_2 \cap V(v_1C_wv_2) \neq \emptyset \).

Suppose \( S_2 \cap V(v_1C_wv_2) = \emptyset \). Then \( S_2 \cap V(v_1Q_1a_1 - v_1) \neq \emptyset \). For, otherwise, in \( H_1 - Q_1 \) there is a path \( Q'_2 \) from \( v_2 \) to \( a_2 \) disjoint from \( S_3 \cap V(Q_2) \) and \( C_w - v_2 \). In \( H_2 - Q_2 \) there is a path \( Q'_3 \) from \( v_3 \) to \( a_3 \) disjoint from \( S_1 \cap V(Q_3) \) and \( C_w - v_3 \) (by (5)). In \( H_3 - Q_3 \) there is a path \( Q'_4 \) from \( v_4 \) to \( a_4 \) disjoint from \( S_1 \cap V(Q_4) \) and \( C_w - v_4 \) (by (4)). In \( H_4 - Q_4 \) there is a path \( Q'_1 \) from \( a_1 \) to \( a_3 \) disjoint from \( C_w - w_1 \) (by (3)). Let \( Q'_i = Q'_1 + \{w, w_1\} \) if \( a_5 \in Q'_1 \), and let \( Q'_i = w_1Q'_1a_5 \). Similarly, for \( i = 2, 3, 4 \), define \( Q'_i = Q'_i \cup wQ_iu_i \) if \( a_5 \notin Q'_i \), and let \( Q'_i = v_3Q'_3a_5 \). Now \( \alpha(Q'_1, Q_2, Q'_3, Q'_4) < \alpha(Q_1, Q_2, Q_3, Q_4) \), contradicting (2).
Thus, let $u_1 \in S_2 \cap V(v_1Q_1a_1 - v_1)$. Then there exists $u_2 \in S_3 \cap V(v_2Q_2a_2 - v_2)$ such that $u_2$ and $u_1$ are cofacial, there exists $u_3 \in S_4 \cap V(v_3Q_3a_3 - v_3)$ such that $u_3$ and $u_2$ are cofacial, there exists $u_4 \in S_4 \cap V(v_4Q_4a_4 - v_4)$ such that $u_4$ and $u_3$ are cofacial, and there exists $v \in V(w_1C_wv_1 - w_1)$ such that $u_4$ and $v$ are cofacial. For $i = 1, 2, 3$, define $H_i'$ as the \{u_i, u_{i+1}\}-bridge of $H_i$ containing $v_1C_wv_{i+1}$. Define $H_i'$ as the \{v, u_4\}-bridge of $H_4$ containing $v_1C_wv$.

Then $H_1'$ contains a path $Q_2'$ from $v_2$ to $u_1$ disjoint from $S_3 \cap V(Q_2)$ and $C_w - v_2$ (since we assume $S_2 \cap V(v_1C_wv_2) = \emptyset$). $H_2' - Q_2'$ contains a path $Q_3'$ from $v_3$ to $u_2$ disjoint from $S_4 \cap V(Q_3)$ and $C_w - v_3$ (by (5)). $H_3' - Q_3'$ contains a path $Q_4'$ from $v_4$ to $u_3$ disjoint from $S_4 \cap V(Q_4)$ and $C_w - v_4$ (by (4)). $H_4' - Q_4'$ contains a path $Q_1'$ from $u_4$ to $u_1$ disjoint from $C_w - w_1$ (by (3)). Let $Q_1^* = Q_1' + \{w, w_1\}$ if $a_5 \in Q_1'$, and let $Q_1^* = w_1Q_1' + \{w, w_1\}$ otherwise. Similarly, for $i = 2, 3, 4$, define $Q_i^* = Q_i' \cup wQ_i'v_i \cup u_{i-1}Q_i\cdot a_{i-1}$ if $a_5 \notin Q_i'$, and let $Q_i^* = v_iQ_i'\cup wQ_i'v_i$. Now $\alpha(Q_1^*, Q_2^*, Q_3^*, Q_4^*) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

By (6) and the definitions of $S_i$ ($1 \leq i \leq 4$), we may let $u \in V(v_1C_wv_2)$ and $u_2 \in V(v_2Q_2a_2 - v_2) \cap S_3$ such that $u$ and $u_2$ are cofacial and, subject to this, $uC_wv_2$ and $v_2Q_2u_2$ are minimal. Let $u_3 \in V(v_3Q_3a_3 - v_3) \cap S_4$ such that $u_2$ and $u_3$ are cofacial and, subject to this, $v_3Q_3u_3$ is minimal. Let $u_4 \in V(v_4Q_4a_4 - v_4) \cap S_1$ such that $u_4$ and $u_3$ are cofacial and, subject to this, $v_4Q_4u_4$ is minimal. Let $v \in V(w_1C_wv_1 - w_1)$ such that $v$ and $u_4$ are cofacial.

Let $H'_1$ denote the \{u, u_2\}-bridge of $H_1$ containing $uC_wv_2 \cup v_2Q_2u_2$; let $H'_2$ denote the \{u_2, u_3\}-bridge of $H_2$ containing $vC_wv_3$; let $H'_3$ denote the \{u_3, u_4\}-bridge of $H_3$ containing $vC_wv_4$; and let $H'_4$ denote the \{u_4, v\}-bridge of $H_4$ containing $vC_wv$.

(7) $a_5 \notin H'_i$ for $1 \leq i \leq 4$.

The proof of (7) is similar to that of (5). First, suppose $a_5 \in H'_1$. Then there is a path $Q_2'$ in $H_1$ from $v_2$ to $a_5$ disjoint from $S_3 \cap V(v_2Q_2u_2 - v_2)$ and $(C_w - v_2) \cup u_2Q_2u_2$ (by minimality of $uC_wv_2$ and $v_2Q_2u_2$). In $H_2 - Q_2'$ there is a path $Q_3'$ from $v_3$ to $a_5$ disjoint from $S_4 \cap V(v_3Q_3u_3 - v_3)$ and $(C_w - v_3) \cup u_3Q_3a_3$ (by (5) and minimality of $v_3Q_3u_3$). In $H_3 - Q_3'$ there is a path $Q_4'$ from $v_4$ to $a_5$ disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$ (by (4) and minimality of $v_4Q_4a_4$). In $H_4 - Q_4'$ there is a path $Q_1'$ from $u_4$ to $a_5$ disjoint from $C_w - u_4$ and $Q_1$ (by (3) and minimality of $w_1C_wv$). Then $\alpha(Q_1^* + \{w, w_1\}, wQ_2v_2 \cup Q_2', wQ_3v_3 \cup Q_3', wQ_4v_4 \cup Q_4') < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Suppose $a_5 \in H'_2$. Then we find a path $Q_2'$ in $H_1$ from $v_2$ to $a_5$ disjoint from $C_w - v_2$ and $S_3 \cap V(v_2Q_2u_2 - \{u_2, v_2\})$ (by minimality of $uC_wv_2$ and $v_2Q_2u_2$). In $H_2 - Q_2'$ we find a path $Q_3'$ from $v_3$ to $a_5$ disjoint from $S_4 \cap V(v_3Q_3u_3 - v_3)$ and $(C_w - v_3) \cup u_3Q_3a_3$ (by (5) and minimality of $v_3Q_3u_3$). In $H_3 - Q_3'$ we find a path $Q_4'$ from $v_4$ to $a_5$ disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$ (by (4) and minimality of $v_4Q_4a_4$). In $H_4 - Q_4'$, we find a path $Q_1'$ from $u_4$ to $a_5$ disjoint from $C_w - u_4$ and $Q_1$ (by (3) and minimality of $w_1C_wv$). Then $\alpha(Q_1^* + \{w, w_1\}, wQ_2v_2 \cup Q_2', wQ_3v_3 \cup Q_3', wQ_4v_4 \cup Q_4') < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Now assume $a_5 \notin H'_3$. Then there is a path $Q_2'$ in $H_1$ from $v_2$ to $a_5$ disjoint from $C_w - v_2$ and $S_3 \cap V(v_2Q_2u_2 - \{u_2, v_2\})$ (by minimality of $uC_wv_2$ and $v_2Q_2u_2$). In $H_2 - Q_2'$ there is a path $Q_3'$ from $v_3$ to $a_5$ disjoint from $C_w - v_3$ and $S_4 \cap V(v_3Q_3u_3 - v_3)$ (by (5) and minimality of $v_3Q_3u_3$). In $H_3 - Q_3'$ there is a path $Q_4'$ from $v_4$ to $a_5$ disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$ (by (4) and minimality of $v_4Q_4a_4$). In $H_4 - Q_4'$, we find a path
$Q_1'$ from $w_1$ to $a_4$ disjoint from $C_w - w_1$ and $Q_1$ (by (3) and minimality of $w_1 C_w v$). Then 
$\alpha(Q_1'+\{w,wv\},wQ_2v_2\cup Q_2',wQ_3v_3\cup Q_3',wQ_4v_4\cup Q_4') < \alpha(Q_1,Q_2,Q_3,Q_4)$, contradicting (2).

Finally, assume $a_5 \in H_1'$. Then we find a path $Q_2'$ in $H_1$ from $v_2$ to $a_2$ disjoint from $C_w - v_2$ and $S_3 \cap V(v_2Q_2w_2 - \{v_2,v_3\})$ (by minimality of $uC_w v_2$ and $v_2Q_2w_2$). In $H_2 - Q_2'$ we find a path $Q_3'$ from $v_3$ to $a_3$ disjoint from $C_w - v_3$ and $S_4 \cap V(v_3Q_3w_3 - v_3$) (by (5) and minimality of $v_3Q_3w_3$). In $H_3 - Q_3'$ we find a path $Q_4'$ from $v_4$ to $a_4$ disjoint from $C_w - v_4$ and $S_1 \cap V(v_4Q_4w_4 - v_4$) (by (4) and minimality of $v_4Q_4w_4$). In $H_4 - Q_4'$, we find a path $Q_1'$ from $w_1$ to $a_5$ disjoint from $C_w - w_1$ and $Q_1$ (by (3) and minimality of $w_1 C_w v$). Again, 
$\alpha(Q_1'+\{w,wv\},wQ_2v_2\cup Q_2',wQ_3v_3\cup Q_3',wQ_4v_4\cup Q_4') < \alpha(Q_1,Q_2,Q_3,Q_4)$, contradicting (2) and completing the proof of (7).

Thus there exists $w' \in N(w) \cap V(v_1 C_w u - u)$; for, otherwise, it follows from (7) that 
$\{u,u_2,u_3,u_4,v\}$ is a cut in $G$ separating $A$ from $w$, contradicting (1). We choose such $w'$
that $v_1 C_w w'$ is minimal. We now apply the arguments (3) – (7), using $uw' \cup v_1 C w' \cup v_1Q_1a_1$
(instead of $Q_1$), $Q_2,Q_3,Q_4$, and using counter-clockwise order instead of clockwise order. As
a consequence and by planarity, there exist $u' \in V(v_1 C w')$, $u_2' \in V(v_2Q_2a_2)$, $u_3' \in V(u_3Q_3a_3)$,
$u_4' \in V(u_4Q_4a_4)$, and $v' \in V(vC v_1)$ such that $u'$ and $u_2'$ are cofacial, $u_3'$ and $u_4'$ are cofacial,
and $u_3'$ and $u_4'$ are cofacial. Moreover, if $H_1''$ denote the $\{u'_i,u'_{i+1}\}$-bridge of $H_1$
containing $v_i C w v_{i+1}$, $1 \leq i \leq 4$, with $u'_1 = u'$, $u'_5 = v'$ and $v_5 = v$, then $a_5 \notin H_1''$. However,
$\{u'_i,u'_2,u'_3,u'_4,v'\}$ is a cut in $G$ separating $A$ from $w$, contradicting (1).

**Theorem 4.3** Let $G$ be a graph drawn in a closed disc in the plane with no edge crossings,
let $a_1,a_2,a_3,a_4,a_5$ be distinct vertices of $G$ on the boundary of the disc in clockwise order, and
let $A = \{a_1,a_2,a_3,a_4,a_5\}$. Suppose $G$ is $(5,A)$-connected and $|V(G)| \geq 7$. Then there exist
$w \in V(G) - A$, a cycle $C_w$ in $(G - A) - w$, and four paths $P_1,\ldots,P_4$ from $w$ to $A$ such that

(i) $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, and

(ii) there exist $1 \leq i \neq j \leq 4$ such that $a_1$ is an end of $P_i$ and $a_5$ is an end of $P_j$.

**Proof.** Assume the assertion is false, and let $G$ be a counterexample with $|V(G)|$ minimal.
Then

(1) $G$ has no 5-separation $(G_1,G_2)$ such that $A \subseteq G_1$ and $|V(G)| > |V(G_2)| \geq 7$.

For, suppose such a separation $(G_1,G_2)$ does exist. By Menger’s theorem, there are five disjoint
paths $R_1,R_2,R_3,R_4,R_5$ in $G_1$ from $V(G_1 \cap G_2)$ to $A$. By choosing notation appropriately,
we may assume $a_i \in R_i$ for $i = 1,\ldots,5$. Let $b_i$ be the other end of $R_i$. Note that $G_2$ is
$(5,\{b_1,b_2,b_3,b_4,b_5\})$-connected. Then by the choice of $G$ and by appropriate notation change,
there exist $w \in V(G_2) - \{b_1,b_2,b_3,b_4,b_5\}$, a cycle $C_w$ in $G_2 - \{w,b_1,b_2,b_3,b_4,b_5\}$, and
four paths $Q_1,Q_2,Q_3,Q_4$ in $G_2$ from $w$ to $\{b_1,b_2,b_3,b_4,b_5\}$, such that $V(Q_i \cap Q_j) = \{w\}$
for $1 \leq i \neq j \leq 4$, $|V(Q_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, and $b_1 \in P_i$ and $b_5 \in P_j$ for some $1 \leq i \neq j \leq 4$.
Now $P_i := Q_i \cup R_i$, $i = 1,\ldots,4$, are the desired paths.

By Lemma 4.1, $G - A$ is 2-connected. So $|V(G) - A| \geq 3$. Hence by (1), each $a_i$ has
at least two neighbors in $G - A$, and so $G$ is 2-connected. Let $C,C'$ denote the outer cycles
of $G,G - A$, respectively. By Lemma 4.1 again, $(G - A) - C'$ is nonempty, and for each
$w \in V(G - A) - V(C')$, the vertices of $G$ that are cofacial with $w$ induce a cycle $C_w$, and $C_w \subseteq G - A$. Thus, we may choose $w$ so that whenever possible the following hold:

(2) if both $a_1$ and $a_5$ have exactly two neighbors on $C'$ and $a_1$ and $a_5$ share a common neighbor $x$ with $d(x) = 5$, then $wx \notin E(G)$, and

(3) $w$ and $a_1$ have a common neighbor, or $w$ and $a_5$ have a common neighbor.

By Lemma 4.2, there exist paths $P_1, P_2, P_3, P_4$ from $w$ to $A$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$. Let $V(P_i \cap C_w) = \{w_i\}$ for $i = 1, 2, 3, 4$. If for some $1 \leq i \leq 4$, $|V(P_i) \cap A| = 2$ then we may replace it with its subpath between $w_i$ and the vertex that is in $A \cap V(P_i)$ but is not an end of $P_i$. So we may assume that $A \not\subseteq \bigcup_{i=1}^4 P_i$.

If $a_1 \in P_i$ and $a_5 \in P_j$ for some $i \neq j$, then the assertion of the theorem holds. So we may assume by symmetry (between $a_1$ and $a_5$) that $a_1 \notin P_i$ for $1 \leq i \leq 4$. By changing notation if necessary we may assume $a_2 \in P_2, a_3 \in P_3, a_4 \in P_4$ and $a_5 \in P_1$. See Figure 3 for an illustration.

![Figure 3: The regions divided by $P_1, P_2, P_3, P_4$.](image)

Note that $P_1, P_2, P_3, P_4$ divide the disc into four closed regions. Let $H_i$ (for each $1 \leq i \leq 4$) denote the maximal subgraph of $G - w$ contained in the closed region which has $P_i$ and $P_{i+1}$ in its boundary, where $P_5 = P_1$. Then $a_1 \in H_1$ (by planarity). We may further assume that the paths $P_1, P_2, P_3, P_4$ are chosen so that

(4) $H_1$ is maximal.

Since $G$ is $(5, A)$-connected, $G - (A - \{a_1\})$ has a path $P$ from $a_1$ to $a \in P_1 \cup P_2 \cup w_1 C w_2$ such that $P - a$ is disjoint from $P_1 \cup P_2 \cup w_1 C w_2$. By planarity, any such path is contained in $H_1$. Moreover, we may assume that for any such choice of $P$, we have $a \notin P_2$; for otherwise, $P_1, w P_2 a \cup P, P_3, P_4$ are the desired paths. Then by planarity and the existence of $P_1$, there exist $v \in V(w_1 C w_2 - w_2)$ and a path $P$ in $H_1$ from $a_1$ to $v$ disjoint from $P_2 \cup (C w - v)$. We choose such $P$ that $v C w_2$ is minimal. Then $v \in a_1 C a_2$. Also, we may assume $w v \notin E(G)$; or else $P_1, P + \{w, wv\}, P_3, P_4$ give the desired paths. We claim that
(5) \( a_4C_5 \cap (w_4C_ww_1 - w_1) = \emptyset \).

For, otherwise, let \( b \in V(a_4C_5) \cap V(w_4C_ww_1 - w_1) \). Then \( \{b, w, v\} \) is a cut in \( G \) separating \( \{w_1, a_1, a_5\} \) from \( \{a_2, a_3, a_4\} \). So by (1), \( b \) and \( w_1 \) are the only neighbors of \( a_5 \) on \( C \), \( v \) and \( w_1 \) are the only neighbors of \( a_1 \) on \( C \), and \( N(w_1) = \{a_1, a_5, b, v, w\} \). Let \( w' \in N(v) \cap V(C_w - w_1) \). Since \( d(v) \geq 5 \) and \( w \notin E(G) \), \( w' \notin C \). It is easy to see that \( w' \) contradicts the choice of \( w \) in (2) (but satisfies (3)), completing the proof of (5).

**Case 1.** Suppose there exists a path \( Q \) from \( a_1 \) to \( a \in V(P_1) \) such that \( (Q - a) \cap C_w = \emptyset \), \( Q \cap P_2 = \emptyset \), and \( Q \cap P_1 = \{a\} \).

Choose \( Q \) so that \( aP_1w_1 \) is minimal. If \( a_4C_5 \cap w_1P_1a = \emptyset \), then by (5), \( P_4 \cup a_4C_5 \) contains a path \( P'_4 \) from \( a_5 \) to \( w \) such that \( P'_4 \cap C_w = \{w_4\} \); and so \( Q \cup aP_1w, P_2, P_3, P_4 \) give the desired paths.

Hence, we may assume \( a_4C_5 \cap w_1P_1a \neq \emptyset \). Then by (4), \( aP_1a_5 = aC_5 \). By (1), \( \{a, v\} \) cannot separate \( \{a_5, a_1\} \) from \( C_w \cup \{a_2, a_3, a_4\} \). Hence by the minimality of \( aP_1w_1 \), there is a path \( R \) in \( H_1 - aC_5 \) from \( a_1 \) to some \( u \in V(w_1C_wv) - \{w_1, v\} \). We choose such \( u \) that \( w_1C_wu \) is minimal. We may assume \( wu \notin E(G) \); or else \( R + \{w, wu\}, P_2, P_3, P_4 \) give the desired paths.

Suppose \( w \) has a neighbor, say \( w' \), in \( wC_wv - \{u, v\} \). Note that \( \{a, u, v, w\} \) is a cut of \( G \); and let \( H' \) denote the \( \{a, u, v, w, w'\} \)-bridge of \( G \) containing \( a_1 \) and \( a_5 \). Then since \( G \) is \( (5, A) \)-connected and by planarity (and also because of \( P \) and \( R \)), \( H' \) contains a path \( P' \) from \( a_1 \) disjoint from \( (C_w - w') - aC_5 \). Now the assertion of the theorem holds with \( C_w \) and the paths \( P_1, R' + \{w, w'\}, P_2, P_3 \).

Therefore we may assume that such \( w' \) does not exist. Then \( \{a, u, v\} \) is a cut in \( G \) separating \( \{a_1, a_5\} \) from \( \{a_2, a_3, a_4, w\} \). So by (1), there is a vertex \( x \) such that \( x \) and \( a \) are the only neighbors of \( a_5 \) on \( C \), \( x \) and \( v \) are the only neighbors of \( a_1 \) on \( C \), and \( N(x) = \{a_1, a_5, a, u, v\} \). Since \( d(u) \geq 5 \) and \( wu \notin E(G) \), we see that \( a \neq w_1 \). So \( w \) has no common neighbor with any of \( a_1 \) and \( a_5 \). Let \( w' \in N(a) \) such that \( w' \notin C \) and \( w \neq u \), which exists because \( d(a) \geq 5 \). Then by planarity, \( w'x \notin E(G) \), and \( w' \) has a common neighbor with \( a_1 \). So \( w' \) contradicts the choice of \( w \) in (3).

**Case 2.** There exists \( u \in V(w_1C_ww_2) - \{w_1, w_2\} \) such that all paths from \( a_1 \) to \( C_w \) must intersect \( uC_wv \).

We choose such \( u \) so that \( uC_wv \) is minimal. Then \( \{u, v, w\} \) is a 3-cut in \( G \) separating \( a_1 \) from \( A - \{a_1\} \). Since \( G \) is \( (5, A) \)-connected, the component of \( G - \{u, v, w\} \) containing \( a_1 \) has exactly one vertex. So by planarity, \( u \) and \( v \) are the only neighbors of \( a_1 \) in \( G \), and \( uv \) is an edge of \( C_w \). Thus \( u, v \in C \).

If \( wu \in E(G) \) then \( wwa_1, P_2, P_3, P_4 \) give the desired paths, and if \( ww \in E(G) \) then \( P_1, wva_1, P_3, P_4 \) give the desired paths. So we may assume \( wu, ww \notin E(G) \). Without loss of generality, we may assume \( w_1, u, v, w_2 \) occur on \( C_w \) in this clockwise order.

Clearly, \( w \) and \( a_1 \) have no common neighbors. Moreover, \( w \) and \( a_5 \) have no common neighbor. For, otherwise, let \( b \in N(w) \cap N(a_5) \). Then since \( d(u) \geq 5 \) and \( wu \notin E(G) \), \( \{b, u, w, a_5\} \) is a cut in \( G \), contradicting the assumption that \( G \) is \( (5, A) \)-connected.

Let \( v_1 \in V(C_w) - \{u, v\} \) such that \( v_1v \in E(G) \). Since \( G \) is \( (5, A) \)-connected and \( vw \notin E(G) \), \( v_1 \notin C \). By Lemma 4.1, the vertices of \( G \) which are cofacial with \( v_1 \) induce a cycle in \( G - A \). Note that \( v_1 \) has no common neighbor with \( a_5 \) (i.e. satisfying (2)); however, \( v_1 \) and \( a_1 \) have \( v \) as a common neighbor. So \( v_1 \) contradicts the choice of \( w \) in (3).

\( \blacksquare \)
Proof of Theorem 1.2. Let \((G_1, G_2)\) be a 5-separation in \(G\) such that \(V(G_1 \cap G_2) = \{a_1, a_2, a_3, a_4, a_5\}\) and \(|V(G)| > |V(G_2)| \geq 7\). Moreover, assume that \(G_2\) may be drawn in a closed disc in the plane without edge crossings such that \(a_1, a_2, a_3, a_4, a_5\) occur on the boundary of the disc in clockwise order. Note that \(|G_1| \geq 2\) (since \(G\) is not planar). Let \(A = \{a_1, a_2, a_3, a_4, a_5\}\). We may choose \((G_1, G_2)\) such that \(G_2\) is maximal. Then each \(a_i\) has at least two neighbors in \(G_1\), and \(A\) is an independent set in \(G_1\). Hence \(G_1 - a_i\) is 2-connected for \(1 \leq i \leq 5\).

By Theorem 4.3, there exist \(w_1 \in V(G_2) - A\), a cycle \(C_{w_1}\) in \((G_2 - A) - w_1\), and four paths \(P_1, P_2, P_3, P_4\) in \(G_2\) from \(w_1\) to \(A\) such that \(V(P_i \cap P_j) = \{w_1\}\) for \(1 \leq i < j \leq 4\), and \(|V(P_i \cap C_{w_1})| = 1\) for \(1 \leq i \leq 4\). Without loss of generality, we may assume that \(a_i\) is an end of \(P_i\), \(1 \leq i \leq 4\).

If \(G_1 - a_5\) contains disjoint paths \(A_1, A_2\) from \(a_1, a_2\) to \(a_3, a_4\), respectively, then \(C_{w_1} \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup A_1 \cup A_2\) is a TK\(_5\) in \(G\).

So we may assume that such \(A_1, A_2\) do not exist. By Theorem 2.2 and by the fact that \(G_1\) is \((5, A)\)-connected, \(G_1 - a_5\) can be drawn in a closed disc in the plane with no edge crossings such that \(a_1, a_2, a_3, a_4\) occur on the boundary of the disc in this cyclic order. Let \(C\) denote the outer cycle of \(G_1 - a_5\). Since \(G\) is nonplanar, \(a_5\) has at least one neighbor, say \(a\), such that \(a \notin a_4Ca_1\).

By Theorem 4.3 there exist \(w_2 \in V(G_2) - A\), a cycle \(C_{w_2}\) in \((G_2 - A) - w_2\), and paths \(Q_1, Q_2, Q_3, Q_4\) in \(G_2\) from \(w_2\) to \(A\) such that \(V(Q_i \cap Q_j) = \{w_2\}\) for \(1 \leq i < j \leq 4\), \(|V(Q_i \cap C_{w_2})| = 1\) for \(1 \leq i \leq 4\), \(a_4\) is an end of \(Q_3\), and \(a_5\) is an end of \(Q_4\). Let \(a_s, a_t\) be the ends of \(Q_1, Q_2\) with \(1 \leq s < t \leq 3\). If \((G_1 - a_5) - a_4Ca_s\) has a path \(R\) from \(a\) to \(a_t\), then \(C_{w_2} \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup (R \cup a_5a) \cup a_tCa_s\) is a TK\(_5\) in \(G\).

So we may assume that such a path \(R\) does not exist. Then \(s = 2\) and \(a \in a_3Ca_s\). By Theorem 4.3 there exist \(w_3 \in V(G_2) - A\), a cycle \(C_{w_3}\) in \((G_2 - A) - w_3\), and paths \(R_1, R_2, R_3, R_4\) in \(G_2\) from \(w_3\) to \(A\) such that \(V(R_i \cap R_j) = \{w_3\}\) for \(1 \leq i < j \leq 4\), \(|V(R_i \cap C_{w_3})| = 1\) for \(1 \leq i \leq 4\), \(a_1\) is an end of \(R_1\), and \(a_5\) is an end of \(R_4\). Let \(a_s, a_t\) be the ends of \(R_2, R_3\) with \(2 \leq s < t \leq 4\). Now \(C_{w_3} \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup a_1Ca_1 \cup (a_5a \cup aCa_s)\) is a TK\(_5\) in \(G\). 

Acknowledgment. We thank an anonymous referee for helpful suggestions.

References


