Abstract

Thomassen proved that every $(k + 3)$-connected graph $G$ contains an induced cycle $C$ such that $G - V(C)$ is $k$-connected, establishing a conjecture of Lovász. In general, one could ask the following question: For any positive integers $k, l$, does there exist a smallest positive integer $g(k, l)$ such that for any $g(k, l)$-connected graph $G$, any $X \subseteq V(G)$ with $|X| = k$, and any $e \in E(G - X)$, there is an induced cycle $C$ in $G - X$ such that $e \in E(C)$ and $G - V(C)$ is $l$-connected? The case when $k = 0$ is a well-known conjecture of Lovász which is still open for $l \geq 3$. In this paper, we prove $g(k, 1) \leq 10k + 1$ and $g(k, 2) \leq 10k + 11$. We also consider a weaker version: For any positive integers $k, l$, is there a smallest positive integer $f(k, l)$ such that for every $f(k, l)$-connected graph $G$ and any $X \subseteq V(G)$ with $|X| = k$, there is an induced cycle $C$ in $G - X$ such that $G - V(C)$ is $l$-connected? The case when $k = 0$ was studied by Thomassen. We prove $f(k, l) \leq 2k + l + 2$ and $f(k, 1) = k + 3$.

Keywords: connectivity, non-separating cycle, $k$-contractible edge
1 Introduction

We begin with notation necessary for describing problems and results in this paper. Let $G$ be a graph; we use $V(G)$ and $E(G)$ to denote its vertex and edge set, respectively. For any $e \in E(G)$, $V(e)$ denotes the set of vertices of $G$ incident with $e$. By $H \subseteq G$ we mean that $H$ is a subgraph of $G$, and we view any subset of vertices as a subgraph with no edges. For any $U \subseteq G$, the neighborhood of $U$, denoted by $N_G(U)$, is the set of vertices in $V(G) - V(U)$ adjacent to at least one vertex in $U$; and $N_G[U] := N_G(U) \cup U$ is the closed neighborhood of $U$ in $G$. The degree $d_G(u)$ of a vertex $u$ in $G$ is $|N_G(\{u\})|$. If the graph $G$ is clear from the context, the reference to $G$ is usually omitted. For any $U \subseteq G$, $G[U]$ denotes the subgraph of $G$ induced by $V(U)$ and we write $G - U := G[V(G) - V(U)]$. Let $k$ be a positive integer. A graph $G$ is $k$-connected if $|V(G)| \geq k + 1$ and $G - U$ is connected for any $U \subseteq V(G)$ with $|U| < k$.

Lovász (see [14]) conjectured the existence of a function $g(k)$ such that for any positive integer $k$, any $g(k)$-connected graph $G$, and any distinct $s,t \in V(G)$, there exists a path $P$ in $G$ between $s$ and $t$ such that $G - V(P)$ is $k$-connected. This is equivalent to the problem for asking the existence of such a cycle $C$ through a specified edge. A result of Tutte [16] shows that $g(1) = 3$; and that $g(2) = 5$ was proved independently by Chen, Gould and Yu [2] and by Kriesell [9]. Lovász’s conjecture remains open for $k \geq 3$.

The result of Tutte [16] showing $g(1) = 3$ is actually stronger: For every 3-connected graph $G$, $e \in E(G)$, and $u \in G - V(e)$, there exists an induced cycle $C$ in $G - u$ such that $e \in E(C)$ and $G - V(C)$ is connected. It is conjectured in [8] that there exists a function $f(k)$ such that every $f(k)$-connected graph $G$ and every $s,t,u \in V(G)$, there is a path $P$ between $s$ and $t$ in $G$ and a $k$-connected subgraph $H$ of $G$ such that $u \in V(H)$ and $V(H) \cap V(P) = \emptyset$. It is further shown that this conjecture implies the above conjecture of Lovász. We feel that in potential applications one may need $P$ and $C$ to avoid certain vertices (see [17] for another example), and believe the following is true.

**Conjecture 1.1.** For any positive integers $k,l$, there exists a smallest positive integer $g(k,l)$ such that for any $g(k,l)$-connected graph $G$, any $e \in E(G)$, and any $X \subseteq V(G) - V(e)$ with $|X| = k$, there exists an induced cycle $C$ in $G - X$ such that $e \in E(C)$ and $G - V(C)$ is $l$-connected.

When $k = 0$, Conjecture 1.1 is simply Lovász’s conjecture mentioned above. We provide evidence to Conjecture 1.1 by proving

**Theorem 1.2.** For any positive integer $k$, $g(k,1) \leq 10k + 1$ and $g(k,2) \leq 10k + 11$.

The proof of Theorem 1.2 is given in Section 2, where we also mention a result about connectivity of $k$-linked graphs that is needed in our proof.

It turns out that asking the cycle in Lovász’s conjecture to go through a specified edge is what makes the conjecture difficult. Thomassen [13] proved that for any positive integer $k$, every $(k + 3)$-connected graph contains a cycle $C$ such that $G - V(C)$ is $k$-connected, establishing a conjecture of Lovász [10]. (This result was further strengthened by Egawa [3,4] for graphs with girth at least 4 or 5.) We consider a similar relaxation of Conjecture 1.1: For any positive integers $k,l$, there exists a smallest positive integer $f(k,l)$ such that for any $f(k,l)$-connected graph $G$ and any $X \subseteq V(G)$ with $|X| = k$, there is an induced cycle $C$ in
A solution to every linkage problem in a graph is strongly $k$-linked. To prove Theorem 1.2 we need a result about connectivity of $k$-linked graphs. A linkage problem in a graph $G$ is a set of pairs of vertices of $G$, written as $L = \{\{s_1,t_1\},\ldots,\{s_k,t_k\}\}$. A solution to $L$ is a set of paths $\{P_1, \ldots, P_k\}$ such that $s_i, t_i$ are the ends of $P_i$ and, for any $i \neq j$ and any $x \in V(P_i) \cap V(P_j)$, $x \in \{s_i, t_i\} \cap \{s_j, t_j\}$. A graph $G$ is called $k$-linked if every linkage problem with $k$ pairwise disjoint pairs of vertices has a solution. A graph $G$ is strongly $k$-linked if every linkage problem in $G$ consisting of $k$ pairs has a solution. Bollobás and Thomason [1] proved that every $22k$-connected graph is $k$-linked. In [15] Thomas and Wollan improve this bound to $10k$.

**Lemma 2.1.** (Thomas and Wollan). Every $10k$-connected graph is $k$-linked.

A result of Mader [11] implies that any $k$-linked graph on at least $2k$ vertices is strongly $k$-linked. Thus the following statement follows trivially from Lemma 2.1.

**Corollary 2.2.** Every $10k$-connected graph is strongly $k$-linked.

For a path $P$ and $u, v \in V(P)$, we use $uPv$ to denote the subpath of $P$ between $u$ and $v$, and we view $P$ as a sequence of vertices. Let $G$ be a graph and $B \subseteq G$. A $B$-bridge of $G$ is a subgraph of $G$ induced by all edges in a component of $G - V(B)$ and all edges from that component to $B$.

**Proof of Theorem 1.2.** We break this proof into two cases. In Case 1 we prove $g(k,1) \leq 10k + 1$; and in Case 2, we show $g(k,2) \leq 10k + 11$.

**Case 1.** Let $G$ be a $(10k + 1)$-connected graph, $e = st \in E(G)$, and $X = \{x_1, \ldots, x_k\} \subseteq G - V(e)$. We need to show that there is an induced cycle $C$ in $G - X$ such that $e \in E(C)$ and $G - V(C)$ is connected.

Note that $G - e$ is $10k$-connected. Consider the linkage problem $L = \{\{s, t\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}\}$ in $G - e$, which has $k$ pairs of vertices. By Corollary 2.2, there is a solution $\{P, Q_1, \ldots, Q_{k-1}\}$ to $L$ such that $P$ is from $s$ to $t$, and $Q_i$ is from $x_i$ to $x_{i+1}$ for $i = 1, \ldots, k-1$. We may assume all paths are induced in $G - e$. Note that $X \subseteq \bigcup_{i=1}^{k-1} Q_i$, and $\bigcup_{i=1}^{k-1} Q_i$ is a connected subgraph of $G - e - V(P)$.

Thus $G - e$ has an induced path $P$ between $s$ and $t$ such that $X$ is contained in a connected component $C_0$ of $G - e - V(P)$. Let $C_1, C_2, \ldots, C_q$ be the other components of $G - V(P)$ (if any) such that $|V(C_1)| \geq |V(C_2)| \geq |V(C_q)|$, and let $S(P) := (|V(C_0)|, |V(C_1)|, \ldots, |V(C_q)|)$. We choose $P$ so that $S(P)$ is maximal with respect to the lexicographic ordering.
If \( q = 0 \) then \( G - V(P) \) is connected; so \( C := G[V(P)] \) is the desired cycle showing that \( g(k, 1, 1) \leq 10k+1 \). We may thus assume that \( q > 0 \). Note that \( |N(C_q) \cap V(P)| \geq 10k+1 \). Choose two vertices \( u, v \in N(C_q) \cap V(P) \) such that \( uPv \) is maximum. Without loss of generality we may assume that \( s, u, v, t \) occur on \( P \) in this order. Let \( Q \) be an induced path in \( G[C_q \cup \{u, v\}] \) between \( u \) and \( v \). Then \( P' := sPuQvPt \) is an induced path in \( G - X \) between \( s \) and \( t \). Since \( G \) is \((10k + 1)\)-connected, \( uPv - \{u, v\} \) has a neighbor in \( \bigcup_{i=0}^{q-1} V(C_i) \). So \( S(P') \) is larger than \( S(P) \), a contradiction.

**Case 2.** Let \( G \) be a \((10k + 11)\)-connected graph, \( e = st \in E(G) \), and \( X = \{x_1, \ldots, x_k\} \subseteq V(G) - V(e) \). We show that \( G - X \) contains an induced cycle \( C \) such that \( e \in E(C) \) and \( G - V(C) \) is 2-connected.

Note that \( G - e \) is \((10(k + 1)\)-connected. Consider the linkage problem \( \mathcal{L} = \{\{s, t\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_1\}\} \), which has \( k+1 \) pairs of vertices. By Corollary 2.2, there is a solution \( \{P, Q_1, \ldots, Q_k\} \) to \( \mathcal{L} \) such that \( P \) is from \( s \) to \( t \) and \( Q_i \) is from \( x_i \) to \( x_{i+1} \) for \( i = 1, \ldots, k \) (with \( x_{k+1} = x_1 \)).

Since \( X \subseteq \bigcup_{i=1}^k Q_i \) which is a cycle, \( X \) is contained in a 2-connected block \( B_0 \) of \( G - V(P) \). Let \( B_1, \ldots, B_q \) be the \( B_0 \)-bridges of \( G - V(P) \) (if any) such that \( |V(B_1)| \geq |V(B_2)| \geq \ldots \geq |V(B_q)| \), and let \( S(P) := (|V(B_0)|, |V(B_1)|, \ldots, |V(B_q)|) \). Note that \( |V(B_1 \cap B_0)| \leq 1 \). We choose \( P \) so that \( S(P) \) is maximal with respect to the lexicographic ordering. We may assume that \( P \) is induced in \( G - e \).

If \( q = 0 \) then \( G - V(P) \) is 2-connected; so \( C := G[V(P)] \) is the desired cycle showing that \( g(k, 2) \leq 10k + 11 \). Hence we may assume \( q > 0 \). Since \( |V(B_q \cap B_0)| \leq 1 \) and \( G - e \) is \((10(k + 1)\)-connected, we may let \( u, v \in N(B_q - V(B_q \cap B_0)) \) such that \( uPv \) is maximal. Without loss of generality we may assume that \( s, u, v, t \) occur on \( P \) in order. Let \( Q \) be an induced path in \( G[B_q \cup \{u, v\}] - V(B_q \cap B_0) \) between \( u \) and \( v \). Then \( P' := sPuQvPt \) is an induced path in \( G - e - X \) between \( s \) and \( t \). Since \( G \) is \((10k + 11)\)-connected, \( uPv - \{u, v\} \) has a neighbor in \( \bigcup_{i=1}^{q-1} B_i \), or at least two neighbors in \( B_0 \). So \( S(P') \) is larger than \( S(P) \), a contradiction.

The bounds in Theorem 1.2 are probably far from being best possible. One way to reduce these bounds is to find out the minimum connectivity of \( G \) that guarantees the existence of disjoint connected subgraphs \( P \) and \( H \) such that \( \{s, t\} \subseteq V(P) \) and \( X \subseteq V(H) \). When \( |X| = 2 \), \( G \) needs to be 6-connected by a result of Jung [6]; but this is not known for \( |X| \geq 3 \). The problem is more difficult if, in addition, one requires \( H \) to be \( l \)-connected for some \( l \geq 2 \).

### 3 Contractible edges

For our proof of Theorem 1.3, we need the concept of a contractible edge. Let \( G \) be a \( k \)-connected graph and \( e \in E(G) \). We say that \( e \) is \( k \)-contractible if the graph obtained from \( G \) by contracting \( e \), denoted by \( G/e \), is \( k \)-connected.

Clearly every edge in a 1-connected graph (other than \( K_2 \)) is 1-contractible. An edge \( e \) in a 2-connected graph \( G \) is 2-contractible iff \( G - V(e) \) is connected; from this one can see that any 2-connected graph other than \( K_3 \) contains a lot of 2-contractible edges. Tutte [16] showed that \( K_4 \) is the only 3-connected graph which does not admit any 3-contractible edge. Fontet [5] and, independently, Martinov [12] proved that if a 4-connected graph contains no 4-contractible edge then it is the square of a cycle of length at least 5 or it is the line graph of
a cyclically 4-edge-connected cubic graph. For general \(k\), Thomassen [13] proved that if \(G\) is a \(k\)-connected graph with no triangles then \(G\) admits a \(k\)-contractible edge. This is then used in [13] to prove the following result (establishing a conjecture of Lovász [10]).

**Lemma 3.1.** (Thomassen). For \(k \geq 4\), every \(k\)-connected graph \(G\) contains an induced cycle \(C\) such that \(G - V(C)\) is \((k - 3)\)-connected and \(|N(u) \cap V(C)| \leq 3\) for all \(u \in V(G) - V(C)\).

For our proof of Theorem 1.3, we need to prove the following lemma about contractible edges avoiding a given set of vertices.

**Lemma 3.2.** Let \(G\) be a \(k\)-connected graph, where \(k \geq 4\), and let \(X \subseteq V(G)\) such that \(|X| \leq k/2 - 1\), \(G - X\) has girth at least 5, and \(|V(G) - N[X]| \geq |X| + 1\). Then there exists \(u \in V(G) - N[X]\) such that \(u\) is incident with a \(k\)-contractible edge of \(G\).

**Proof.** Suppose that no vertex in \(V(G) - N[X]\) is incident with a \(k\)-contractible edge of \(G\). Let \(u \in V(G) - N[X]\). Then for any \(e \in E(G)\) incident with \(u\), \(V(e)\) is contained in a \(k\)-cut \(S_e\) of \(G\). Let \(Q_u\) denote the collection of all quadruples \((e, S_e, A_e, B_e)\) such that \(e \in E(G)\) is incident with \(u\), \(S_e\) is a \(k\)-cut of \(G\) with \(V(e) \subseteq S_e\), \(A_e\) is a component of \(G - S_e\), and \(B_e := G - S_e - A_e\). We choose \(e \in E(G)\) incident with \(u\) such that

1. \((e, S_e, A_e, B_e) \in Q_u\) and \(|A_e|\) is minimal.

We claim that

2. for each \(u \in V(G) - N[X]\) and each \((e, S_e, A_e, B_e) \in Q_u\) satisfying (1), \(|A_e| \leq k - 2\).

For, suppose that there exist \(u \in V(G) - N[X]\) and \((e, S_e, A_e, B_e) \in Q_u\) satisfying (1) such that \(|A_e| \geq k - 1\). Let \(e = uv\). Since \(G\) is \(k\)-connected, \(u\) is adjacent to some \(w \in V(A_e)\). Let \(f = uw\). Then there exists a quadruple \((f, S_f, A_f, B_f) \in Q_u\).

We claim that \(A_e \cap A_f = \emptyset\) or \(B_e \cap B_f = \emptyset\). For suppose \(A_e \cap A_f \neq \emptyset\) and \(B_e \cap B_f \neq \emptyset\). Then \(T_1 := (S_e \cap V(A_f)) \cup (S_e \cap S_f) \cup (V(A_e) \cap S_f)\) is a cut of \(G\) and contains \(V(f)\), and \(T_2 := (S_e \cap V(B_f)) \cup (S_e \cap S_f) \cup (V(B_e) \cap S_f)\) is a cut of \(G\). Thus \(|T_1| \geq k + 1\) (by (1)) and \(|T_2| \geq k\) (since \(G\) is \(k\)-connected). So

\[
2k = |S_e| + |S_f| = |T_1| + |T_2| \geq 2k + 1,
\]
a contradiction.

Similarly, we can show that \(A_e \cap B_f = \emptyset\) or \(A_f \cap B_e = \emptyset\).

Suppose \(B_e \cap B_f = \emptyset\). If \(A_f \cap B_e = \emptyset\) then by (1), \(k - 1 \leq |A_e| \leq |B_e| = |V(B_e) \cap S_f| \leq k - 2\), a contradiction. So \(A_f \cap B_e \neq \emptyset\). Thus \(A_e \cap B_f = \emptyset\), and \(S_e \cap V(A_f) \neq \emptyset\) as \(G\) is \(k\)-connected. Hence, by (1), \(k - 1 \leq |A_e| \leq |B_f| = |V(B_f) \cap S_e| \leq k - 2\), a contradiction.

Therefore, \(B_e \cap B_f \neq \emptyset\). Similarly, \(B_e \cap A_f \neq \emptyset\). So \(A_e \cap A_f = \emptyset\) and \(A_e \cap B_f = \emptyset\). Moreover, \(S_e \cap V(A_f) \neq \emptyset\) as \(G\) is \(k\)-connected; so \(S_f \cap V(B_e) \neq \emptyset\) as \(G\) is \(k\)-connected. Hence \(|A_e| \leq k - 2\), a contradiction which completes the proof of (2).

Since \(G\) is \(k\)-connected, \(u\) has a neighbor in \(A_e\); hence, since \(u \in V(G) - N[X]\), \(A_e - X \neq \emptyset\). In fact,

3. for any \(u \in V(G) - N[X]\) and for any \((e, S_e, A_e, B_e) \in Q_u\) satisfying (1), we have \(|V(A_e) - X| = 1|.

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First, suppose $|V(A_e) - X| \geq 3$, and let $w_1, w_2, w_3 \in V(A_e) - X$. We wish to estimate $|N(w_1) \cup N(w_2) \cup N(w_3)|$. Applying the principle of inclusion and exclusion, we have

$$|N(w_1) \cup N(w_2) \cup N(w_3)| = \left( \sum_{i=1}^{3} |N(w_i)| \right) - \left( \sum_{1 \leq i < j \leq 3} |N(w_i) \cap N(w_j)| \right) + |N(w_1) \cap N(w_2) \cap N(w_3)|.$$ 

Since $G - X$ has girth at least 5, $|N(w_i) \cap N(w_j) - X| \leq 1$, and if $|N(w_1) \cap N(w_2) \cap N(w_3) - X| \neq 0$ then $N(w_1) \cap N(w_2) - X = N(w_2) \cap N(w_3) - X = N(w_1) \cap N(w_3) - X$. So

$$\left| \bigcup_{1 \leq i < j \leq 3} (N(w_i) \cap N(w_j) - X) \right| + |N(w_1) \cap N(w_2) \cap N(w_3) - X| \leq 3.$$

Note that

$$\left| \bigcup_{1 \leq i < j \leq 3} N(w_i) \cap N(w_j) \cap X \right| + |N(w_1) \cap N(w_2) \cap N(w_3) \cap X| \leq 2|X|.$$

Hence,

$$\left( \sum_{1 \leq i < j \leq 3} |N(w_i) \cap N(w_j)| \right) - |N(w_1) \cap N(w_2) \cap N(w_3)|$$

$$= \left| \bigcup_{1 \leq i < j \leq 3} (N(w_i) \cap N(w_j)) \right| + |N(w_1) \cap N(w_2) \cap N(w_3)|$$

$$\leq 2|X| + 3.$$

To see the equality above, we note that each vertex in exactly two of $N(w_1), N(w_2), N(w_3)$ is counted exactly once on both sides, and each vertex belonging to all three is counted exactly twice on both sides. Therefore, since $|X| \leq k/2 - 1$, we have

$$|N(w_1) \cup N(w_2) \cup N(w_3)| \geq \sum_{i=1}^{3} |N(w_i)| - (2|X| + 3) \geq 3k - (2|X| + 3) \geq 2k - 1.$$

Since $|S_e| = k$ and $N(w_i) \subseteq V(A_e) \cup S_e$ for $i = 1, 2, 3$, we have $|A_e| \geq k - 1$, contradicting (2).

Now suppose $|V(A_e) - X| = 2$, and let $V(A_e) - X = \{w_1, w_2\}$. If $w_1 w_2 \notin E(G)$ then, since $G$ is $k$-connected, each $w_i$ has at least $k - |X|$ neighbors in $S_e - X$; so $w_1$ and $w_2$ have at least $2(k - |X|) - k \geq 2$ common neighbors in $S_e - X$ (since $|X| \leq k/2 - 1$), which forces a cycle in $G - X$ of length at most 4, a contradiction. So $w_1 w_2 \in E(G)$; and thus $N(w_1) \cap N(w_2) \cap (S_e - X) = \emptyset$. Hence,

$$k \geq |S_e - X|$$

$$\geq |(N(w_1) \cup N(w_2)) \cap (S_e - X)|$$

$$= |N(w_1) \cap (S_e - X)| + |N(w_2) \cap (S_e - X)|$$

$$\geq 2(k - |X| - 1)$$

$$\geq k \text{ (since } |X| \leq k/2 - 1).$$
This inequality shows that \((N(w_1) \cup N(w_2)) \cap (S_e - X) = S_e - X = S_e\). Since \(u, v \in S_e - X\), 
\(G\{u, v, w_1, w_2\}\) contains a cycle, a contradiction, which completes the proof of (3).

So for any \(u \in V(G) - N[X]\) and for any \((e, S_e, A_e, B_e) \in Q_u\) satisfying (1), we let \(V(A_e) - X = \{w_u\}\). Let \(X_u = V(A_e) \cap X\). Since \(u \in V(G) - N[X]\), we have

\[(4) \ uv_u \in E(G), \ N(X_u) = (S_e - \{u\}) \cup \{w_u\}, \text{ and } N(w_u) - N[X] = \{u\}.\]

We claim that

\[(5) \text{ for any distinct } u, u' \in V(G) - N[X], \text{ any } (e, S_e, A_e, B_e) \in Q_u \text{ satisfying } (1), \text{ and any } (f, S_f, A_f, B_f) \in Q_{u'} \text{ satisfying } (1), \text{ we have } X_u \cap X_{u'} = \emptyset, \text{ where } X_u = X \cap V(A_e) \text{ and } X_{u'} = X \cap V(A_f).\]

Suppose \(X_u \cap X_{u'} \neq \emptyset\). If \(w_u = w_{u'}\), then, by (4), \(\{u\} = N(w_u) - N[X] = N(w_{u'}) - N[X] = \{u'\}\), a contradiction. So \(w_u \neq w_{u'}\).

Since \(u, u' \notin N[X]\), \(N[X_u \cup X_{u'}] \neq V(G)\). Therefore, since \(G\) is \(k\)-connected, \(|N(X_u \cup X_{u'})| \geq k\) and, by (4),

\[2k \leq |N(X_u \cup X_{u'})| + |N(X_u \cap X_{u'})| \leq |N(X_u)| + |N(X_{u'})| = 2k.\]

It follows that \(|N(X_u \cup X_{u'})| = k\). Let \(F_u := N(w_u) - (X \cup \{u\})\) and \(F_{u'} := N(w_{u'}) - (X \cup \{u'\})\). Then \(F_u \subseteq N(X_u) - X\) (by (4)) and \(|F_u| \geq k - |X| - 1\), and \(F_{u'} \subseteq N(X_{u'}) - X\) (by (4)) and \(|F_{u'}| \geq k - |X| - 1\). Note that \(w_u \neq u'\) and \(w_{u'} \neq u\), since \(u, u' \notin N[X]\).

If \(w_u w_{u'} \notin E(G)\) then \(w_u, w_{u'} \notin F_u \cup F_{u'}\). Hence, since \(w_u \in N(X_u) - X\) and \(w_{u'} \in N(X_{u'}) - X\), \(|F_u \cup F_{u'}| \leq |N(X_u \cup X_{u'}) - X| - 2 \leq k - 2\) (as \(|N(X_u \cup X_{u'})| = k\)). So \(|F_u \cup F_{u'}| = |F_u| + |F_{u'}| - |F_u \cap F_{u'}| \geq 2(k - |X| - 1) - (k - 2) = 2\) (as \(|X| \leq k/2 - 1\), which forces a cycle of length 4 in \(G - X\), \(u\), a contradiction.

So \(w_u w_{u'} \in E(G)\). Then \(F_u \cap F_{u'} = \emptyset\), since the girth of \(G - X\) is at least 5. Thus \(|F_u \cup F_{u'}| = |F_u| + |F_{u'}| \geq 2(k - |X| - 1) \geq k\) (as \(|X| \leq k/2 - 1\)). On the other hand, \(|F_u \cup F_{u'}| \leq |N(X_u \cup X_{u'})| = k\). So \(F_u \cup F_{u'} = N(X_u \cup X_{u'})\). Let \(e = uv\). Then by (4), \(v \in N(X_u) - X \subseteq N(X_u \cup X_{u'})\). Hence \(v \in F_u\) as \(v \notin F_u\) (since \(w_u v \notin E(G)\)). Now \(w_u w_{u'} v u\) is a cycle in \(G - X\), a contradiction, which completes the proof of (5).

Since \(X_u \neq \emptyset\) and because of (5), \(|V(G) - N[X]| \leq |\bigcup_{u \in V(G) - N[X]} X_u| \leq |X|\), contradicting the assumption that \(|V(G) - N[X]| \geq |X| + 1\).

When \(|X| = 1\), we have a stronger version of Lemma 3.2.

**Lemma 3.3.** Let \(G\) be a \(k\)-connected graph, where \(k \geq 4\), and let \(x \in V(G)\) such that \(G - x\) has girth at least 5. Then there is a \(k\)-contractible edge not incident with \(x\).

**Proof.** If \(|V(G) - N[x]| \geq 2\) then, by Lemma 3.2, there is a \(k\)-contractible edge in \(G\) not incident with \(X\). So we may assume \(|V(G) - N[x]| \leq 1\). Then \(d_G(x) \geq |V(G - x)| - 1\). Let \(u, v, w \in V(G - x)\) be distinct such that \(uv, vw \in E(G - x)\). Since \(G - x\) has girth at least 5 and \(G - x\) is at least \((k - 1)\)-connected, \(|V(G - x)| \geq (d_{G - x}(u) - 1) + (d_{G - x}(v) - 2) + (d_{G - x}(w) - 1) + 3 \geq 3k - 4\). Since \(k \geq 4\),

\[(1) \ d_G(x) \geq |V(G - x)| - 1 \geq 3k - 5 \geq k + 3.\]
Suppose that all $k$-contractible edges in $G$ are incident with $x$. Then for any edge $e$ not incident with $x$, there is a $k$-cut $S_e$ containing $V(e)$. Let $Q$ denote the set of all quadruples $(e,S_e,A_e,B_e)$ such that $e \in E(G)$ is not incident with $x$, $S_e$ is a $k$-cut of $G$ containing $V(e)$, $A_e$ is a component of $G-S_e$, and $B_e = G-S_e-V(A_e)$. We may choose an edge $e$ not incident with $x$ and choose $(e,S_e,A_e,B_e) \in Q$, such that

(2) $|A_e|$ is minimal.

By (1) and by our assumption that $G - x$ has girth at least 5, we see that $|A_e| \geq 2$. Furthermore,

(3) $|A_e| \geq k - 1$.

Since $|A_e| \geq 2$, $A_e - \{x\} \neq \emptyset$. If $|A_e - \{x\}| = 1$, then $x \in A_e$ and thus $d_G(x) \leq |S_e \cup A_e - \{x\}| = k + 1$, contradicting (1). Hence $|A_e - \{x\}| \geq 2$. Let $w_1,w_2 \in V(A_e) - \{x\}$. Since $G - x$ has no cycle of length less than 5, $|(N(w_1) \cap N(w_2)) - \{x\}| \leq 1$ if $w_1w_2 \not\in E(G)$ and $|(N(w_1) \cap N(w_2)) - \{x\}| = 0$ if $w_1w_2 \in E(G)$. So $|N[w_1] \cup N[w_2]| \geq 2k - 1$. Hence $|A_e| \geq k - 1$, since $|S_e| = k$ and $N[w_1] \cup N[w_2] \subseteq V(A_e) \cup S_e$, which completes the proof of (3).

Since $|A_e| \geq k - 1 \geq 3$, it is easy to see that there is an edge $f$ in $G - x$ such that $V(f) \cap V(A_e) \neq \emptyset$ and $V(f) \cap S_e \neq \emptyset$. Then the same proof for (2) in the proof of Lemma 3.2 works here and gives a contradiction.

We now can prove the following result from which Theorem 1.3 follows directly. Let $\mathcal{F}$ be a family of graphs. A graph $G$ is said to be $\mathcal{F}$-free if no induced subgraph of $G$ is isomorphic to a graph in $\mathcal{F}$.

**Theorem 3.4.** For any positive integers $k,l$, any $(2k+l+2)$-connected graph $G$, and any $X \subseteq V(G)$ with $|X| = k$, there is an induced cycle $C$ in $G-X$ such that $G-V(C)$ is $l$-connected. Moreover, if $G - X$ is $\{K_3,K_{2,3}\}$-free then $|N(u) \cap V(C)| \leq 1$ for any $u \not\in V(C) \cup X$.

**Proof.** Let $G$ be a $(2k+l+2)$-connected graph and let $X \subseteq V(G)$ with $|X| = k$. We proceed by induction on $|V(G)|$. If $|V(G)| = 2k+l+3$, then $G$ is complete; so any triangle in $G - X$ gives the desired cycle. We may thus assume that $|V(G)| \geq 2k+l+4$ and the assertion holds for graphs of order less than $|V(G)|$. We may also assume that

(1) the girth of $G - X$ is at least 5.

For, suppose the girth of $G - X$ is at most 4, and let $C$ be a cycle in $G - X$ with $|C|$ minimum. Then $|C| \leq 4$, $C$ is induced and $G-V(C)$ has connectivity at least $2k+l+2-4 = 2(k-1)+l \geq l$, as $k \geq 1$. Now assume that $G - X$ is $\{K_3,K_{2,3}\}$-free. Then $|C| = 4$, and $|N(u) \cap V(C)| \leq 1$ for each $u \not\in V(C) \cup X$.

Next, we show that

(2) when $k \geq 2$, we may assume that $|V(G) - N[X]| \geq |X| + 1 = k + 1$. 

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Suppose \( k \geq 2 \) and \(|V(G) - N[X]| \leq k\). For each \( x \in X \), \( G - x \) has connectivity at least \( 2k + l + 2 - 1 > 2(k - 1) + l + 2 \); so by induction there is an induced cycle \( C_x \) in \( G - X \) such that \((G - x) - V(C_x)\) is \( l \)-connected and \(|N(u) \cap V(C_x)| \leq 1\) for any \( u \notin V(C_x) \cup X \). Hence

\[
|V(G - X) - V(C_x)| \geq \sum_{u \in V(C_x)} (d_G(u) - k - 2) \geq (k + l)|V(C_x)|.
\]

It follows that \(|V(C_x)| \leq |V(G - X)|/(k + l + 1)\). If \(|N(x) \cap V((G - x) - V(C_x))| \geq l\) then \( G - V(C_x) \) is \( l \)-connected. So we may assume that for any \( x \in X \),

\[
l > |N(x) \cap V((G - x) - V(C_x))| = d_G(x) - |N(x) \cap V(C_x)| \geq d_G(x) - |V(C_x)| \geq d_G(x) - |V(G - X)|/(k + l + 1).
\]

Hence,

\[
|V(G - X)| > (d_G(x) - l)(k + l + 1).
\]

So

\[
k|V(G - X)| \geq \left( \sum_{x \in X} d_G(x) \right) - kl (k + l + 1).
\]

Since we assume \(|V(G) - N[X]| \leq k\), \( \sum_{x \in X} d_G(x) \geq |N(X)| \geq |V(G - X)| - k\). Thus \( k|V(G - X)| \geq (|V(G - X)| - k - kl)(k + l + 1)\), and hence \(|V(G - X)| \leq k(k + l + 1)\). Therefore,

\[
(2k + 2)(k + l + 1) \leq (d_G(x) - l)(k + l + 1) < |V(G - X)| \leq k(k + l + 1),
\]

a contradiction which completes the proof of (2).

By (2), we may apply Lemma 3.2 or Lemma 3.3 to conclude that there is a \((2k + l + 2)\)-contractible edge \( e = uv \in E(G - X) \). Thus \( G/e \) is \((2k + l + 2)\)-connected and \( X \subseteq V(G/e) \). Let \( z \) denote the vertex of \( G/e \) resulted from the contraction of \( e \). By (1), \( G/e \) is also \( \{K_3, K_{2,3}\}\)-free. So by induction, \( G/e - X \) contains an induced cycle \( C' \) such that \( G/e - V(C') \) is \( l \)-connected and \(|N(u) \cap V(C')| \leq 1\) for all \( u \in V(G/e) - (X \cup V(C'))\).

We may view the edges of \( C' \) as edges of \( G \), and let \( C \) be an induced cycle in the subgraph of \( G \) induced by \( E(C') \cup \{e\} \). So \( C \subseteq G - X \).

Let \( v \in V(G) - (X \cup V(C)) \). We claim that \(|N_G(v) \cap V(C)| \leq 1\). If \( v \notin V(e) \) then \(|N_G(v) \cap V(e)| \leq 1\) (by (1)); so \(|N_G(v) \cap V(C)| \leq |N_{G/e}(v) \cap V(C')| \leq 1\). Now assume \( v \in V(e) \). If \( z \notin V(C') \) then \(|N_G(v) \cap V(C)| \leq |N_{G/e}(z) \cap V(C')| \leq 1\). So assume \( z \in V(C') \). Then, since \( C' \) is induced in \( G/e \), it follows from (1) that \(|N_G(v) \cap V(C)| \leq 1\).

It remains to show that \( G - V(C) \) is \( l \)-connected. For, suppose \( S \) is cut in \( G - V(C) \) with \(|S| \leq l - 1\). Then \(|S \cap V(e)| = 1\) as otherwise \( S \) or \((S - V(e)) \cup \{z\}\) would be a cut in \( G/e - V(C') \) (which is \( l \)-connected). For the same reason, \( G - S \) has exactly two components, one of which consists of the vertex in \( V(e) - S \), say \( v \). But \( v \) has degree at most \( 1 + (l - 1) = l < 2k + l + 2 \), a contradiction. \( \blacksquare \)
4 Determining $f(k, 1)$

In this section, we prove Theorem 1.4. Actually we prove a result stronger than Theorem 1.4, and our approach is to find a connected subgraph $T$ of $G$ such that $X \subseteq V(T)$, and then find a cycle in $G - T$. This leads to the following concept.

Let $G$ be a graph and $X \subseteq V(G)$. We define an $X$-tree in $G$ as a minimal connected induced subgraph of $G$ containing $X$. When $|X| = 1$, there is a unique $X$-tree, namely $G[X]$. When $|X| = 2$, an $X$-tree is simply an induced path in $G$ between the vertices in $X$. However, for $|X| \geq 3$, an $X$-tree need not be a tree.

Let $T$ be an $X$-tree in $G$. Then by minimality of $T$, any vertex in $V(T) - X$ is a cut vertex of $T$ and, in particular, every vertex of degree at most 1 in $T$ must be in $X$. Given any subgraph $H$ of $G - V(T)$, we define a partition of $V(T)$ as follows. Recall the definition of a bridge in Section 2 (following Corollary 2.2). For each $u \in V(T)$, let $C_u$ denote the maximal union of $u$-bridges of $T$ such that $N(C_u - u) \cap V(H) = \emptyset$ and let $C_u = \emptyset$ if no such $u$-bridge of $T$ exists. (Thus, by definition, $C_u \subseteq T$; so $C_u \cap H = \emptyset$.) We say that $u$ is $H$-maximal if for any $v \in V(T) - \{u\}$, $C_u$ is not contained in $C_v$. Define

\[
V_1 = \bigcup V(C_u - u), \text{ where the union is taken over all } H\text{-maximal } u;
\]
\[
V_2 = (X - V_1) \cup \{u : u \text{ is } H\text{-maximal}\};
\]
\[
V_3 = V(T) - (V_1 \cup V_2).
\]

We say that $V_1, V_2, V_3$ is the $H$-partition of $V(T)$. The following lemma summarizes a few properties about $H$-partitions and $X$-trees. In particular, property (1) implies that $V_1 \cap V_2 = \emptyset$; so $V_1, V_2, V_3$ is a partition of $V(T)$ (here we allow sets in a partition to be empty).

**Lemma 4.1.** Let $G$ be a connected graph, $X \subseteq V(G)$, $T$ an $X$-tree in $G$, $H \subseteq G - V(T)$, and $V_1, V_2, V_3$ the $H$-partition of $V(T)$. Then

1. if $u, v \in V_2$ are distinct and $H$-maximal then $C_u \cap C_v = \emptyset$,
2. $N(V_1) \cap (V(H) \cup V_3) = \emptyset$,
3. $X \subseteq V_1 \cup V_2$,
4. for any $u \in V_3$, each component of $T - u$ contains a neighbor of $H$ (so if $H$ is connected then $G[T \cup H] - u$ is connected),
5. $|V_2| \leq |X|$, and if $|V_2| = |X|$ then every component of $G[V_1 \cup V_2] - E(G[V_2])$ is a path between $X$ and $V_2$ with internal vertices (if any) in $V_1$.

**Proof.** Let $u, v \in V(T)$ be distinct and $H$-maximal. If $u \notin V(C_v)$ and $v \notin V(C_u)$ then $u$ belongs to a $v$-bridge of $T$ not contained in $C_v$, and $v$ belongs to a $u$-bridge of $T$ not contained in $C_u$; in this case it is easy to see that $C_u \cap C_v = \emptyset$. So by symmetry, we may assume $u \in V(C_v)$. By the maximality of $C_v$, any $v$-bridge of $T$ not contained in $C_v$ has a neighbor in $H$, which shows that $C_u \subseteq C_v$, contradicting the $H$-maximality of $u$. So we have (1).

We have (2) and (3) from the definition of the $H$-partition of $V(T)$. Now let $u \in V_3$. Then by definition, every component of $T - u$ contains a neighbor of $H$; so if $H$ is connected then $G[T \cup H] - u$ is connected, and we have (4).
If $V_2 \subseteq X$ then $|V_2| \leq |X|$. Now assume $V_2 \not\subseteq X$, and let $u_1, \ldots, u_t \in V_2 - X$. Then by definition, each $u_i$ is $H$-maximal. Thus, by (1), $C_{u_i} \cap C_{u_j} = \emptyset$ whenever $i \neq j$. Moreover, since $T$ is an $X$-tree, every component of $C_{u_i} - u_i$ intersects $X$. Therefore, $|V_2| \leq |X|$. Next, assume $|V_2| = |X|$. If $V_2 = X$ then by definition $V_1 = \emptyset$; so (5) holds (by viewing each vertex in $X = V_2$ as a trivial path between $X$ and $V_2$). Hence, let $u_1, \ldots, u_t \in V_2 - X$. Then each $C_{u_i} - u_i$ is connected and contains only one vertex from $X$; hence by the minimality of $T$, $C_{u_i}$ must be a path from $u_i \in V_2$ to some vertex in $X$ and with all internal vertices in $V_1$. Thus we have (5), viewing each vertex in $X \cap V_2$ as a trivial path between $X$ and $V_2$.

In order to state our next result, we need the concept of a \emph{k-fold wheel with center $X$}, which is defined as a graph obtained from the disjoint union of a cycle $C$ and a disconnected graph $X$ with $|V(X)| = k$ by adding all possible edges between $V(C)$ and $V(X)$; and $V(X)$ (or $X$) is called the \emph{center} of the wheel. A 2-fold wheel is also called a \emph{double wheel}. Note that a $k$-fold wheel with center $X$ is not ($|X| + 3$)-connected, but if the cycle $G - X$ has at least $|X| + 2$ vertices then it is ($|X| + 2$)-connected.

Kawarabayashi, Lee and Yu [7] proved that if $G$ is a 4-connected graph and $u,v \in V(G)$ are distinct, then either $G$ is a 2-fold wheel with center $\{u,v\}$, or $G$ has a path $P$ between $u$ and $v$ such that $G - V(P)$ is 2-connected. This result together with a result in [2,9] implies $f(2,1) = 5$.

Recall the result of Tutte that if $G$ is a 3-connected graph and $x \in V(G)$ then there is a cycle $C$ in $G - x$ such that $G - V(C)$ is connected. From this we can deduce $f(1,1) = 3$. Therefore, to prove Theorem 1.4 it suffices to consider $k \geq 3$.

**Lemma 4.2.** Let $G$ be a $(k + 2)$-connected graph and $X \subseteq V(G)$ with $|X| = k \geq 3$. Suppose $G$ is not a $k$-fold wheel with center $G[X]$. Then there exists an induced cycle $C$ in $G$ such that $X$ is contained in a component of $G - V(C)$.

**Proof.** For an $X$-tree $T$ in $G$, let $D_1, \ldots, D_r$ be the components of $G - V(T)$ such that $|V(D_1)| \geq |V(D_2)| \geq \ldots \geq |V(D_r)|$. We choose $T$ such that

1. $|V(T)|$ is minimum, and
2. subject to (1), $S(T) := (|V(D_1)|, |V(D_2)|, \ldots, |V(D_r)|)$ is maximal with respect to the lexicographic ordering.

Suppose that the assertion of Lemma 4.2 is false for $G,X$. Then each $D_1$ is a tree. Let $x \in V(D_r)$ with degree at most 1 in $D_r$, and let $V_1, V_2, V_3$ be the $x$-partition of $V(T)$. (So Lemma 4.1 holds for $V_1, V_2, V_3$.) Then $N(x) \cap V(T) \subseteq V_2 \cup V_3$. (In this proof $N$ without subscript is used to denote the neighborhood in $G$.) Since $d_G(x) \geq k + 2$ and $|V_2| \leq k$ (by Lemma 4.1(5)), we see that $N(x) \cap V_3 \neq \emptyset$; in particular, $V_3 \neq \emptyset$. We claim that

3. each $u \in V_3$ has at most one neighbor in $V(D_r) - \{x\}$ and no neighbor in $V(D_i)$ for $i = 1, \ldots, r - 1$.

Since $G[V(T) \cup \{x\}] - u$ is connected (by Lemma 4.1(4)), $G[V(T) \cup \{x\}] - u$ contains an $X$-tree, say $T'$. If $u$ has two neighbors in $V(D_r) - \{x\}$, say $y_1, y_2$, then the edges $uy_1, uy_2$ and a path in $D_r - x$ between $y_1$ and $y_2$ form a cycle disjoint from $T'$, a contradiction to our assumption.
that Lemma 4.2 fails with \( G, X \). If \( u \) has a neighbor in \( D_i \) for some \( i \leq r - 1 \), then \( S(T') \) is larger than \( S(T) \), a contradiction to (1) or (2). So we have (3).

We will show \( |V_3| = 1 \). Let \( \delta \) denote the minimum degree of \( G[V_3] \), and let \( u \in V_3 \) have degree \( \delta \) in \( G[V_3] \). By (3), \( u \) has at most 2 neighbors outside \( T \) (\( u \) may be adjacent to \( x \) and a vertex in \( D_r - x \)); so by Lemma 4.1(2), \( |N(u) \cap V_2| \geq (k + 2) - \delta - 2 = k - \delta \). Let \( A := N(u) \cap V_2 \) and \( B := V_2 - A \). Thus

\[
(4) \quad |A| \geq k - \delta \quad \text{and} \quad |B| \leq |V_2| - (k - \delta) \leq \delta.
\]

(5) For any edge \( wz \) of \( G[V_3] - u \), \( \{w, z\} \) is a 2-cut of \( T \), and \( T - \{w, z\} \) has a component \( F_{wz} \) such that \( V(F_{wz}) \cap V_3 = \emptyset \), \( N_T(F_{wz}) = \{w, z\}, V(F_{wz}) \cap A = \emptyset \), and \( V(F_{wz}) \cap B \neq \emptyset \).

Since \( X \subseteq V_1 \cup V_2 \) (by Lemma 4.1(3)), if follows from (1) that \( G[V(T) \cup \{x\}] - \{w, z\} \) is not connected and hence has a component \( F_{wz} \) disjoint from \( A \cup \{u, x\} \). So \( V(F_{wz}) \cap A = \emptyset \) and \( F_{wz} \) is a component of \( T - \{w, z\} \). Moreover, \( w \in N_T(F_{wz}) \) as, otherwise, \( G[V(T) \cup \{x\}] - z \) is not connected, contradicting Lemma 4.1(4) (since \( z \in V_3 \)). Similarly, \( z \in N_T(F_{wz}) \). So we have \( N_T(F_{wz}) = \{w, z\} \).

If there exists \( z' \in V(F_{wz}) \cap V_3 \), then, since \( z' \notin X \), \( T - z' \) has a component, say \( F \), which is inside \( F_{wz} \). Now \( F \) is also a component of \( G[V(T) \cup \{x\}] - z' \), contradicting Lemma 4.1(4). Hence, \( V(F_{wz}) \cap V_3 = \emptyset \).

Therefore, since \( V(F_{wz}) \cap A = \emptyset \), \( V(F_{wz}) \cap B \neq \emptyset \) (by Lemma 4.1(2)), completing the proof of (5).

(6) If \( |V_3| \geq 2 \) then there exist \( v \in V_3 - \{u\} \) and a component \( F_v \) of \( T - v \) such that \( V(F_v) \cap V_3 = \emptyset \), \( N_T(F_v) = \{v\} \), \( V(F_v) \cap A = \emptyset \), and \( V(F_v) \cap B \neq \emptyset \).

Suppose \( |V_3| \geq 2 \). Then \( T - u \) has a component, say \( F \), containing at least one vertex in \( V_3 \). Now, for any \( v \in V(F) \cap V_3 \), at least one component of \( T - v \), say \( F_v \), is contained in \( F \). Choose \( v \) and \( F_v \) so that \( |V(F_v)| \) is minimum. Then \( V(F_v) \cap V_3 = \emptyset \) and \( N_T(F_v) = \{v\} \). So \( u \notin F_v \), which implies \( V(F_v) \cap A = \emptyset \). Hence, \( V(F_v) \cap B \neq \emptyset \) by Lemma 4.1(2), completing the proof of (6).

(7) \( |V_2| = |X| \), and \( G[V_3] \cong K_s \) for some \( s \in \{1, 2\} \).

Let \( |V_3| = t \) and \( |E(G[V_3])| = m \). Suppose \( t = 1 \). Then \( G[V_3] \cong K_1 \) and \( V_3 = \{u\} \). So by Lemma 4.1(2) and by (3), \( |V_2| \geq d_T(u) \geq d_G(u) - 2 \geq (k + 2) - 2 = k \). It follows from Lemma 4.1(5) that \( |V_2| = k \), and we have (7).

Thus we may assume \( t \geq 2 \). Let \( e, f \in S := \{wz : wz \text{ is an edge of } G[V_3] - u \} \cup \{v\} \) be arbitrary (\( v \) is given in (6)). By (5) and (6), \( V(F_e) \cap V_3 = \emptyset = V(F_f) \cap V_3 \), and \( N_T(F_e) = V(e) \) and \( N_T(F_f) = V(f) \). So \( F_e \cap F_f = \emptyset \) when \( e \neq f \). Hence, since there are \( m - \delta \) edges in \( G[V_3] - u \) and \( V(F_e) \cap B \neq \emptyset \neq V(F_f) \cap B \) (by (5) and (6)), it follows from (4) that

\[
\delta \geq |B| \geq |S| = m - \delta + 1.
\]

So \( \delta \geq 1 \) and \( m \leq 2\delta - 1 \). Since \( m \geq t\delta/2 \), \( t = 2 \) or \( t = 3 \).

If \( t = 2 \) then \( \delta = |B| = 1 \), and \( G[V_3] \cong K_2 \); so by Lemma 4.1(5) and by (4), \( k \geq |V_2| = |B| + |A| \geq 1 + (k - 1) = k \), and (7) holds.
Thus we may assume $t = 3$. Then $\delta \geq m - \delta + 1 \geq 3\delta/2 - \delta + 1$. Therefore, $\delta = 2$, $G[V_3] \cong K_3$, and $|B| = 2$. Assume $V_3 = \{u, v, w\}$ and $B = \{b_1, b_2\}$. By (5) and (6), we may assume that $b_1 \in F_v$ and $b_2 \in F_{uv}$, and, by Lemma 4.1(2), that $b_1v, b_2v, b_2w \in E(T)$. Then $T - w$ is connected (as $N_T(w) \subseteq N_T(\{u, v\})$), contradicting (1) and completing the proof of (7).

We may assume

(8) $|V_3| = 1$.

For, suppose $|V_3| \geq 2$. Then by (7), $G[V_3] \cong K_2$. So let $V_3 = \{u, v\}$. If $V_2 \subseteq N(u)$ then $T - v$ is connected (as $N_T(v) \subseteq V_2 \cup \{u\}$ by Lemma 4.1(2)), contradicting (1). Thus $V_2 \not\subseteq N(u)$. Therefore, since $|N(u) - T| \leq 2$ (by (3)) and $|V_2| = k$ (by (7)), it follows from $|N(u)| \geq k + 2$ that $x \in N(u)$.

Similarly, we have $x \in N(v)$. Then $xuvx$ is a cycle in $G - X$ (as $X \subseteq V_1 \cup V_2$) by Lemma 4.1(3). Since $k \geq 3$, $G - \{u, v, x\}$ is $(k - 1)$-connected and contains $X$; so the assertion of the lemma holds. Thus we may assume (8).

(9) $N_T(u) = V_2$, $|N(u) - V(T)| = 2$, $|V(D_r)| \geq 2$, and $N_T[u] = N(x) \cap V(T) = V_2 \cup V_3$.

Since $|N(u)| \geq k + 2$ and $N_T(u) \subseteq V_2$ (by Lemma 4.1(2)), it follows from (3) and (7) that $N_T(u) = V_2$ and $|N(u) - V(T)| = 2$. Since $N(x) \cap V(T) \subseteq V_2 \cup V_3$ (by definition of $x$-partition) and $|N(x) \cap V(D_r)| \leq 1$, we see that $V_2 \subseteq N(x)$, $|V(D_r)| \geq 2$, and $N_T[u] = N(x) \cap V(T) = V_2 \cup V_3$. This proves (9).

Let $V_2 = \{v_1, \ldots, v_k\}$. By (7) and Lemma 4.1(5), let $P_i$, for each $1 \leq i \leq k$, be the path which is the component of $G[V_1 \cup V_2] - E(G[V_2])$ containing $v_i$, and let $X = \{x_1, \ldots, x_k\}$ such that $x_i \in V(P_i)$. Since $|V(D_r)| \geq 2$, there exists $y \in V(D_r - x)$ with degree 1 in $D_r$. Then

(10) $u \in N(y)$ and $|N(y) \cap V_2| \geq k - 1$.

Consider the $y$-partition $V'_1, V'_2, V'_3$ of $V(T)$. Then (3)-(9) holds for $V'_1, V'_2, V'_3$. In particular, $|V'_2| = k$ and $|V'_3| = 1$. Let $V'_3 = \{u'\}$. Then by (9), $N_T(u') = V'_2$, $|N(u') - V(T)| = 2$, $N_T[u'] = N(y) \cap V(T) = V'_2 \cup V'_3$.

If $u' = u$, then $V'_2 = N_T(u') = N_T(u) = V_2$, $u \in N(y)$ and $|N(y) \cap V_2| = |N(y) \cap V'_3| = |V'_2| = k > k - 1$. So assume $u' \neq u$.

Since $G[V_1 \cup V_2] - E(G[V_2])$ is the disjoint union of $P_1, \ldots, P_k$, every vertex in $V_1 - V_2$ has degree at most 2 in $T$. Since $u'$ has degree at least $k \geq 3$ in $T$, $u' \in V_2$. Without loss of generality, let $u' = v_1$. Then $v_1 \not\in X$; so let $v'_1$ be the neighbor of $u' = v_1$ on $P_1$. Then $N_T[u'] \subseteq V_2 \cup \{v'_1, u\}$. Thus

$|N(y) \cap V_2| \geq |N(y) \cap N_T[u']| - 2 = |N(y) \cap V(T)| - 2 = |V'_2 \cup V'_3| - 2 = (k + 1) - 2 = k - 1$,

and $u \in N(y)$ (since $u \in N_T(u') \subseteq N(y)$). So we may assume (10).

From (10) and without loss of generality, we may assume that $v_1, \ldots, v_{k-1} \in N(y)$. Let $F = V(D_r) - \{x, y\}$. We may assume

(11) $N(F) \cap V_1 = \emptyset$. 


For, suppose that there exists $v_i' \in V(P_i) \cap V_1$ such that $v_i' \in N(F)$ for some $1 \leq i \leq k$. If $i = k$ then $v_k \notin X$, $xw_kx$ is a cycle and, since $v_1, \ldots, v_{k-1} \in N(y)$, $G[V(T) \cup V(D_r)] - \{u, v_k, x\}$ is a connected subgraph of $G - \{u, v_k, x\}$ containing $X$; so the assertion of the lemma holds. If $i < k$ then $v_i \notin X$, $yw_iy$ is a cycle and, since $N(x) \cap V(T) = V_2 \cup V_3$ (by (9)), $G[V(T) \cup V(D_r)] - \{u, v_i, y\}$ is a connected subgraph of $G - \{u, v_i, y\}$ containing $X$; again the assertion of the lemma holds. So we may assume (11).

By (11), $N(F \cup \{x, u\}) = V_2 \cup \{y\}$ which has $k + 1$ vertices. Since $G$ is $(k + 2)$-connected, $V_2 \cup \{y\}$ cannot be a cut in $G$; so $V(G) = V(D_r) \cup V_2 \cup \{u\}$, and $|V(T)| = |V_2 \cup V_3| = k + 1$. Hence $X = V_2$. Moreover, $V(T) \subseteq N(y)$ (as $|N(y)| \geq k + 2$ and $|N(y) - V(T)| = 1$).

Since $|N(u) - V(T)| \leq 2$, we see that $D_r$ has exactly two vertices of degree 1, and hence $D_r$ is a path between $x$ and $y$. Furthermore, $u \not\in N(F)$ (by (3)), and therefore each vertex in $F$ is adjacent to all of $V_2$. Finally, note that $G[X] = G[V_2]$ is not connected; as otherwise, $T - u = G[X]$ contradicts the choice of $T$ in (1). So $G$ is a $k$-fold wheel with center $G[X]$. 

Theorem 1.4 follows from the following result.

**Theorem 4.3.** Let $G$ be a $(k + 2)$-connected graph, where $k \geq 2$, and let $X \subseteq V(G)$ with $|X| = k$. Then either $G$ is a $k$-fold wheel with center $G[X]$, or there exists an induced cycle $C$ in $G - X$ such that $G - V(C)$ is connected.

**Proof.** Suppose $G$ is not a $k$-fold wheel with center $G[X]$. By Lemma 4.2, there is a connected subgraph $T$ of $G$ containing $X$ and there is an induced cycle $C$ in $G - V(T)$. So $T$ is contained in some component of $G - V(C)$, say $U_0$. Let $U_1, \ldots, U_r$ be the components of $G - V(C) - V(U_0)$. We may select $C$ and $T$ so that $S(C) := (|U_0|, |U_1|, \ldots, |U_r|)$ is maximal with respect to the lexicographic ordering.

If $r = 0$ then $C$ is the desired cycle. So we may assume $r \geq 1$. Since $G$ is $(k + 2)$-connected, $U_r$ has at least $k + 2$ neighbors on $C$. Choose $u, v \in N(U_r) \cap V(C)$ such that there is an $u$-$v$ path $P$ in $C$ whose internal vertices have no neighbor in $U_r$. Let $Q$ be an induced path in $G[V(U_r) \cup \{u, v\}]$ between $u$ and $v$. Then $C' = P \cup Q$ is an induced cycle in $G$. Since $G$ is $(k + 2)$-connected and $C$ is induced, $V(C) - V(P)$ has at least one neighbor in $V(G) - V(C \cup U_r)$. Thus $S(C')$ is larger than $S(C)$, a contradiction. 

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References


