Cycles in 4-Connected Planar Graphs

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Abstract

Let G be a 4-connected planar graph on n vertices. Previous results show that G contains a cycle of length k for each $k \in \{n, n-1, n-2, n-3\}$ with $k \geq 3$. These results are proved using the "Tutte path" technique, and this technique alone cannot be used to obtain further results in this direction. One approach to obtain further results is to combine Tutte paths and contractible edges. In this paper, we demonstrate this approach by showing that G also has a cycle of length k for each $k \in \{n-4, n-5, n-6\}$ with $k \geq 3$. This work was partially motivated by an old conjecture of Malkevitch.

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1 Introduction and notation

In 1931, Whitney [10] proved that every 4-connected planar triangulation contains a Hamilton cycle, and hence, is 4-face-colorable. In 1956, Tutte [8] extended Whitney's result to all 4-connected planar graphs.

There are many 3-connected planar graphs which do not contain Hamilton cycles (see [1]). On the other hand, Plummer [4] conjectured that any graph obtained from a 4-connected planar graph by deleting one vertex has a Hamilton cycle. This conjecture follows from a theorem of Tutte as observed by Nelso (see [7]). Plummer [4] also conjectured that any graph obtained from a 4-connected planar graph by deleting two vertices has a Hamilton cycle. This conjecture was proved by Thomas and Yu [6]. Note that deleting three vertices from a 4-connected planar graph may result in a graph which is not 2-connected (and hence, has no Hamilton cycle). However, Sanders [5] showed that in any 4-connected planar graph with at least six vertices there are three vertices whose deletion results in a Hamiltonian graph.

The above results can be rephrased as follows. Let G be a 4-connected planar graph on n vertices. Then G has a cycle of length k for every $k \in \{n, n-1, n-2, n-3\}$ with $k \geq 3$. (In fact, the results in [7] and [6] are slightly stronger.) So it is natural to ask whether G contains a cycle of length $n - l$ for $l \geq 4$. The following conjecture of Malkevitch ([2], Conjecture (6.1)) says that this is the case for almost all l.

(1.1) Conjecture. *Let G be a 4-connected planar graph on n vertices. If G contains a cycle of length 4, then G contains a cycle of length k for every $k \in \{n, n-1, \ldots, 3\}$.*

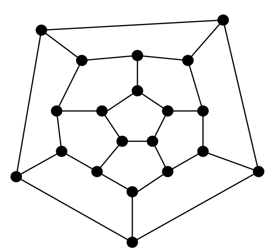

Figure 1: A cyclically 4-edge-connected cubic graph with girth 5

Note that there are 4-connected planar graphs with no cycles of length 4. For example, the line graph of a cyclically 4-edge-connected cubic planar graph with girth at least 5 contains no cycle of length 4. An example of a cyclically 4-edge-connected cubic graph is shown in Figure 1. For this example, its line graph has 30 vertices. Hence, we propose the following weaker conjecture.

(1.2) Conjecture. *Let G be a 4-connected planar graph on n vertices. Then G contains a cycle of length k for every $k \in \{n, n-1, \ldots, n-25\}$ with $k \geq 3$.*

One may also ask whether (1.2) holds for sufficiently large n if we replace the number 25 by a non-constant function of n. We will see that the "Tutte path" method used in

[8], [7], [6] and [5] cannot be extended to show the existence of cycles of length $n-l$ for $l \geq 4$. We believe that a possible approach to attack the above conjectures is to combine Tutte paths and contractible edges (to be defined later). We will demonstrate this approach by proving the following result.

(1.3) Theorem. *Let G be a 4-connected planar graph with n vertices. Then G contains a cycle of length k for every $k \in \{n-4, n-5, n-6\}$ with $k \geq 3$.*

This paper is organized as follows. In the rest of this section, we describe notation and terminology that are necessary for stating and proving results. In Section 2, we will define Tutte paths and show how they can be applied to obtain results on Hamilton paths and cycles. We also explain why this technique cannot be generalized. In Section 3, we study contractible edges in 4-connected planar graphs and prove our main result.

We consider only simple graphs. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. For an edge e of G with incident vertices x and y, we also use xy or yx to denote e. A graph H is a *subgraph* of G, denoted by $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We will use $\emptyset$ to denote the empty graph (as well as the empty set). For two subgraphs G and H of a graph, $G \cup H$ (respectively, $G \cap H$) denotes the graph with vertex set $V(G) \cup V(H)$ (respectively, $V(G) \cap V(H)$) and edge set $E(G) \cup E(H)$ (respectively, $E(G) \cap E(H)$).

Let G be a graph, let $X \subset V(G)$, and let $Y \subset E(G)$. The subgraph of G *induced by* X, denoted by $G[X]$, is the graph with vertex set X and edge set $\{xy \in E(G) : x, y \in X\}$. The subgraph of G *induced by* Y, denoted by $G[Y]$, is the graph with edge set Y and vertex set $\{x \in V(G) : x \text{ is incident with some edge in } Y\}$. Let H be a subgraph of G. We use $H + X$ to denote the graph with vertex set $V(H) \cup X$ and edge set $E(H)$, and if $X = \{x\}$ then let $H + x := H + X$. Let $H - X := G[V(H) - X]$, and let $H - Y$ denote the graph with vertex set $V(H)$ and edge set $E(H) - Y$. If $X = \{x\}$ then let $H - x := H - \{x\}$, and if $Y = \{y\}$ then let $H - y := H - \{y\}$. Let Z be a set of 2-element subsets of $V(G)$; then we use $G + Z$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \cup Z$, and if $Z = \{\{x, y\}\}$, then let $G + xy := G + Z$.

Let G be a graph and let $H \subset G$. Then G/H denotes the graph with vertex set $(V(G) - V(H)) \cup \{h\}$ (where $h \notin V(G)$) and edge set $(E(G) - E(H)) \cup \{hy : y \in V(G) - V(H) \text{ and } yy' \in E(G) \text{ for some } y' \in V(H)\}$. We say that G/H is obtained from G by *contracting* H to the vertex h. If H is induced by an edge $e = xy$, then we write G/e or G/xy instead of G/H. A graph X is a *minor* of G or G contains an X-*minor* if X can be obtained from a subgraph of G by contracting edges.

Let G be a graph. For any $X \subset V(G)$, let $N_G(X) := \{u \in V(G) - X : u \text{ is adjacent to some vertex in } X\}$. For any $H \subset G$, we write $N_G(H) := N_G(V(H))$. If $X \subset V(G)$ such that $|X| = k$ (where k is a positive integer) and $G - X$ is not connected, then X is called a k-*cut* of G. If $\{x\}$ is a 1-cut of G, then x is called a *cut vertex* of G. We say that G is n-connected, where n is a positive integer, if $|V(G)| \geq n + 1$ and G has no k-cut with $k < n$.

A graph G is *planar* if G can be drawn in the plane with no pair of edges crossing, and such a drawing is called a *plane representation* of G (or a *plane graph*). Let G be a plane graph. The *faces* of G are the connected components (in topological sense) of the complement of G in the plane. Two vertices of G are *cofacial* if they are incident with a common face of G. The *outer* face of G is the unbounded face. The boundary of the outer face is called the *outer walk* of the graph, or the *outer cycle* if it is a cycle. A cycle is a *facial cycle* in a plane graph if it bounds a face of the graph. A *closed disc* in the plane is a homeomorphic image of $\{(x, y) : x^2 + y^2 \le 1\}$ (and the image of $\{(x, y) : x^2 + y^2 = 1\}$ is the *boundary* of the disc).

Note that a graph is planar iff it has no K_5-minor or $K_{3,3}$-minor. It is well known that if G is a 2-connected plane graph then every face of G is bounded by a cycle. Also note that if G is a plane graph and a, b, c, d occur on a facial cycle in this cyclic order, then G contains no vertex disjoint paths from a to c and from b to d, respectively.

For any path P and $x, y \in V(P)$, we use xPy to denote the subpath of P between x and y. Given two distinct vertices x and y on a cycle C in a plane graph, we use xCy to denote the path in C from x to y in clockwise order.

2 Tutte paths

In this section, we will show how Tutte paths can be used to derive cycles of length $n, n-1, n-2, n-3$ in 4-connected planar graphs on n vertices. We will also explain why Tutte paths alone cannot give further results in this direction.

(2.1) Definition. Let G be a graph and let P be a path in G. A *P-bridge* of G is a subgraph of G which either (1) is induced by an edge of $G - E(P)$ with both incident vertices in $V(P)$ or (2) is induced by the edges in a component D of $G - V(P)$ and all edges from D to P. For a P-bridge B of G, the vertices of $B \cap P$ are the *attachments* of B on P. We say that P is a *Tutte path* in G if every P-bridge of G has at most three attachments on P. For any given $C \subset G$, P is called a *C-Tutte path* in G if P is a Tutte path in G and every P-bridge of G containing an edge of C has at most two attachments on P.

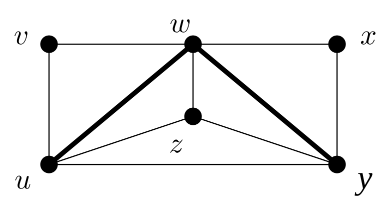

Figure 2: A Tutte path and its bridges

Let G be the graph in Figure 2, let $P = uwy$, and let $C = uvwxy$. Then the P-bridges of G are: $G[\{uv, vw\}]$, $G[\{wx, xy\}]$, $G[\{zu, zw, zy\}]$, and $G[\{uy\}]$. It is easy to check that P is a C-Tutte path in G.

Note that if P is a Tutte path in a 4-connected graph and $|V(P)| \ge 4$, then P is in fact a Hamilton path. The following result is the main theorem in [7], where a P-bridge is called a "P-component".

(2.2) Theorem. *Let G be a 2-connected plane graph with a facial cycle C, let $x \in V(C)$, $e \in E(C)$, and $y \in V(G) - \{x\}$. Then G contains a C-Tutte path P from x to y such that $e \in E(P)$.*

Theorem (2.2) immediately implies that every 4-connected planar graph is Hamiltonian (by requiring $xy \in E(G) - \{e\}$). The following result was proved by Thomas and Yu ([6], Theorem (2.6)). In [6], a C-Tutte path is called an "$E(C)$-snake".

(2.3) Theorem. *Let G be a 2-connected plane graph with a facial cycle C. Let $x, y \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that x, y, e, f occur on C in this clockwise order. Then there exists a yCx-Tutte path P between x and y in G such that $\{e, f\} \subset E(P)$.*

We mention that (2.3) was proved independently by Sanders [5]. Before deriving consequences of the above two results, let us introduce several concepts. A *block* of a graph H is either (1) a maximal 2-connected subgraph of H or (2) a subgraph of H induced by an edge of H not contained in any cycle. An *end block* of a graph H is a block of H containing at most one cut vertex of H.

(2.4) Definition. We say that a graph H is a *chain of blocks from x to y* if one of the following holds:

(1) H is 2-connected and x and y are distinct vertices of H; or

(2) H is connected but not 2-connected, H has exactly two end blocks, neither x nor y is a cut vertex of H, and x and y belong to different end blocks of H.

Remark. If H is not a chain of blocks from x to y, then there exist an end block B of H and a cut vertex b of H such that $b \in V(B)$ and $(V(B) - \{b\}) \cap \{x, y\} = \emptyset$.

(2.5) Definition. Let G be a graph and $\{a_1, \ldots, a_l\} \subset V(G)$, where l is a positive integer. We say that $(G, a_1, \ldots, a_l)$ is *planar* if G can be drawn in a closed disc with no pair of edges crossing such that $a_1, \ldots, a_l$ occur on the boundary of the disc in this cyclic order. We say that G is *$(4, \{a_1, \ldots, a_l\})$-connected* if $|V(G)| \geq l + 1$ and for any $T \subset V(G)$ with $|T| \leq 3$, every component of $G - T$ contains some element of $\{a_1, \ldots, a_l\}$.

Note that if G is 4-connected, then G is $(4, S)$-connected for all $S \subset V(G)$ with $S \neq V(G)$. Using the above results on Tutte paths, we can prove the following result which will be used extensively in the remainder of this paper.

(2.6) Lemma. *Let G be a graph and $\{a_1, \ldots, a_l\} \subset V(G)$, where $3 \leq l \leq 5$. Assume that $(G, a_1, \ldots, a_l)$ is planar, G is $(4, \{a_1, \ldots, a_l\})$-connected, and $G - \{a_3, \ldots, a_l\}$ is a chain of blocks from a_1 to a_2. Then*

(1) $G - \{a_3, \ldots, a_l\}$ has a Hamilton path between a_1 and a_2, and

(2) if $j \in \{3, \ldots, l\}$ and a_j has at least two neighbors contained in $V(G) - \{a_3, \ldots, a_l\}$, then $G - (\{a_3, \ldots, a_l\} - \{a_j\})$ has a Hamilton path between a_1 and a_2.

Proof: (1) Let $H := (G - \{a_3, \ldots, a_l\}) + a_1a_2$. Because $G - \{a_3, \ldots, a_l\}$ is a chain of blocks from a_1 to a_2, either $V(H) = \{a_1, a_2\}$ or H is 2-connected. If $V(H) = \{a_1, a_2\}$ then clearly (1) holds. So we may assume that H is 2-connected. Since $(G, a_1, \ldots, a_l)$ is planar, we may assume that $G + a_1a_2$ is drawn in a closed disc with no pair of edges crossing so that $a_1, a_2, \ldots, a_l$ occur in this clockwise order on the boundary of the disc. See Figure 3. Then H is a 2-connected plane graph. Let C denote the outer cycle of H. Note that for each $i \in \{3, \ldots, l\}$, those neighbors of a_i contained in $V(H)$ are all contained in $V(a_2Ca_1)$. Choose $u, v \in V(C)$ such that a_1, a_2, u, v occur on C in this clockwise order, $N_G(a_3) \cap V(H) \subset V(a_2Cu)$, $N_G(a_4) \cap V(H) \subset V(uCv)$ (if $l \geq 4$), and $N_G(a_5) \cap V(H) \subset V(vCa_1)$ (if $l = 5$). This can be done since $l \leq 5$. Pick $e, f \in E(C)$ such that e is incident with u and f is incident with v. By applying (2.3) to H (with H, a_1, a_2 as G, x, y, respectively), we find an a_2Ca_1-Tutte path P between a_1 and a_2 in H such that $e, f \in E(P)$ (and hence, $u, v \in V(P)$).

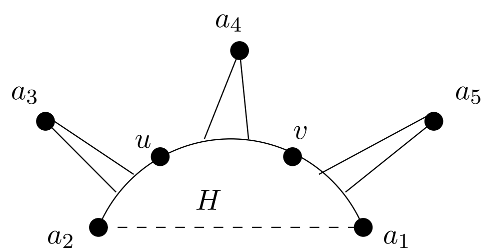

Figure 3: Lemma (2.6)

Next we show that P is a Hamilton path in H. Suppose for a contradiction that P is not a Hamilton path in H. Then there is a P-bridge B of H such that $V(B) - V(P) \neq \emptyset$. If $V(B) - V(P)$ contains no vertex of C, then $B - V(P)$ is a component of $H - (V(B) \cap V(P))$ containing no vertex of C. Therefore, by planarity, $B - V(P)$ is a component of $G - (V(B) \cap V(P))$ containing no element of $\{a_1, \ldots, a_l\}$. This contradicts the assumption that G is (4,$\{a_1, \ldots, a_l\}$)-connected (since $|V(B) \cap V(P)| \leq 3$). So assume that $V(B) - V(P)$ contains a vertex of C. Then $|V(B) \cap V(P)| = 2$ since P is a C-Tutte path. By the choice of u and v and because $u, v \in V(P)$, at most one element of $\{a_3, \ldots, a_l\}$ has a neighbor in $V(B) - V(P)$. Hence, $T := (V(B) \cap V(P)) \cup \{a_j : N_G(a_j) \cap (V(B) - V(P)) \neq \emptyset\}$ is a k-cut of G with $k \leq 3$, and $B - V(P)$ is a component of $G - T$ containing no element of $\{a_1, \ldots, a_l\}$. This contradicts the assumption that G is (4,$\{a_1, \ldots, a_l\}$)-connected. Therefore, P is a Hamilton path in H, and (1) holds.

(2) Let $H := (G - (\{a_3, \ldots, a_l\} - \{a_j\})) + a_1a_2$. Then H is 2-connected because $G - \{a_3, \ldots, a_l\}$ is a chain of blocks from a_1 to a_2 and $G - \{a_3, \ldots, a_l\}$ contains at least two neighbors of a_j. Because $(G, a_1, \ldots, a_l)$ is planar, we may assume that $G + a_1a_2$ is drawn in a closed disc with no pair of edges crossing so that $a_1, \ldots, a_l$ occur on the boundary of the disc in this clockwise oder. Then H is a 2-connected plane graph. Let C denote the outer cycle of H.

First, assume that $j = 4$ or $l \leq 4$. Pick $e \in E(C)$ such that e is incident with a_j. By applying (2.2) to H (with H, a_1, a_2 as G, x, y, respectively), we find a C-Tutte path P

between a_1 and a_2 in H such that $e \in E(P)$. As in the second paragraph in the proof of (1), we can show that P is a Hamilton path in H between a_1 and a_2, and so, (2) holds.

Now assume that $j = 3$ and $l = 5$. Let $u = a_3$, and choose $v \in V(a_3Ca_1)$ such that $N_G(a_4) \cap V(H) \subset V(a_3Cv)$ and $N_G(a_5) \cap V(H) \subset V(vCa_1)$. Pick $e, f \in E(C)$ such that e is incident with u and f is incident with v. By applying (2.3) (with H, a_1, a_2 as G, x, y, respectively), we find an a_2Ca_1-Tutte path P in H between a_1 and a_2 such that $e, f \in E(P)$. As in the second paragraph in the proof of (1), we can show that P is a Hamilton path between a_1 and a_2 in H, and so, (2) holds.

Finally assume that $j = 5$. Let $v = a_5$, and choose $u \in V(a_2Ca_5)$ such that $N_G(a_3) \cap V(H) \subset V(a_2Cu)$ and $N_G(a_4) \cap V(H) \subset V(uCa_5)$. Pick $e, f \in E(C)$ such that e is incident with u and f is incident with v. By applying (2.3) (with H, a_1, a_2 as G, x, y, respectively), we find an a_2Ca_1-Tutte path P in H between a_1 and a_2 such that $e, f \in E(P)$. As in the second paragraph in the proof of (1), we can show that P is a Hamilton path in H between a_1 and a_2, and so, (2) holds. □

We comment here that the condition $l \leq 5$ in (2.6) is necessary. For otherwise, we would need a result about Tutte paths between two given vertices and through three given edges, in the same sense of (2.3). But this is not possible as shown by the graph in Figure 4. In that graph, we see that there is no Tutte path from x to y and containing edges e, f, g. Therefore, additional structural information of the graph is needed in order to find cycles avoiding more vertices in 4-connected planar graphs, and this is our motivation to study (in Section 3) contractible edges in 4-connected planar graphs.

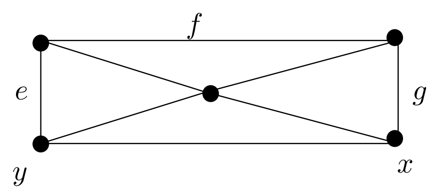

Figure 4: No Tutte path through e, f and g.

Below we derive some known results as consequences of (2.6). The first is a combination of a result of Thomassen [7] and a result of Thomas and Yu [6]. The second is due to Sanders [5].

(2.7) Corollary. *Let G be a 4-connected planar graph and let $u \in V(G)$. Then for each $l \in \{1, 2\}$ there exists a set $S_l \subset V(G)$ such that $u \in S_l$, $|S_l| = l$, and $G - S_l$ has a Hamilton cycle.*

Proof: Since G is 4-connected, $|V(G)| \geq 5 \geq l + 3$. Without loss of generality, we work with a plane representation of G. To show the existence of S_1, we pick three vertices a_1, a_2, a_3 on a facial cycle C of G such that $a_1a_2 \in E(C)$ and $a_3 = u$. Clearly, (G, a_1, a_2, a_3) is planar. Because G is 4-connected, G is $(4, \{a_1, a_2, a_3\})$-connected and $G - a_3$ is 3-connected (and hence, is a chain of blocks from a_1 to a_2). So by (1) of (2.6),

$G - a_3$ contains a Hamilton path P between a_1 and a_2. Let $S_1 = \{u\}$; then $u \in S_1$, $|S_1| = 1$, and $P + a_1a_2$ is a Hamilton cycle in $G - S_1$.

Next we show the existence of S_2. If there is a facial cycle C of G containing u such that $|V(C)| \geq 4$, then we pick vertices a_1, a_2, a_3, a_4 in clockwise order on C such that $a_1a_2 \in E(C)$ and $u \in \{a_3, a_4\}$, and in this case we let $G' = G$. (Clearly, (G', a_1, a_2, a_3, a_4) is planar.) If all facial cycles of G containing u has length three, then let $a_2a_3a_4a_2$ and $a_1a_2a_4a_1$ be facial cycles of G such that $u = a_4$, and in this case, we let $G' := G - a_2a_4$. (Clearly, (G', a_1, a_2, a_3, a_4) is planar.) Since G is 4-connected, G' is $(4,\{a_1, a_2, a_3, a_4\})$-connected and $G' - \{a_3, a_4\}$ is 2-connected (and hence, is a chain of blocks from a_1 to a_2). So by (1) of (2.6), $G' - \{a_3, a_4\}$ contains a Hamilton path Q between a_1 and a_2. Let $S_2 = \{a_3, a_4\}$; then $u \in S_2$, $|S_2| = 2$, and $Q + a_1a_2$ gives a Hamilton cycle in $G - S_2$. □

(2.8) Corollary. *Let G be a 4-connected planar graph with $|V(G)| \geq 6$, and let S_3 be the vertex set of a triangle in G. Then $G - S_3$ has a Hamilton cycle.*

Proof: Let $S_3 = \{a_3, a_4, a_5\}$. We claim that $G - \{a_3, a_4, a_5\}$ is 2-connected. For otherwise, G has a 4-cut S containing S_3. Let $S := \{a_3, a_4, a_5, x\}$, and let A be a component of $G - S$. Since G is 4-connected, contracting A to a single vertex and contracting $G - (V(A) \cup \{a_3, a_4, a_5\})$ to a single vertex, we produce a K_5-minor in G, a contradiction. So $G - \{a_3, a_4, a_5\}$ is 2-connected.

Let D be the cycle which bounds the face of $G - \{a_3, a_4, a_5\}$ containing $\{a_3, a_4, a_5\}$. Pick an edge $a_1a_2 \in E(D)$ such that a_2 is adjacent to a_3 and a_5 is cofacial with both a_1 and a_2. Let $G' := G - \{a_2a_5, a_3a_5\}$. Then $(G', a_1, a_2, a_3, a_4, a_5)$ is planar. Since G is 4-connected, G' is $(4, \{a_1, \ldots, a_5\})$-connected. Note that $G' - \{a_3, a_4, a_5\} = G - \{a_3, a_4, a_5\}$ is 2-connected (and hence, is a chain of blocks from a_1 to a_2). So by (1) of (2.6) (with G' as G in (2.6)), $G' - \{a_3, a_4, a_5\}$ contains a Hamilton path P between a_1 and a_2. Now $P + a_1a_2$ is a Hamilton cycle in $G - S_3$. □

Because every 4-connected planar graph contains a triangle (by Euler's formula), (2.8) implies that if G is a 4-connected planar graph on $n \geq 6$ vertices, then G has a cycle of length $n - 3$. We conclude this section by proving a convenient lemma.

(2.9) Lemma. *Let G be a graph and $\{a_1, a_2, a_3, a_4\} \subset V(G)$ such that G is $(4, \{a_1, a_2, a_3, a_4\})$-connected. Then $G - \{a_3, a_4\}$ is a chain of blocks from a_1 to a_2.*

Proof: Suppose for a contradiction that $G - \{a_3, a_4\}$ is not a chain of blocks from a_1 to a_2. Then there exist an end block B and a cut vertex b of $G - \{a_3, a_4\}$ such that $b \in V(B)$ and $(V(B) - \{b\}) \cap \{a_1, a_2\} = \emptyset$. Then $B - b$ is a component of $G - \{a_3, a_4, b\}$. Because $B - b$ contains no element of $\{a_1, a_2, a_3, a_4\}$, we have reached a contradiction to the assumption that G is $(4, \{a_1, a_2, a_3, a_4\})$-connected. □

3 Long cycles

As we have discussed in the previous section, the Tutte path technique alone cannot be used to produce cycles of length $n-l$ for $l \geq 4$. In this section, we will demonstrate a possible approach by considering contractible edges.

An edge e in a k-connected graph G is said to be *k-contractible* if the graph G/e is also k-connected. Tutte [9] has shown that K_4 is the only 3-connected graph with no 3-contractible edges. On the other hand, there are infinitely many 4-connected graphs with no 4-contractible edges, and in fact, all such graphs are characterized in the following result of Martinov [3].

(3.1) Theorem. *If G is a 4-connected planar graph with no 4-contractible edges, then G is either the square of a cycle of length at least 4 or the line graph of a cyclically 4-edge-connected cubic graph.*

The square of a cycle C is a graph obtained from C by adding edges joining vertices of C with distance two apart. It is not hard to see that if G is the square of a cycle, then G has cycles of length k for all $3 \leq k \leq |V(G)|$. However, (3.1) does not provide information about 4-contractible edges incident with a specific vertex. We show below that for a 4-connected planar graph G and a vertex u of G, either G contains a 4-contractible edge incident with u or there is a "useful" structure around u in G. From now on, by "contractible" we mean 4-contractible.

(3.2) Theorem. *Let G be a 4-connected planar graph and let $u \in V(G)$. Then one of the following holds:*

(1) G has a contractible edge incident with u; or

(2) there are 4-cuts S and T of G such that $1 \leq |S \cap T| \leq 2$, S contains u and a neighbor of u, T contains u and a neighbor of u, and $G-S$ has a component consisting of only one vertex which is also contained in T.

Proof: If G has a contractible edge incident with u, then (1) holds. So we may assume that G has no contractible edge incident with u. Hence, for every edge of G incident with u, both its incident vertices are contained in some 4-cut of G. Let $\mathcal{F}$ denote the set of those 4-cuts of G containing u and a neighbor of u. Note that $\mathcal{F} \neq \emptyset$. Select $S \in \mathcal{F}$ and a component A of $G-S$ such that

(i) for any $S' \in \mathcal{F}$ and for any component A' of $G-S'$, $|V(A)| \leq |V(A')|$.

Let $B = G-(V(A) \cup S)$. Let a be a neighbor of u contained in $V(A)$. Since the edge ua is not contractible, there is some $T \in \mathcal{F}$ such that $\{u, a\} \subset T$. Let C be a component of $G-T$, and let $D := G-(V(C) \cup T)$. This situation is illustrated in Figure 5.

(ii) We claim that $A \cap C = \emptyset = A \cap D$.

Suppose for a contradiction that (ii) is false. Without loss of generality, we may assume that $A \cap C \neq \emptyset$. For convenience, let $X := (S \cap V(C)) \cup (S \cap T) \cup (V(A) \cap T)$ and $Y := (S \cap V(D)) \cup (S \cap T) \cup (V(B) \cap T)$. Clearly, $G-X$ has a component contained in $A \cap C$. So $X \notin \mathcal{F}$ by (i). Since G is 4-connected, $|X| \geq 4$. Because $\{a, u\} \subset X$ and $X \notin \mathcal{F}$,

	A	S	B
C	$A \cap C$	$S \cap V(C)$	$B \cap C$
T	a ●	u ●	$V(B) \cap T$
D	$A \cap D$	$S \cap V(D)$	$B \cap D$

Figure 5: S, T, A, B, C, D

$|X| \geq 5$. Since $|X|+|Y| = |S|+|T| = 8$, $|Y| \leq 3$. Therefore, Y cannot be a cut set of G, and so, $B \cap D = \emptyset$. Assume for the moment that $|S \cap T| = |S \cap V(D)| = |V(B) \cap T| = 1$. This implies that $|S \cap V(C)| = 2 = |V(A) \cap T|$. Let $Z := (S \cap V(D)) \cup (S \cap T) \cup (V(A) \cap T)$; then $|Z| = 4$. If $A \cap D \neq \emptyset$, then $A \cap D$ contains a component of $G - Z$, and so, Z contradicts the choice of S in (i) (since $\{u, a\} \subset Z$). Thus $A \cap D = \emptyset$, and hence, $|V(D)| = 1$. But then T and D contradict the choice of S and A in (i). So at least one of $|S \cap T|$, $|S \cap V(D)|$, or $|V(B) \cap T|$ is at least 2. Since $|Y| = 3$, either $|S \cap V(D)| = 0$ or $|V(B) \cap T| = 0$. If $|S \cap V(D)| = 0$ then $D = D \cap A \neq \emptyset$, and hence, T and D contradict the choice of S and A in (i). So $|S \cap V(D)| \neq 0$. Then $|V(B) \cap T| = 0$ and $|S - V(D)| \leq 3$. Hence, $B \cap C = B \neq \emptyset$ contains a component of $G - (S - V(D))$, contradicting the assumption that G is 4-connected. This completes the proof of (ii).

By (ii), $V(A) = V(A) \cap T$. If $S \cap V(D) = \emptyset$, then by (ii), $B \cap D = D \neq \emptyset$ contains a component of $G - (T - V(A))$, a contradiction (because $|T - V(A)| \leq 3$ and G is 4-connected). Similarly, if $S \cap V(C) = \emptyset$, then by (ii), $B \cap C = C \neq \emptyset$ contains a component of $G - (T - V(A))$, a contradiction. So we have

(iii) $S \cap V(D) \neq \emptyset \neq S \cap V(C)$.

(iv) We further claim that $V(B) \cap T \neq \emptyset$.

Suppose on the contrary that $V(B) \cap T = \emptyset$. Then since $B \neq \emptyset$, $B \cap D \neq \emptyset$ or $B \cap C \neq \emptyset$. If $B \cap D \neq \emptyset$ then $S - V(C)$ is a cut of G, and if $B \cap C \neq \emptyset$ then $S - V(D)$ is a cut of G. Since $|S - V(C)| \leq 3 \geq |S - V(D)|$ (by (iii)), we have a contradiction to the assumption that G is 4-connected. This proves (iv).

By (ii) and (iv) and because $u \in S \cap T$, $|V(A)| = |V(A) \cap T| \leq 2$. By (iii), $|S \cap T| \leq 2$. If $|V(A)| = 1$ then $V(A) = \{a\} \subset T$, and we have (2). So we may assume that $|V(A)| = 2$. Then by (iv), $|S \cap T| = 1 = |V(B) \cap T|$. Since $|S| = 4$, $|S \cap V(C)| \leq 1$ or $|S \cap V(D)| \leq 1$. By the symmetry between C and D, we may assume that $|S \cap V(C)| \leq 1$. Then since G is 4-connected, $B \cap C = \emptyset$. Hence by (ii), $V(C) = S \cap V(C)$. This means $|V(C)| = 1$, and so, T, C contradict the choice of S, A in (i). □

When dealing with the structures in (2) of (3.2) in the proof of (1.3), we need to find two paths between vertices of $S \cup T$, one in $G - (V(D) \cup \{a\})$ and the other in $G - (V(C) \cup \{a\})$, such that the union of these two paths gives the desired cycle. The following two technical lemmas will be useful for this purpose.

(3.3) Lemma. *Let H be a graph and $\{a_1, a_2, a_3, a_4\} \subset V(H)$. Assume that (H, a_1, a_2, a_3, a_4) is planar, H is $(4, \{a_1, a_2, a_3, a_4\})$-connected, and a_1 has at least two neighbors contained in $V(H) - \{a_1, a_2, a_3, a_4\}$. Then one of the following holds:*

(1) $H - \{a_2, a_3, a_4\}$ is 2-connected; or

(2) both $H - \{a_1, a_3, a_4\}$ and $H - \{a_1, a_2, a_3\}$ are 2-connected.

Proof: Without loss of generality we may assume that H is drawn in a closed disc with no pair of edges crossing such that a_1, a_2, a_3, a_4 occur in this clockwise order on the boundary of the disc. By planarity,

(i) H contains no disjoint paths from a_1 to a_3 and from a_2 to a_4, respectively.

If $H' := H - \{a_2, a_3, a_4\}$ is 2-connected, then (1) holds. So we may assume that H' is not 2-connected. We need to show that (2) holds. Let $H_2, \ldots, H_m$ denote the end blocks of H' and let $v_2, \ldots, v_m$ denote the cut vertices of H' such that for $k = 2, \ldots, m$, $v_k \in V(H_k)$ and $a_1 \notin V(H_k) - \{v_k\}$. Note that $m \geq 2$ because H' is not 2-connected. We claim that

(ii) for any $k \in \{2, \ldots, m\}$ and any $j \in \{2, 3, 4\}$, a_j has a neighbor in $V(H_k) - \{v_k\}$.

Suppose (ii) fails for some $k \in \{2, \ldots, m\}$ and for some $j \in \{2, 3, 4\}$. Then $H_k - v_k$, and hence $H - ((\{a_2, a_3, a_4\} - \{a_j\}) \cup \{v_k\})$, has a component containing no element of $\{a_1, a_2, a_3, a_4\}$, contradicting the assumption that H is a $(4, \{a_1, a_2, a_3, a_4\})$-connected. So (ii) holds.

If $m \geq 3$, then by (ii) we can find a path P from a_4 to a_2 in $H[V(H_2) \cup \{a_2, a_4\}] - v_2$ and find a path Q from a_1 to a_3 in $H - ((V(H_2) - \{v_2\}) \cup \{a_2, a_4\})$. Note that P and Q are disjoint paths in H, contradicting (i). So $m = 2$. Therefore, H' has exactly two end blocks. Let H_1 denote the other end block of H', and let v_1 denote the cut vertex of H' contained in $V(H_1)$. See Figure 6.

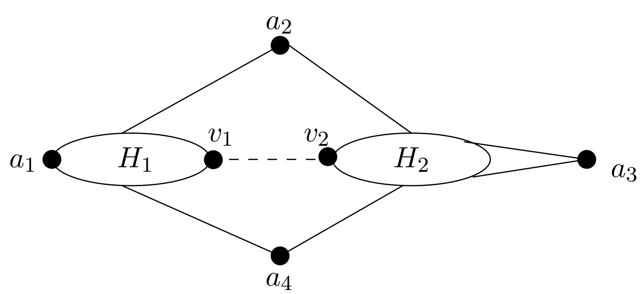

Figure 6: H and end blocks H_1, H_2 of H'.

By the definitions of H_k for $k = 2, \ldots, m$, $a_1 \in V(H_1) - \{v_1\}$. Since a_1 has at least two neighbors in $V(H) - \{a_1, a_2, a_3, a_4\}$, a_1 has at least two neighbors in $V(H_1)$. Hence $|V(H_1)| \geq 3$. Because a_2, a_4 have neighbors in $V(H_2) - \{v_2\}$ (by (ii)) and by planarity, we conclude that

(iii) a_3 has no neighbor in $V(H_1)$.

We further claim that

(iv) each element of $\{a_2, a_4\}$ has a neighbor in $V(H_1) - \{a_1, v_1\}$.

Suppose (iv) fails. By symmetry between a_2 and a_4, we may assume that a_2 has no neighbor in $V(H_1) - \{a_1, v_1\}$. Then by (iii), $H_1 - \{a_1, v_1\}$, and hence, $H' - \{a_1, v_1, a_4\}$, has a component containing no element of $\{a_1, a_2, a_3, a_4\}$, contradicting the assumption that H is a $(4, \{a_1, a_2, a_3, a_4\})$-connected. So (iv) holds.

By (ii) and (iv), each element of $\{a_2, a_4\}$ has at least two neighbors in $V(H) - \{a_1, a_2, a_3, a_4\}$. We consider $H'' := H - \{a_1, a_3, a_4\}$. Suppose that H'' is not 2-connected. Note that a_2, $N_H(a_2)$, and $V(H_2)$ are all contained in one end block of H''. Let H^* denote another end block of H'', and let v^* denote the cut vertex of H'' contained in $V(H^*)$. Then a_2 has no neighbor in $V(H^*) - \{v^*\}$ and $H^* \subset H_1$. By (iii) and since $H^* \subset H_1$, a_3 has no neighbor in $V(H^*) - \{v^*\}$. Hence, $H^* - \{v^*\}$ is a component of $H - \{a_1, a_4, v^*\}$ containing no element of $\{a_1, a_2, a_3, a_4\}$, contradicting the assumption that H is $(4, \{a_1, a_2, a_3, a_4\})$-connected. Therefore, $H'' := H - \{a_1, a_3, a_4\}$ is 2-connected. By the same argument (using symmetry between a_2 and a_4), we can prove that $H - \{a_1, a_2, a_3\}$ is 2-connected. □

(3.4) Lemma. *Let H be a graph and $\{a_1, a_2, a_3, a_4\} \subset V(H)$. Assume that (H, a_1, a_2, a_3, a_4) is planar, H is $(4, \{a_1, a_2, a_3, a_4\})$-connected, and $|V(H)| \geq 6$. Then there is a vertex $z \in V(H) - \{a_1, a_2, a_3, a_4\}$ such that $H - \{z, a_3, a_4\}$ has a Hamilton path from a_1 to a_2.*

Proof: Without loss of generality, we may assume that H is drawn in a closed disc with no pair of edges crossing such that a_1, a_2, a_3, a_4 occur in this clockwise order on the boundary of the disc. By (2.9), we have

(i) $H - \{a_3, a_4\}$ is a chain of blocks from a_1 to a_2.

(ii) We further claim that $H - \{a_3, a_4\}$ has a non-trivial block.

For otherwise, $H - \{a_3, a_4\}$ is a path. Because $|V(H)| \geq 6$, $H - \{a_3, a_4\}$ has at least four vertices. Since H is $(4, \{a_1, a_2, a_3, a_4\})$-connected, every vertex in $V(H) - \{a_1, a_2, a_3, a_4\}$ is adjacent to both a_3 and a_4. But this implies that H has disjoint paths from a_1 to a_3 and from a_2 to a_4, respectively, contradicting the assumption that (H, a_1, a_2, a_3, a_4) is planar.

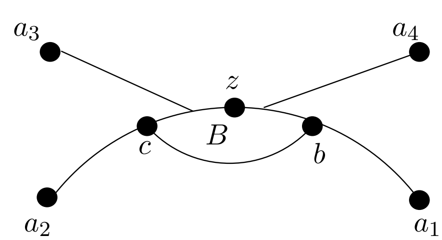

Figure 7: The graph H

By (ii), let B be a non-trivial block of $H - \{a_3, a_4\}$. Let C denote the outer cycle of B. Let $b = a_1$ if $a_1 \in V(B)$, and otherwise let $b \in V(C)$ denote the cut vertex of $H - \{a_3, a_4\}$ separating a_1 from B. Let $c = a_2$ if $a_2 \in V(B)$, and otherwise let $c \in V(C)$ denote the cut vertex of $H - \{a_3, a_4\}$ separating a_2 from B. See Figure 7.

Note that both a_3 and a_4 have neighbors in $V(cCb) - \{b, c\}$. Otherwise, $B - \{b, c\}$ contains a component of $H - \{a_3, b, c\}$ or a component of $H - \{a_4, b, c\}$. Since $B - \{b, c\}$ contains no element of $\{a_1, a_2, a_3, a_4\}$, we have a contradiction to the assumption that H is $(4, \{a_1, a_2, a_3, a_4\})$-connected.

By planarity, we can pick $z \in V(cCb) - \{b, c\}$ such that $N_H(a_3) \cap V(B) \subset V(cCz)$ and $N_H(a_4) \cap V(B) \subset V(zCb)$. We see that

(iii) $(H, a_1, a_2, a_3, z, a_4)$ is planar.

In order to apply (2.6), we need to show that

(iv) $H - \{a_3, a_4, z\}$ is a chain of blocks from a_1 to a_2.

Suppose on the contrary that (iv) is false. Then by (i) and (ii), there is an end block B_1 of $B - z$ such that $(V(B_1) - \{v_1\}) \cap \{b, c\} = \emptyset$, where v_1 is the cutvertex of $B - z$ contained in $V(B_1)$. Suppose both a_3 and a_4 have neighbors in $B_1 - v_1$. Then by planarity, all neighbors of z are contained in $V(B_1)$. This implies that the component of $G - \{a_3, a_4, v_1\}$ containing z contains no element of $\{a_1, a_2, a_3, a_4\}$, contradicting the assumption that G is $(4, \{a_1, a_2, a_3, a_4\})$-connected. So either a_3 or a_4 has no neighbor contained in $V(B_1) - \{v_1\}$. Hence, $B_1 - v_1$ is a component of $H - \{v_1, z, a_3\}$ or a component of $H - \{v_1, z, a_4\}$. Since $B_1 - v_1$ contains no element of $\{a_1, a_2, a_3, a_4\}$, we have a contradiction to the assumption that H is $(4, \{a_1, a_2, a_3, a_4\})$-connected. This proves (iv).

By (iii) and (iv), we can apply (1) of (2.6) (with H, a_1, a_2, a_3, z, a_4 as $G, a_1, a_2, a_3, a_4, a_5$ in (2.6), respectively), and we find the desired Hamilton path between a_1 and a_2 in $H - \{z, a_3, a_4\}$. $\square$

In order to prove our main result, we prove a stronger result for $l \leq 5$.

(3.5) Theorem. *Let G be a 4-connected planar graph and let $u \in V(G)$. Then for each $l \in \{1, \ldots, 5\}$ there is a set $S_l \subset V(G)$ such that $u \in S_l$, $|S_l| = l$, and if $|V(G)| \geq l + 3$ then $G - S_l$ has a Hamilton cycle.*

Proof: Suppose that this theorem is not true. Let G be a counter example such that $|V(G)|$ is minimum. We will derive a contradiction by finding a set $S_l \subset V(G)$ for each $l \in \{1, 2, 3, 4, 5\}$ such that $u \in S_l$, $|S_l| = l$, and if $|V(G)| \geq l + 3$ then $G - S_l$ has a Hamilton cycle.

We claim that G contains no contractible edge incident with u. Otherwise, let $e = uv$ be a contractible edge of G incident with u. Then G/e is also a 4-connected planar graph. Let u^* denote the vertex of G/e resulted from the contraction of e. By the choice of G, for each $l \in \{1, \ldots, 5\}$, there is a set $S_l^* \subset V(G/e)$ such that $u^* \in S_l^*$, $|S_l^*| = l$, and if $|V(G/e)| \geq l + 3$ then $G/e - S_l^*$ has a Hamilton cycle. For $l = 1, 2, 3, 4$, let $S_{l+1} = (S_l^* - \{u^*\}) \cup \{u, v\}$. Then $G - S_{l+1} = G/e - S_l^*$ has a Hamilton cycle for $l \in \{1, \ldots, 4\}$. Let $S_1 = \{u\}$. By (2.7), $G - S_1$ has a Hamilton cycle. Therefore, G is not a counter example, a contradiction.

Hence by (3.2) there are 4-cuts S and T of G such that $1 \leq |S \cap T| \leq 2$, S contains u and a neighbor of u, T contains u and neighbor of u, and $G - S$ has a component A consisting of only one vertex which is also in T. Let a be the only vertex in $V(A)$, and let $B := G - (\{a\} \cup S)$. Let C be a component of $G - T$ and let $D := G - (V(C) \cup T)$. (See Figure 5.)

We claim that $S \cap V(C) \neq \emptyset \neq S \cap V(D)$. For if $S \cap V(C) = \emptyset$, then $B \cap C = C \neq \emptyset$ is a component of $G - (T - \{a\})$, contradicting the assumption that G is 4-connected. Similarly, if $S \cap V(D) = \emptyset$ then $B \cap D = D \neq \emptyset$ is a component of $G - (T - \{a\})$, a contradiction.

We consider two cases.

Case 1. The above S and T may be chosen such that $|S \cap T| = 2$.

In this case, $|S \cap V(C)| = 1 = |S \cap V(D)|$ (because $S \cap V(C) \neq \emptyset \neq S \cap V(D)$). By symmetry, we may assume that $|V(B) \cap V(C)| \leq |V(B) \cap V(D)|$. Recall that $u \in S \cap T$. Let v denote the other vertex in $S \cap T$, let w denote the vertex in $S \cap V(C)$, let b denote the vertex in $S \cap V(D)$, and let c denote the vertex in $V(B) \cap T$. See Figure 8. Note that $\{a, u\}$ is contained in a triangle of G because S contains u and some neighbor of u. So by (2.7) and (2.8), there exists $S_l \subset V(G)$ for each $l \in \{1, 2, 3\}$ such that $u \in S_l$, $|S_l| = l$, and if $|V(G)| \geq l + 3$ then $G - S_l$ has a Hamilton cycle.

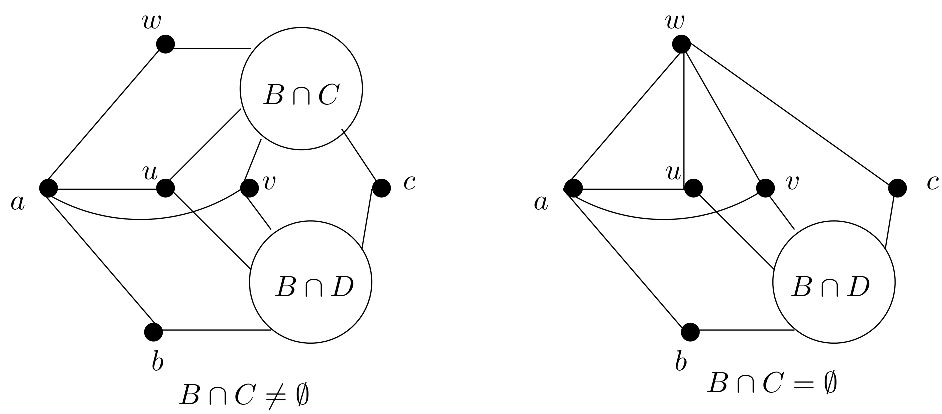

Figure 8: Case 1

To derive a contradiction, we need to find S_l for $l = 4, 5$ and $V(G)| \geq l + 3$. Let $H_1 := G[V(C) \cup \{u, v, c\}]$ and $H_2 := G[V(D) \cup \{u, v, c\}]$. Since $au, av \in E(G)$, in any plane representation of G, a and v are cofacial, and a and u are cofacial. Because T is a cut set of G, we see that in any plane representation of G, c and v are cofacial, and c and u are cofacial. Therefore, since a is adjacent to both b and w, (H_1, c, v, w, u) is planar and (H_2, c, v, b, u) is planar. Since G is 4-connected, H_1 is $(4, \{c, v, w, u\})$-connected (if $B \cap C \neq \emptyset$) and H_2 is $(4, \{c, v, b, u\})$-connected (if $B \cap D \neq \emptyset$). Therefore by (2.9), $H_1 - \{u, w\}$ is a chain of blocks from c to v, and $H_2 - \{u, b\}$ is a chain of blocks from c to v. Then by applying (1) of (2.6) (with H_1, c, v, w, u as G, a_1, a_2, a_3, a_4 in (2.6), respectively), we have that

(i) if $B \cap C \neq \emptyset$ then $H_1 - \{u, w\}$ has a Hamilton path P_1 from c to v.

Similarly, by applying (1) of (2.6) (with H_2, c, v, b, u as G, a_1, a_2, a_3, a_4 in (2.6), respectively), we have that

(ii) if $B \cap D \neq \emptyset$ then $H_2 - \{u, b\}$ has a Hamilton path P_2 from c to v.

By applying (3.4) (with H_2, c, v, b, u as H, a_1, a_2, a_3, a_4 in (3.4), respectively), we have that

(iii) if $|V(B) \cap V(D)| \geq 2$ then there is a vertex $z \in V(B) \cap V(D)$ such that $H_2 - \{z, b, u\}$ has a Hamilton path P_2' from c to v.

(iv) We may assume that $B \cap C = \emptyset$.

Suppose that $B \cap C \neq \emptyset$. Because $|V(B) \cap V(D)| \geq |V(B) \cap V(C)|$, $B \cap D \neq \emptyset$. Let $S_4 := \{a, b, u, w\}$; then by (i) and (ii), $P_1 \cup P_2$ is a Hamilton cycle in $G - S_4$. If $|V(B) \cap V(D)| \geq 2$ then let $S_5 := \{a, b, u, w, z\}$; and by (i) and (iii), $P_2' \cup P_1$ is a Hamilton cycle in $G - S_5$. So $|V(B) \cap V(D)| = 1$. Then $|V(B) \cap V(C)| = 1$ since $1 \leq |V(B) \cap V(C)| \leq |V(B) \cap V(D)|$. Therefore $|V(G)| = 8$. Let y denote the vertex in $V(B) \cap V(C)$, and let z denote the vertex in $V(B) \cap V(D)$. Then $N_G(y) = \{c, u, v, w\}$. Because c is not adjacent to a and the degree of c is at least 4, c is adjacent to at least one element of $\{b, v, w\}$. If c is adjacent to v then let $S_5 := \{a, b, u, w, y\}$, if c is adjacent to b then let $S_5 := \{a, u, v, w, y\}$, and if c is adjacent to w then let $S_5 := \{a, b, u, v, z\}$. It is then easy to see that $G - S_5$ has a Hamilton cycle. This completes the proof of (iv).

By (iv), $N_G(w) = T$. We may assume that $|V(B) \cap V(D)| \geq 1$; otherwise there is nothing to prove. We may further assume that

(v) $|V(B) \cap V(D)| \geq 2$.

Otherwise, $|V(B) \cap V(D)| = 1$. In this case, we only need to find S_4. Let z denote the vertex in $V(B) \cap V(D)$. Then $N_G(z) = \{b, c, u, v\}$. Because c is not adjacent to a and the degree of c is at least 4, c is adjacent to at least one element of $\{b, v\}$. If c is adjacent to b then let $S_4 := \{a, u, v, w\}$, and if c is adjacent to v then let $S_4 := \{a, b, u, w\}$. It is easy to check that $G - S_4$ has a Hamilton cycle.

(vi) We may assume that c is not adjacent to v.

Suppose c is adjacent to v. Let $S_4 := \{a, b, u, w\}$; then $P_2 + cv$ is a Hamilton cycle in $G - S_4$. Let $S_5 := \{a, b, u, w, z\}$; then by (v) and (iii), $P_2' + cv$ is a Hamilton cycle in $G - S_5$.

(vii) We may further assume that c is not adjacent to b.

If c is adjacent to b, then by deleting aw, by contracting ab, and by contracting $B \cap D$ to a single vertex, we produce a $K_{3,3}$-minor of G, a contradiction. So we have (vii).

(viii) We may assume that b has at least two neighbors in $V(B) \cap V(D)$.

If b is not adjacent to v, then (viii) follows from (vii). So we may assume that b is adjacent to v. Recall that (H_2, v, b, u, c) is planar. By (v), H_2 is $(4, \{b, c, u, v\})$-connected. So by (2.9) (with H_2, v, b, u, c as G, a_1, a_2, a_3, a_4 in (2.9), respectively), $H_2 - \{u, c\}$ is a chain of blocks from b to v. Hence we can apply (1) of (2.6) (with H_2, v, b, u, c as G, a_1, a_2, a_3, a_4 in (2.6), respectively) to find a Hamilton path Q in $H_2 - \{u, c\}$ between v and b. We can also apply (3.4) (with $H_2, v, b, u, c,$ as G, a_1, a_2, a_3, a_4 in (3.4), respectively) to find a vertex $z' \in V(B) \cap V(D)$ and a Hamilton path Q' in $H_2 - \{u, c, z'\}$ between v and b. Let $S_4 := \{a, u, c, w\}$; then $Q + vb$ is a Hamilton cycle in $G - S_4$. Let $S_5 := \{a, u, c, w, z'\}$; then $Q' + vb$ is a Hamilton cycle in $G - S_5$. So (viii) holds.

Because c is not adjacent to a and by (vi) and (vii), c is adjacent to none of $\{a, b, v\}$. Hence,

(ix) c has at least two neighbors in $V(B) \cap V(D)$.

By (viii) and (ix) and by (3.3) (with H_2, c, v, b, u as H, a_1, a_2, a_3, a_4 in (3.3), respectively), there is some $x \in \{v, c\}$ such that $H_2 - (\{b, c, u, v\} - \{x\})$ is 2-connected. Pick a vertex x' of $H_2 - (\{b, c, u, v\} - \{x\})$ such that $x'x$ is an edge and H_2 can be drawn in a closed disc so that xx' lies on the boundary and $x, x', \{b, c, u, v\} - \{x\}$ occur in this cyclic order on the boundary of the disc. Note that x' exists because c is adjacent to none of $\{a, b, v\}$. By (1) of (2.6) (with $H_2 - (\{b, c, u, v\} - \{x\}), x, x', \{b, c, u, v\} - \{x\}$ as $G, a_1, a_2, \{a_3, a_4, a_5\}$ in (2.6), respectively), $H_2 - (\{b, c, u, v\} - \{x\})$ has a Hamilton path R from x to x'. Because b has at least two neighbors in $V(B) \cap V(D)$, we can apply (2) of (2.6) (with $H_2 - (\{b, c, u, v\} - \{x\}), x, x', \{b, c, u, v\} - \{x\}$ as $G, a_1, a_2, \{a_3, a_4, a_5\}$ in (2.6), respectively), to find a Hamilton path R' in $H - (\{c, u, v\} - \{x\})$ from x to x'. Now let $S_4 := \{a, u, w\} \cup (\{v, c\} - \{x\})$; then $R' + xx'$ is a Hamilton cycle in $G - S_4$. Let $S_5 := \{a, u, w\} \cup (\{b, v, c\} - \{x\})$; then $R + xx'$ is a Hamilton cycle in $G - S_5$.

Case 2. For all choices of S and T, we have $|S \cap T| = 1$.

Then $S \cap T = \{u\}$. Let a be the only vertex in A, and let $B := G - (\{a\} \cup S)$. Let C be a component of $G - T$ and let $D := G - (V(C) \cup T)$ such that $|S \cap V(C)| = 2$ and $|S \cap V(D)| = 1$. This can be done because $S \cap V(C) \neq \emptyset \neq S \cap V(D)$. Let v, w denote the vertices in $S \cap V(C)$, let b denote the only vertex in $S \cap V(D)$, and let c, d denote the vertices in $V(B) \cap T$. See Figure 9.

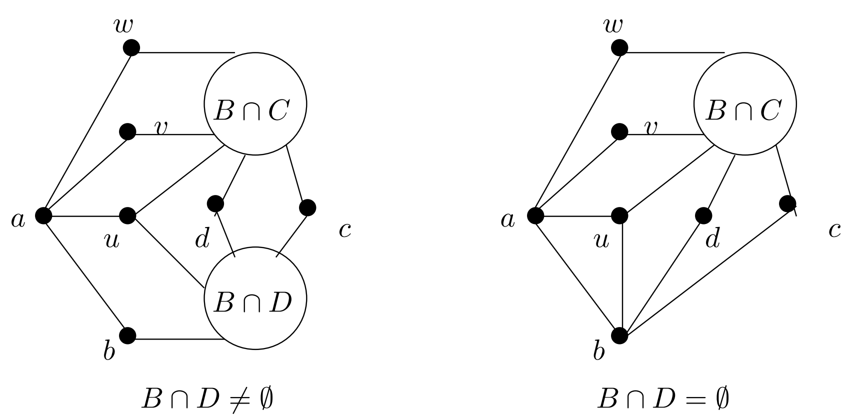

Figure 9: Case 2

Let $H_1 := G[V(C) \cup \{u, c, d\}]$ and let $H_2 := G[V(D) \cup \{u, c, d\}]$. Because a is adjacent to u and T is a 4-cut of G, c and d are cofacial. Likewise, v and w are cofacial. Without loss of generality, assume that (H_1, c, d, u, v, w) is planar. Then (H_2, c, d, u, b) is planar. We claim that

(i) $B \cap C \neq \emptyset$.

Suppose $B \cap C = \emptyset$. Then one element of $\{v, w\}$ is not adjacent to some element of $\{c, d\}$; otherwise, by contracting $G[V(D) \cup \{u\}]$ to a single vertex, we produce a $K_{3,3}$-minor of G, a contradiction. If v is not adjacent some element of $\{c, d\}$, then $T' := N_G(v) \in \mathcal{F}$ and $|S \cap T'| = 2$, a contradiction (since we are in Case 2). Similarly, if w is not adjacent some element of $\{c,d\}$, then $T' := N_G(w) \in \mathcal{F}$ and $|S \cap T'| = 2$, a contradiction.

(ii) We claim that $H_1 - \{u,v,w\}$ is a chain of blocks from c to d.

Otherwise, by (i), let K be an end block of $H_1 - \{u,v,w\}$ and let r be the cut vertex of $H_1 - \{u,v,w\}$ contained in $V(K)$ such that $(V(K) - \{r\}) \cap \{c,d\} = \emptyset$. Since G is 4-connected, each element of $\{u,v,w\}$ has a neighbor in $V(K) - \{r\}$. Since (H_1, c, d, u, v, w) is planar, $T' := \{a, u, r, w\} \in \mathcal{F}$ and $|S \cap T'| = 2$, a contradiction (since we are in Case 2). So (ii) holds.

Since (H_1, c, d, u, v, w) is planar and by (ii), we may apply (1) of (2.6) (with H_1, c, d, u, v, w as $G, a_1, a_2, a_3, a_4, a_5$ in (2.6), respectively). Hence,

(iii) there is a Hamilton path P in $H_1 - \{u,v,w\}$ from c to d.

We may assume that

(iv) $B \cap D = \emptyset$ for all choices of S, T, A, B, C, D with $|S \cap V(D)| = 1$ and $|S \cap V(C)| = 2$.

Suppose $B \cap D \neq \emptyset$ for some choice of S, T, A, B, C, D. Since (H_2, c, d, u, b) is planar and by (2.9) (with H_2, c, d, u, b as G, a_1, a_2, a_3, a_4, respectively), $H_2 - \{b, u\}$ is a chain of blocks from c to d. By applying (1) of (2.6) (with H_2, c, d, u, b as G, a_1, a_2, a_3, a_4 in (2.6), respectively), we find a Hamilton path R from c to d in $H_2 - \{b, u\}$. Because the degree of b is at least 4, b has at least two neighbors in $V(H_2) - \{u\}$. Therefore, by applying (2) of (2.6) (with H_2, c, d, u, b as G, a_1, a_2, a_3, a_4 in (2.6), respectively), we find a Hamilton path Q between c and d in $H_2 - u$. Let $S_4 := \{a, u, v, w\}$ and let $S_5 := \{a, b, u, v, w\}$. Then $P \cup Q$ is a Hamilton cycle in $G - S_4$ and $P \cup R$ is a Hamilton cycle in $G - S_5$. This completes the proof of (iv).

Let $S_4 := \{a, u, v, w\}$; then by (iii) and (iv), $(P + b) + \{bc, bd\}$ is a Hamilton cycle in $G - S_4$. Next we construct S_5. If c is adjacent to d, then let $S_5 := \{a, b, u, v, w\}$, and by (ii), $P + cd$ is a Hamilton cycle in $G - S_5$. So we may assume that c is not adjacent to d.

(v) We may assume that d has at least two neighbors in $V(B) \cap V(C)$.

Otherwise, assume that d has at most one neighbor in $V(B) \cap V(C)$. Since (H_1, c, d, u, v, w) is planar and because c is not adjacent to d, d is adjacent to both u and v, u is adjacent to v, u has no neighbor in $V(B) \cap V(C)$, and d has exactly one neighbor in $V(B) \cap V(C)$. Let $H' := H_1 - u$. Then (H', c, d, v, w) is planar and H' is $(4, \{c, d, v, w\})$-connected (since G is 4-connected). Hence by (2.9) (with H', d, v, w, c as G, a_1, a_2, a_3, a_4 as in (2.9), respectively), $H' - \{c, w\}$ is a chain of blocks from d to v. By (1) of (2.6) (with H', d, v, w, c as G, a_1, a_2, a_3, a_4 in (2.6), respectively), $H' - \{c, w\}$ contains a Hamilton path P' from d to v. Let $S_5 := \{a, b, c, u, w\}$. Then $P' + dv$ is a Hamilton cycle in $G - S_5 = H' - \{c, w\}$. This proves (v).

(vi) We claim that $H_1 - \{c, d, u\}$ is a chain of blocks from v to w.

Otherwise, let K denote an end block of $H_1 - \{c, d, u\}$ and let r be the cut vertex of $H_1 - \{c, d, u\}$ contained in $V(K)$ such that $(V(K) - \{r\}) \cap \{v, w\} = \emptyset$. Since G is 4-connected and (H_1, c, d, u, v, w) is planar, each of $\{c, d, u\}$ has a neighbor in $V(K) - \{r\}$. Since (H_1, c, d, u, v, w) is planar, $T' := \{a, c, u, r\} \in \mathcal{F}$. Let C' be the component of $G - T'$ containing $\{v, w\}$, and let $D' := G - (V(C') \cup T')$. Then $|S \cap V(D')| = 1$, $|S \cap V(C')| = 2$, and $B \cap D' \neq \emptyset$, contradicting (iv). This completes the proof of (vi).

If v is adjacent to w, then let $S_5 := \{a, b, c, d, u\}$. By (1) of (2.6) (with H_1, v, w, c, d, u as $G, a_1, a_2, a_3, a_4, a_5$ in (2.6), respectively), we find a Hamilton path P' in $H_1 - \{c, d, u\}$ from v to w. Then $P' + vw$ is a Hamilton cycle in $G - S_5$. So we may assume that

(vii) v is not adjacent to w.

By (vii) and by the same argument as for (v) (by exchanging the roles of d and v and by exchanging the roles of c and w), we may assume that

(viii) v has at least two neighbors in $V(B) \cap V(C)$.

(ix) We claim that $H_1 - \{c, u, w\}$ is 2-connected.

Suppose on the contrary that $H_1 - \{c, u, w\}$ is not 2-connected. Let $J_1, \ldots, J_m$ denote the end blocks of $H_1 - \{c, u, w\}$, and let v_i be the cutvertex of $H_1 - \{c, u, w\}$ contained in $V(J_i)$ (for $i = 1, \ldots, m$). Then for any $i \in \{1, \ldots, m\}$, either $v \in V(J_i) - \{v_i\}$ or $d \in V(J_i) - \{v_i\}$; otherwise, each element of $\{c, u, w\}$ has a neighbor in $V(J_i) - \{v_i\}$ (because G is 4-connected), and this contradicts the assumption that (H_1, c, d, u, v, w) is planar. Hence $m = 2$, and we may assume that $d \in V(J_1) - \{v_1\}$ and $v \in V(J_2) - \{v_2\}$. By (v) and (viii), $|V(J_1)| \geq 3$ and $|V(J_2)| \geq 3$. Since G is 4-connected and by planarity, $w, u \in N_G(V(J_2) - \{v_2\})$ and $u, c \in N_G(V(J_1) - \{v_1\})$. Since (H_1, c, d, u, v, w) is planar, $T' := \{a, u, v_2, w\} \in \mathcal{F}$ or $T'' := \{a, u, v_1, c\} \in \mathcal{F}$. If $T' \in \mathcal{F}$, then $|S \cap T'| = 2$, a contradiction (since we are in Case 2). So $T'' \in \mathcal{F}$. Let C' be the component of $G - T''$ containing $\{v, w\}$, and let $D' := G - (V(C') \cup T'')$. Then $|S \cap V(D')| = 1$, $|S \cap V(C')| = 2$, and $B \cap D' \neq \emptyset$, contradicting (iv). This proves (ix).

So let F denote the outer cycle of $H_1 - \{u, c, w\}$. Let d' be a neighbor of d on F such that d', d, v occur on F in this clockwise order. Let $y \in V(vFd')$ such that $N_{H_1}(w) \subset V(vFy)$ and $N_{H_1}(c) \subset V(yFd)$. Let e and f be edges of F incident with v and y, respectively. Applying (2.3) (with $H_1 - \{c, u, w\}, F, d', d$ as G, C, x, y in (2.3), respectively), we find an F-Tutte path P^* in $H_1 - \{c, u, w\}$ from d to d' such that $e, f \in E(P^*)$. Since G is 4-connected, we can show (as in the proof of (2.6)) that P^* is a Hamilton path in $H_1 - \{c, u, w\}$. Let $S_5 := \{a, b, c, u, w\}$; then $P^* + dd'$ is a Hamilton cycle in $G - S_5$. □

Proof of (1.3). Suppose this theorem is not true. Let G be a counter example such that $|V(G)|$ is minimum. If G contains a contractible edge e, we consider G/e. Let u be the vertex resulted from the contraction of e. Applying (3.5), we see that for each $l \in \{1, \ldots, 5\}$, there is some $S_l \subset V(G/e)$ such that $u \in S_l$, $|S_l| = l$, and if $|V(G/e)| \geq l + 3$ then $G/e - S_l$ has a Hamilton cycle. Hence, for each $l \in \{1, \ldots, 6\}$, if $n \geq l + 3$ then G has a cycle of length $n - l$. By (2.7), G also has a cycle of length n.

So G contains no contractible edge. Then by (3.1), either G is the square of a cycle or G is the line graph of a cyclically 4-edge-connected cubic graph. Because G is a counter example, G is the line graph of a cyclically 4-edge-connected cubic graph. Therefore G is 4-regular, every vertex is contained in exactly two triangles, and no two triangles share an edge. Using these properties and by planarity, it is easy to show that every triangle T in G is contractible, that is, G/T is 4-connected and planar. Let u denote the new vertex resulted from the contraction of T. Now by (3.5), for each $l \in \{1, \ldots, 5\}$, there is some $S_l \subset V(G/T)$ such that $u \in S_l$, $|S_l| = l$, and if $|V(G/T)| \geq l + 3$ then $G/T - S_l$ has a Hamilton cycle. Hence, G has cycles of length $n - l$ for each $l \in \{4, \ldots, 8\}$ with

$n - l \geq 3$. That G has a cycle of length $n, n-1, n-2, n-3$ follows from (2.7) and (2.8). $\square$

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