The Circumference of a Graph with no $K_{3,t}$-minor, II

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Abstract

The class of graphs with no $K_{3,t}$-minors, $t ≥ 3$, contains all planar graphs and plays an important role in graph minor theory. In 1992, Seymour and Thomas conjectured the existence of a function $α(t) > 0$ and a constant $β > 0$, such that every 3-connected $n$-vertex graph with no $K_{3,t}$-minors, $t ≥ 3$, contains a cycle of length at least $α(t)n^β$. The purpose of this paper is to confirm this conjecture with $α(t) = (1/2)^{(t-1)}$ and $β = \log_{1729} 2$.

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1 Introduction

Let \( G \) be a graph. The circumference of \( G \), denoted by \( c(G) \), is the length of a longest cycle in \( G \). The problem of determining \( c(G) \) is a classical \( NP \)-hard problem, so the focus of extensive research has been on the lower bound of \( c(G) \). While studying paths in polytopes, Moon and Moser [11] implicitly conjectured that for every 3-connected planar graph \( G \) on \( n \) vertices, \( c(G) = \Omega(n^{\log_2 3}) \); Grünbaum and Walther [8] later made (explicitly) the same conjecture for 3-connected cubic planar graphs. Over the past four decades various authors have obtained several theorems related to the Moon-Moser conjecture; see, for instance, [7, 9]. This conjecture was eventually established by Chen and Yu [5], where the same bound (within a constant factor) was also derived for 3-connected graphs embeddable in the torus or the Klein bottle. In [3] this result was applied to prove that \( c(G) \geq \epsilon(n) n^{\log_3 2} \) for every 3-connected \( n \)-vertex graph \( G \) of orientable genus \( g \), where \( \epsilon(n) \) is a positive function dependent on \( g \); we refer to [14] for an improved bound (with a positive constant in place of \( \epsilon(n) \)) for “locally planar” graphs.

It is well known that a 3-connected graph with no \( K_{3,3} \)-minor is planar, with the exception of \( K_5 \). So the result obtained by Chen and Yu [5] can be extended to graphs with no \( K_{3,3} \)-minors; and thus a natural question is to ask whether a similar result holds for graphs with no \( K_{3,1,t} \)-minors, where \( t \geq 4 \). As discovered by Robertson and Seymour [13], the class of graphs with no \( K_{3,1,t} \)-minors plays an important role in graph minor theory: if a minor-closed class of graphs does not contain all graphs, then every graph in it is glued together in a tree-like fashion from graphs that can almost be embedded in a fixed surface. Moreover, if a graph is embeddable in a given surface, then it contains no \( K_{3,1,t} \) as a minor for some \( t > 0 \); see Lovász [10] for a comprehensive survey of graph minor theory. Although a structural characterization of all graphs with no \( K_{3,1,t} \)-minors is still unavailable and seems extremely hard to obtain, Oporowski, Oxley, and Thomas [12] proved that if a 3-connected graph contains no \( K_{3,1,t} \)-minors, then it must contain a large wheel. Motivated by this result, Thomas and Seymour [15] made the following two conjectures.

**Conjecture 1.1.** (Thomas) There exist two functions \( \alpha(t) > 0 \) and \( \beta(t) > 0 \) such that, for any integer \( t \geq 3 \) and any 3-connected \( n \)-vertex graph \( G \) with no \( K_{3,1,t} \)-minor, \( c(G) \geq \alpha(t)n^{\beta(t)} \).

**Conjecture 1.2.** (Seymour and Thomas) There exist a function \( \alpha(t) > 0 \) and a constant \( \beta > 0 \) such that, for any integer \( t \geq 3 \) and any 3-connected \( n \)-vertex graph \( G \) with no \( K_{3,1,t} \)-minor, \( c(G) \geq \alpha(t)n^{\beta} \).

Jointly with Sheppardson, we have obtained a proof of Conjecture 1.1; see [4]. The purpose of this paper is to confirm the second one.

**Theorem 1.3.** Let \( G \) be a 3-connected \( n \)-vertex graph with no \( K_{3,1,t} \)-minor. Then \( c(G) \geq (1/2)^{t(t−1)}n^{\log_3 2} \).

The base 1729 is chosen because it is the best we can do in the proof of Claim 7.11; it would be interesting to know if this bound is best possible. Nevertheless, we strongly believe that the above result can be strengthened further if \( G \) enjoys higher connectivity.

**Conjecture 1.4.** There exists a function \( \alpha(t) > 0 \) such that, for any integer \( t \geq 4 \) and any 4-connected \( n \)-vertex graph \( G \) with no \( K_{3,1,t} \)-minor, \( c(G) \geq \alpha(t)n \).

Böhme, Maharry, and Mohar [2] have proved that every 7-connected graph with a sufficiently large number of vertices contains a \( K_{3,1} \)-minor for any fixed positive integer \( t \); so to attack Conjecture 1.4 one
may start with 6-connected graphs. The proof of Theorem 1.3 relies heavily on Tutte’s algorithm [20] for decomposing 2-connected graphs into 3-blocks. We envisage that the most difficult step in a proof of Conjecture 1.4, if any, might be to find a counterpart of Tutte’s algorithm for decomposing 3-connected graphs into 4-blocks. In Tutte’s decomposition, the 3-blocks involved form a tree-like structure, yet the situation for higher connectivity seems dramatically different.

The study of the longest cycle problem on 4-connected planar graphs dates back to 1931 when Whitney [21] proved that every 4-connected plane triangulation contains a Hamiltonian cycle; this work was obviously motivated by Tait’s theorem on face 4-colorability of Hamiltonian plane graphs. Whitney’s theorem [21] has been generalized to all 4-connected planar graphs by Tutte [19] and further to all 4-connected projective-planar graphs and 5-connected toroidal graphs by Thomas and Yu [16, 17]; related work can also be found in [18]. Generalizing to other surfaces, Yu [22] showed that every “locally planar” 5-connected triangulation of a surface contains a Hamiltonian cycle. Conjecture 1.4, made in a more general setting, is in the same spirit as that of previous work.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic terminology and notations, present a variant of Tutte’s algorithm for decomposing 2-connected graphs into 3-blocks, define two index functions and notations, formulate the main theorem consisting of three separate statements in terms of $\theta$ and $\phi$, and finally, in Section 3, we deal with rooted $K_{3,1}$-minor, and show that if a 3-connected graph has a $K_{3,1}$-minor, then it contains a $K_{3,\lceil t/3\rceil}$-minor rooted at any three given vertices. Based on this result, we can not only merge minors in different parts of the graph to form a larger minor as desired but have a good control of these minors as well. We also recall some useful properties of the function $f(x) = x^{\log_2 x}$ from [4], which allow us to discard some parts of the graph in our search procedure. In Section 4, we study the longest cycle problem on graphs with weights on edges; in our proof we shall use weights to keep track of the lengths of paths generated in 3-blocks or some of their unions. Finally, in Sections 5-7, we establish the three technical statements stated in Section 3, respectively.

2 Preliminaries

We start this section with some basic terminology and notations.

Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of $G$, respectively. Set $|G| := |V(G)|$; we call it the size of $G$. For each $U \subseteq V(G)$, let $G[U]$ denote the subgraph of $G$ induced by $U$. We call $U$ a connected set of $G$ if $G[U]$ is connected. We shall use $G/U$ to denote the graph obtained from $G$ by contracting $U$ (and deleting the resulting multiple edges and loops) if $U$ is a connected set.

Throughout this paper, we set $G - U := G[V(G) - U]$ and set $G - u := G - U$ if $U = \{u\}$. We say that $U$ is a cutset of $G$ if $G$ is connected and $G - U$ is disconnected. A vertex $u$ is called a cutvertex of $G$ if $\{u\}$ is a cutset. We also set $N_G(U) := \{x \in V(G) - U : x$ is adjacent to some vertex in $U\}$, and set $N_G(u) := N_G(\{u\})$; we shall drop the subscript $G$ if there is no danger of confusion. Let $H$ be a graph with $V(H) \subseteq V(G)$. For notational simplicity, we write $G[H]$, $G/H$, and $G - H$ for $G[V(H)]$, $G/V(H)$, and $G - V(H)$, respectively.

For any two vertices $x, y$ of $G$, an $x-y$ path in $G$ is a path connecting $x$ and $y$ in $G$. If $P$ is a path, we use $\ell(P)$ to denote the length of $P$, which is the number of edges of $P$. For any distinct vertices $x, y$ of a path $P$, we use $P[x, y]$ to denote the subpath of $P$ between $x$ and $y$ (inclusive), and define
$P[x, y] := P[x, y] - y$, $P[x, y] := P[x, y] - x$, and $P(x, y) := P[x, y] - \{x, y\}$. An edge of $G$ with ends $u$ and $v$ is often denoted by $uv$, or $vu$, or $\{u, v\}$. Let $S$ be a family of 2-element subsets of $V(G)$. Then $G + S$ stands for the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. (Note that each edge of $G$ is a 2-element subset of $V(G)$.) If $S = \{\{u_i, v_i\} : i = 1, 2, \ldots, k\}$, then we also write $G + \{u_i v_i : i = 1, 2, \ldots, k\}$ for $G + S$. If $S = \{\{u, v\}\}$, then we set $G + uv := G + S$. Similarly, we can define $G - S$ (with edge set $E(G) - S$).

Given two graphs $G$ and $H$, by $H \subseteq G$ we mean $H$ is a subgraph of $G$; the union of $G$ and $H$, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We call $H$ a minor of $G$ if there exist disjoint connected sets $V_x$ of $G$, indexed by $x \in V(H)$, such that, for any distinct $x, y \in V(H)$ with $xy \in E(H)$, there is at least one edge in $G$ with one end in $V_x$ and the other in $V_y$; we say that the sets $V_x, x \in V(H)$, form a representation of $H$ in $G$, and that $G$ contains an $H$-minor if $H$ is a minor of $G$. We shall not make effort to distinguish between the edges of $H$ and the edges of $G$ if no ambiguity arises; that is, we may view the edges of $H$ as edges of $G$.

As usual, $K_{3, t}$ is the complete bipartite graph with one color class having size $3$ and the other having size $t$. In this paper $\tau(G)$ denotes the maximum number $t$ such that $G$ contains a $K_{3, t}$-minor.

Let $G$ be a graph and let $x, y, z$ be three distinct vertices of $G$. We say that a $K_{3, t}$-minor $H$ of $G$ is rooted at $\{x, y, z\}$ if $H$ has a representation in $G$ such that $x \in V_1$, $y \in V_2$, $z \in V_3$, where $V_1, V_2, V_3$ are connected sets of $G$ representing the vertices of $H$ in the color class of size three. Let $\tau(G; x, y, z)$ denote the largest integer $t$ such that $G$ has a $K_{3, t}$-minor rooted at $\{x, y, z\}$. Clearly, $\tau(G; x, y, z) \leq \tau(G)$.

In our proof we shall use rooted $K_{3, t}$-minors to construct $K_{3, s}$-minors, with $s < t$. The following lemma gives a lower bound on $\tau(G; x, y, z)$ in terms of $\tau(G)$, whose proof will be given in Section 3.

**Lemma 2.1.** Let $G$ be a 3-connected graph and let $x, y, z$ be three distinct vertices of $G$. Then

$$\tau(G; x, y, z) \geq \left\lceil \frac{\tau(G)}{3} \right\rceil.$$
A chain of blocks in a graph $G$ is a sequence $x_0H_0x_1H_1x_2\ldots x_mH_mx_{m+1}$ such that each $H_i$ is a block of $G$, $V(H_i \cap H_{i+1}) = \{x_i\}$ for $1 \leq i \leq m-1$, $H_i \cap H_j = \emptyset$ whenever $|i-j| \geq 2$, $x_0 \neq x_{m+1}$ when $m = 0$, and if $m \geq 1$ then $x_0 \in V(H_0 - x_1)$ and $x_{m+1} \in V(H_m - x_m)$. We say that this chain of blocks is from $x_0$ to $x_{m+1}$.

The proof of our main theorem is based on graph decompositions. A 3-block is a 3-connected graph, or a cycle, or a bond (a set of at least three parallel edges sharing two ends). Let us now present an algorithm for decomposing a 2-connected graph into 3-blocks, which is a variant of Tutte's corresponding algorithm. Since bonds play a very limited role in our search for long cycles, they are merged to other 3-blocks whenever possible in the algorithm.

Algorithm 2.2.

**Input.** A pair $(H;xy)$, where $H$ is a 2-connected graph and $xy$ is an edge of $H$.

**Output.** A decomposition of $(H;xy)$ into 3-blocks, a leading block $H^*$, a set $\Psi(H)$ of virtual edges, and a partial order $\prec$ on $\Psi(H)$.

**Description.** Set $e_0 := xy$. We distinguish among four cases.

**Case 0.** $H$ is a 3-block. In this case, set $H^* := H$ and $\Psi(H) := \{e_0\}$, stop.

**Case 1.** $\{x,y\}$ is a cutset of $H$. In this case, let $B_1,B_2,\ldots,B_m$ be all the nontrivial $(x,y)$-bridges in $H$. For $i = 1,2,\ldots,m$, let $e_i$ be a virtual edge between $x$ and $y$, and let $H_e := B_i + e_i$; we recursively decompose $(H_e;e_i)$ into 3-blocks. Set $V(H^*) := \{x,y\}$, $E(H^*) := \{e_0,e_1,\ldots,e_m\}$, and $\Psi(H) := \{e_0\} \cup (\bigcup_{i=1}^m \Psi(H_e))$. Define $g \prec e_0$ for all $g \in \Psi(H) - \{e_0\}$.

**Case 2.** $\{x,y\}$ is not a cutset of $H$ and $H - e_0$ is a chain of blocks, $x_0H_0x_1H_1x_2\ldots x_mH_mx_{m+1}$, with $m \geq 1$, $x_0 = x$ and $x_{m+1} = y$. In this case, for $i = 0,1,\ldots,m$, let $F_i := \{f_i\}$ and $f_i = x_ix_{i+1}$ if $x_i$ and $x_{i+1}$ are adjacent in $H$ and let $F_i = \emptyset$ otherwise, let $B_{i,1},B_{i,2},\ldots,B_{i,p_i}$ be all the nontrivial $(x_i,x_{i+1})$-bridges in $H_i$, let $e_{i,j}$ be a virtual edge between $x_i$ and $x_{i+1}$, and let $H_{e_{i,j}} := B_{i,j} + e_{i,j}$ for $j = 1,2,\ldots,p_i$; we recursively decompose $(H_{e_{i,j}};e_{i,j})$ into 3-blocks. Set $V(H^*) := \{x_0,x_1,\ldots,x_{m+1}\}$, $E(H^*) := \{e_0\} \cup (\bigcup_{i=1}^m (F_i \cup \{e_{i,1},e_{i,2},\ldots,e_{i,p_i}\}))$ and $\Psi(H) := \{e_0\} \cup (\bigcup_{i,j} \Psi(H_{e_{i,j}}))$. Define $g \prec e_0$ for all $g \in \Psi(H) - \{e_0\}$.

**Case 3.** $\{x,y\}$ is not a cutset of $H$ and $H - e_0$ is 2-connected. In this case, let $\{u_i,v_i\}$, for $i = 1,2,\ldots,m$, be all the vertex pairs of $H$ such that there exists an $xy$-minimal separation $(K_{i-1},K_i)$ of $H$ with $\{u_i,v_i\} = V(K_i) \cap V(H_i)$. For $i = 1,2,\ldots,m$, let $B_{i,1},B_{i,2},\ldots,B_{i,p_i}$ be all the nontrivial $(u_i,v_i)$-bridges in $H_i$, let $e_{i,j}$ be a virtual edge between $u_i$ and $v_i$ for $j = 1,2,\ldots,p_i$, and let $H_{e_{i,j}} := B_{i,j} + e_{i,j}$; we recursively decompose $(H_{e_{i,j}};e_{i,j})$ into 3-blocks. Set $V(H^*) := \bigcap_{i=1}^m V(K_i)$, $E(H^*) := E(H[V(H^*)]) \cup (\bigcup_{i=1}^m \{e_{i,1},e_{i,2},\ldots,e_{i,p_i}\})$, and $\Psi(H) := \{e_0\} \cup (\bigcup_{i,j} \Psi(H_{e_{i,j}}))$. Define $g \prec e_0$ for all $g \in \Psi(H) - \{e_0\}$.

A multicycle is obtained from a cycle by adding parallel edges. By definition, both cycles and bonds are multicycles. So in both Case 1 and Case 2, $H^*$ is a multicycle.

As a large portion of our proof will be concerned with graphs obtained from a 3-connected graph by deleting a vertex, let us now apply Algorithm 2.2 to such graphs and exhibit some properties satisfied by its outputs.
**Lemma 2.3.** Let $G$ be a 3-connected graph, let $x, y, z$ be three distinct vertices of $G$ with $xz, yz \in E(G)$, and let $H = (G - z) + xy$. Then the outputs of Algorithm 2.2, when applied to $(H; xy)$, satisfy the following properties:

(i) $\prec$ induces a partial order on $\Psi(H)$;

(ii) $H^*$ is a minor of $G$;

(iii) $H^*$ is either a multicycle or 3-connected;

(iv) for any virtual edge $f = uv$ in $\Psi(H)$ with $f \neq e_0$, the graph $G_f := G[V(H_f) \cup z] + \{uz, vz\}$ is a 3-connected minor of $G$. In particular, $\tau(G_f) \leq \tau(G)$.

**Proof.** (i) Clearly, the relation $\prec$ defined on $\Psi(H)$ satisfies transitivity and antisymmetry. Hence, $\prec$ induces a partial order on $\Psi(H)$, so $(\Psi(H), \prec)$ is a poset.

(ii) If Case 0 or Case 1 occurs then the statement holds trivially. If Case 2 or Case 3 occurs then $H^*$ can be obtained from $G$ by contracting $xz$ to $x$ and contracting $H_e - u$ to $v$, for each virtual edge $e = uv$ in $H^*$ for which $e \prec xy$ and there is no virtual edge $f$ satisfying $e \prec f \prec xy$. So $H^*$ is a minor of $G$.

(iii) It is easy to see that $H^*$ is 2-connected. Suppose on the contrary that $H^*$ is neither a multicycle nor 3-connected. Then none of Cases 0-2 described in Algorithm 2.2 occurs and $H^*$ contains a cutset $\{a, b\}$. So $\{a, b\}$ is different from $\{x, y\}$ and is also a cutset in $H$. Let $(A, B)$ be a separation of $H$ with $V(A) \cap V(B) = \{a, b\}$ and $xy \in A$. Then Case 3 of Algorithm 2.2 guarantees the existence of an $xy$-minimal separation $(K_i, H_i)$ of $H$ with $K_i \subseteq A$. Let $\{u_i, v_i\} = V(K_i) \cap V(H_i)$. Then $\{a, b\} = \{u_i, v_i\}$, for otherwise $a$ or $b$ would be excluded from $H^*$ by Algorithm 2.2. We thus reach a contradiction because $\{u_i, v_i\}$ is not a cutset of $H^*$.

(iv) By the construction in Algorithm 2.2, $H_f$ is 2-connected. Since $G$ is 3-connected, every 2-cutset of $H_f$ separates $\{u, v\}$ from some neighbor of $z$. It follows that $G_f$ is 3-connected. As $H$ is 2-connected, it contains two disjoint paths $P_1$ and $P_2$ from $\{x, y\}$ to $\{u, v\}$, where $u \in V(P_1)$ and $v \in V(P_2)$. From Algorithm 2.2 we see that $P_i$ contains no vertex in $V(H_f) - \{u, v\}$ for $i = 1, 2$. So there exist two disjoint connected subgraphs $F_1$ and $F_2$ of $H - (V(H_f) - \{u, v\})$ such that $P_i \subseteq F_i$ for $i = 1, 2$ and that $V(F_1) \cup V(F_2) = V(H) - (V(H_f) - \{u, v\})$. If $\{x, y\} = \{u, v\}$ then $xy \in E(G)$; otherwise, since $G$ is 3-connected, there is at least one edge in $G$ between $F_1$ and $F_2$. Thus $G_f$ can be obtained from $G$ by contracting $F_1$ to $u$ and $F_2$ to $v$, and hence is a minor of $G$.

We digress to introduce some important notions before presenting the main result. Let $H$ be a 2-connected graph, let $(e, f)$ be an ordered pair of edges of $H$, and let $A$ and $B$ be two vertex-disjoint connected subgraphs of $H$ (or two disjoint connected sets of $H$). We say that the quadruple $(A, B, e, f)$ is a ladder with top $e$ and bottom $f$ in $H$ if each of $A$ and $B$ contains precisely one end of each of $e$ and $f$. For any family $F$ of 2-element subsets of $V(H)$, we use $F \cap [A, B]$ to denote the subfamily of all 2-element subsets in $F$ with one element in $A$ and the other in $B$.

Let $G$ be a 3-connected graph, let $x, y, z$ be three distinct vertices of $G$ with $xz, yz \in E(G)$, and let $H = (G - z) + xy$. Set $e_0 := xy$, $H_{e_0} := H$, and $G_{e_0} := G$. Suppose we have applied Algorithm 2.2 to $(H; e_0)$. Let us consider an arbitrary virtual edge $f$ in $\Psi(H)$. For each virtual edge $e$ in $\Psi(H_f)$, from Lemma 2.3(d) (with $G_e$ and $G_f$ in place of $G_f$ and $G$ over there, respectively) it follows that
\( \tau(G_e) \leq \tau(G_f) \). We call \( e \) full with respect to \( f \) if \( \tau(G_e) = \tau(G_f) \), and set

\[
\Psi_1(H_f) := \{ e \in \Psi(H_f) : e \text{ is full with respect to } f \} \quad \text{and} \quad \Psi_2(H_f) := \Psi(H_f) - \Psi_1(H_f).
\]

For each \( e \in \Psi(H_f) - \{ f \} \), let \( \theta(e, H_f) \) denote the maximum size of an anti-chain \( X \) (recall Lemma 2.3(a)) in \( \Psi_1(H_f) \cap [A, B] \), taken over all ladders \( (A, B, e, f) \) with top \( e \) and bottom \( f \) in \( H_f + e \), such that

1. if \( e \in \Psi_1(H_f) \) then \( e \in X \), and
2. if \( e \in \Psi_2(H_f) \) (so \( e \notin X \)) then no element of \( X \) is comparable with \( e \).

Notice that \( X \cup \{ e \} \) is always an antichain.

For each \( e \in \Psi(H_f) - \{ f \} \), let \( \phi(e, H_f) \) denote the maximum size of an anti-chain \( Y \) in \( \Psi(H_f) \cap [A, B] \), taken over all ladders \( (A, B, e, f) \) with top \( e \) and bottom \( f \) in \( H_f + e \), such that

1. \( e \in Y \), and
2. \( |Y \cap \Psi_1(H_f)| = \theta(e, H_f) \).

Since \( Y \cap \Psi_1(H_f) \) gives an antichain realizing \( \theta(e, H_f) \), we obtain \( \theta(e, H_f) \leq \phi(e, H_f) \).

Set \( \theta(f, H_f) = \phi(f, H_f) := 0 \), and set

\[
\theta(H_f) := \max_{e \in \Psi(H_f)} \{ \theta(e, H_f) \} \quad \text{and} \quad \phi(H_f) := \max_{e \in \Psi(H_f)} \{ \phi(e, H_f) \}.
\]

By definition, \( \theta(H_f) \leq \phi(H_f) \). Moreover, \( \theta(H_f) = \phi(H_f) = 0 \) if \( H_f \) is 3-connected (in this case \( \Psi(H_f) = \Psi_1(H_f) = \{ f \} \), so \( X = Y = \emptyset \)). For any positive integer \( t \), set

\[
\delta(t, H_f) := \frac{1}{3^3(H_f)} \left( 1 - \frac{\phi(H_f) - \theta(H_f)}{3^t} \right) .
\]

As \( H = H_{e_0} \), we have \( \theta(H) = \theta(H_{e_0}) \), \( \phi(H) = \phi(H_{e_0}) \), and \( \delta(t, H) = \delta(t, H_{e_0}) \).

The following observations aim to give a good estimate of the parameter \( \delta(t, H) \).

**Lemma 2.4.** Let \( G \) be a 3-connected graph, let \( x, y, z \) be three distinct vertices of \( G \) with \( xz, yz \in E(G) \), and let \( H = (G - z) + xy \). Suppose Algorithm 2.2 has been applied to \((H; xy)\). Let \( f = uv \) (possibly \( f = e_0 := xy \)) be a virtual edge in \( \Psi(H) \), and let \( G_f := G[V(H_f) \cup z] + \{ uz, vz \} \). Then the following statements hold:

(i) \( \theta(H_f) \leq \phi(H_f) \leq \tau(G_f) \);
(ii) \( \theta(H_f) \leq 3 \) and equality holds only if \( \phi(H_f) = 3 \);
(iii) for any \( t \geq \tau(G_f) \), we have \( 1/27 \leq \delta(t, H_f) \leq 1 \), and \( \delta(t, H_f) \leq 1/3 \) if \( \theta(H_f) \geq 1 \);
(iv) if \( \tau(G_f) = \tau(G) \), then \( \delta(t, H) \leq \delta(t, H_f) \) for any \( t \geq \tau(G) \).

**Proof.** (i) From the definition it follows instantly that \( \theta(H_f) \leq \phi(H_f) \). To prove that \( \phi(H_f) \leq \tau(G_f) \), we may assume \( \phi(H_f) > 0 \). Thus the definition guarantees the existence of a virtual edge \( e \) in \( \Psi(H_f) \), a ladder \( (A, B, e, f) \) with top \( e \) and bottom \( f \) in \( H_f + e \), and an anti-chain \( Y \in \Psi(H_f) \cap [A, B] \) such that \( e \in Y \subseteq Y \cap \Psi_1(H_f) = \theta(H_f) \), and \( |Y| = \phi(H_f) \).

For each \( g \in Y \), set \( V_g = V(H_f) - (A \cup B) \). Then the sets \( A, B, \{ z \} \) and \( V_g \) (for all \( g \in Y \)) form a representation of a \( K_{3,|Y|\gamma} \)-minor of \( G_f \) rooted at \( \{ u, v, z \} \). So \( \phi(H_f) = |Y| \leq \tau(G_f) \).
(ii) We only need to consider the case when $\theta(H_f) > 0$. Let $e$ be a virtual edge in $\Psi(H_f)$, $(A, B, c, f)$ a ladder with top $e$ and bottom $f$ in $H_f + e$, and $Y$ an anti-chain in $\Psi(H_f) \cap [A, B]$ that $e \in Y$, $|Y \cap \Psi_1(H_f)| = \theta(H_f)$, and $|Y| = \phi(H_f)$. Let $X := Y \cap \Psi_1(H_f)$. Then $|X| = \theta(H_f)$. Suppose $\tau(G_f) = q$. For each $g \in Y$, let $g = u_g v_g$. By Lemma 2.3, $G_g$ is a 3-connected minor of $G_f$. Observe that for each $g \in X$, we have $g \in \Psi_1(H_f)$, so $\tau(G_g) = \tau(G_f) = q$. From Lemma 2.1 we deduce that $G_g$ has a $K_{3, (q/3)}$-minor $\Sigma_g$ rooted at $\{u_g, v_g, z\}$. Clearly, for each $g \in Y - X$, $G_g$ has a $K_{3, 1}$-minor $\Sigma_g$ rooted at $\{u_g, v_g, z\}$. Let $G'$ denote the graph $G[A \cup G[B] \cup (\bigcup_{g \in Y} \Sigma_g)$. Then $G'$ contains a $K_{3, g}$-minor of $G$ (this can be seen by contracting $A$ to a single vertex and $B$ to another vertex), with $p \geq (q/3)|X| + (|Y| - |X|)$. Since $p \leq q$ and $|X| \leq |Y|$, we have $q \geq (q/3)|X|$ (so $|X| \leq 3$), and equality holds only if $|Y| = |X|$ and $q$ is a multiple of 3.

(iii) By (i), we have $\theta(H_f) \leq \phi(H_f) \leq \tau(G_f) \leq t$. So $0 \leq \frac{\phi(H_f) - \theta(H_f)}{3t} \leq \frac{1}{3}$ and hence $\delta(t, H_f) \geq \frac{1}{3\sqrt{\pi t}}$. If $\tau(H_f) \leq 2$, then $\delta(t, H_f) \geq \frac{2}{3\sqrt{\pi t}}$. If $\tau(H_f) = 3$, then $\phi(H_f) = 3$ by (ii) and hence, by definition, $\delta(t, H_f) = \frac{1}{3\sqrt{\pi t}}$. In view of (ii), $\theta(H_f) \leq 3$, so the inequality $\delta(t, H_f) \geq \frac{1}{3\sqrt{\pi t}}$ always holds. As $\delta(t, H_f) \leq \frac{1}{3\sqrt{\pi t}}$, the upper bound follows instantly.

(iv) We may assume that $f \neq e_0 = xy$, for otherwise $H_f = H$, so the statement holds trivially. Since $\tau(G_f) = \tau(G)$, we have $\Psi_1(H_f) \subseteq \Psi_1(H_g)$; and hence $\theta(H_f) \leq \theta(H)$ (by (2.1)). If $\theta(H_f) = \theta(H)$ then, by (2.1), we have $\phi(H_f) \leq \phi(H)$; and so $\delta(t, H) \leq \delta(t, H_f)$ (by (2.2)). If $\theta(H_f) \leq \theta(H) - 1$, then

$$\delta(t, H_f) = \frac{1}{3\phi(H_f)}(1 - \frac{\phi(H_f) - \theta(H_f)}{3t}) \geq \frac{1}{3\phi(H_f)}(1 - \frac{t}{3t}) = \frac{1}{3\phi(H_f)} \frac{2}{3} \geq \frac{2}{3\psi(H)} \geq \delta(t, H),$$

where the first inequality follows from (i). So the statement is established in either case. \hfill \Box

Now we are ready to state the main result of this paper, which implies Theorem 1.3 immediately.

**Theorem 2.5.** Let $t \geq 1$ be an integer, let $G = (V, E)$ be a 3-connected graph with $\tau(G) \leq t$, let $\alpha(t) := (1/2)^{(t-1)}$, and let $\beta := \log_b 2$, where $b = 1729$. Then the following statements hold:

(a) For any distinct $x, y, z \in V$ with $xz, yz \in E$, there exists an $x-y$ path in $G - z$ of length at least $\alpha(t)(\delta(t, H)(|G| - 1))^\beta$, where $H = (G - z) + xy$ and $\delta(t, H)$ is as defined in (2.2).

(b) For any distinct $e, f = xy \in E$, there exists an $x-y$ path in $G$ through $e$ of length at least $\alpha(t)(\frac{|G|}{3\pi t})^\beta + 1$.

(c) For any $xy \in E$, there exists an $x-y$ path in $G$ of length at least $\alpha(t)|G|^\beta$.

**Outline of Proof.** Let $n := |G|$. We prove by double induction on $n$ and $t$. Obviously $G$ contains a path as specified in each of (a), (b) and (c) with length at least 2. If $n \leq b^{(t-1)}$ then $\alpha(t)n^{1/2} \leq 1$. So the lower bounds specified in (a)-(c) are all at most 2 for $\delta(t, H) \leq 1$, and hence (a)-(c) hold simultaneously in this case. If $\tau(G) = 1$ then $G$ is $K_4$ (the complete graph on four vertices). Thus (a)-(c) all hold trivially again. Therefore, we proceed to the induction step and assume that $t \geq 2$, $n > b^{(t-1)}$, and statements (a)-(c) have been established for all graphs with at most $n - 1$ vertices and for all graphs with no $K_{3, t}$-minors. The inductive processes of statements (a)-(c) will take up the last three sections of this paper.

We point out that the proofs of statements (a) and (c) are substantially different from their counterparts in [4].
3 Rooted $K_{3,t}$-Minors and the Function $x^{\log_3 2}$

The purpose of this section is to give a proof of Lemma 2.1, and state several lemmas concerning the function the function $x^{\log_3 2}$, which will be used repeatedly in the proof of Theorem 2.5. 

Before proving Lemma 2.1, we remark that the bound in Lemma 2.1 is sharp. To see this, let $G = (A \cup B, E)$ be a graph such that
- $A = \{a_1, a_2, \ldots, a_5\}$ and $B = \{b_1, b_2, \ldots, b_3t\}$ for any positive integer $t \geq 3$;
- $a_1, a_2, a_3$ are pairwise adjacent;
- $a_i$ is adjacent to vertices $b_{(i-1)t+1}, b_{(i-1)t+2}, \ldots, b_{it}$ for $i = 1, 2, 3$;
- $a_j$ is adjacent to vertices $b_1, b_2, \ldots, b_{3t}$ for $j = 4, 5$.

Clearly, $G$ is 3-connected and contains a $K_{3,5t}$-minor. However, it is easy to verify that $G$ contains no $K_{3,5t+1}$-minor rooted at $\{a_1, a_2, a_3\}$.

In our proof of Lemma 2.1 we shall use contractible edges. An edge $e = uv$ in a 3-connected graph $G$ is called contractible if $G/e$ is also 3-connected and noncontractible otherwise. Obviously, $e = uv$ is noncontractible in $G$ if and only if $\{u, v\}$ is contained in a 3-cutset of $G$. Let $E_\ast(G)$ (resp. $E_n(G)$) denote the set of contractible (resp. noncontractible) edges of $G$, and let $\mathcal{N}(G)$ denote the collection of all triples $(e, S_e, C_e)$, where $e \in E_n(G), S_e$ is a 3-cutset of $G$ containing $V(e)$, and $C_e$ is a component of $G - S_e$. We call $(e, S_e, C_e)$ minimal if there exists no $(f, S_f, C_f) \in \mathcal{N}(G)$ such that $C_f$ is a proper subgraph of $C_e$.

Lemma 3.1. Let $G$ be a 3-connected graph and let $(e, S_e, C_e) \in \mathcal{N}(G)$ be minimal. Then all edges of $C_e$ and all edges from $C_e$ to $S_e - V(e)$ are contractible in $G$.

Proof. Let $f$ be an edge of $C_e$ or an edge from $C_e$ to $S_e - V(e)$. If $f$ is noncontractible, then $V(f)$ is contained in a 3-cutset $S_f$ of $G$. It is thus a routine matter to check that $S_f \subseteq S_e \cup V(C_e)$. Consequently, some component of $G - S_f$ is properly contained in $C_e$, a contradiction. ■

By using similar arguments, Ando et al. [1] obtained the following result.

Lemma 3.2. Let $G$ be a 3-connected graph and let $v$ be a vertex of $G$ with degree 3. Then
(i) $G$ has a contractible edge incident with $v$, and
(ii) if there is exactly one contractible edge incident with $v$, then the noncontractible edges incident with $v$ induce a triangle $T$ whose vertices all have degree 3 in $G$.

Observe that the triangle $T$ specified in (ii) is contractible; that is, $G/T$ is 3-connected.

Proof of Lemma 2.1. Let $\tau(G) = t$ and let $V_1, V_2, \ldots, V_{t+3}$ denote a representation of a $K_{3,t}$-minor in $G$ with color classes $\{V_1, V_2, V_3\}$ and $\{V_4, V_5, \ldots, V_{t+3}\}$. Clearly, we may assume that $\bigcup_{i=1}^{t+3} V_i = V(G)$. Since $G$ is 3-connected, it has a $K_{3,1}$-minor rooted at $\{x, y, z\}$. Thus the statement holds for $t \leq 3$. It remains to consider the case when $t \geq 4$.

Since the statement is trivial if $|V_s| = 1$ for $s = 1, 2, \ldots, t+3$, we may assume that $|V_s| \geq 2$ for some $s$ with $1 \leq s \leq t+3$, and that the assertion has been established for smaller graphs.

(1) We may choose $V_1, V_2, \ldots, V_{t+3}$ so that for every $1 \leq s \leq t+3$, if $|V_s| \geq 2$ then $|V_s \cap \{x, y, z\}| \geq 2$. 


To justify this, let us assume that $|V_s| \geq 2$ while $|V_s \cap \{x, y, z\}| \leq 1$ for some subscript $s$ with $1 \leq s \leq t + 3$. Then no edge $e$ in $G[V_s]$ is contractible, for otherwise, $G/e$ is 3-connected and $\tau(G/e) = t$ (as both ends of $e$ are in $V_s$). It follows from induction hypothesis that $\tau(G; x, y, z) \geq \tau(G/e; x, y, z) \geq t/3$, we are done. Hence, there exists $(e, S_c, C_e) \in \mathcal{N}(G)$. Let $\{a\} = S_c - V(e)$. Then $a \in V_r$ for some $r$ (possibly $r = s$). By the structure of $K_{3,t}$-minor, $G - (V_s \cup V_r)$ is connected. Hence we can choose $(e, S_c, C_e)$ so that $V(C_e) \subseteq V_s \cup V_r$. For technical reasons, we further assume that $V_1, V_2, \ldots, V_{t+3}$ and $(e, S_c, C_e)$ are chosen so that $|C_e|$ is maximized.

Let $V'_r = V_s \cup V(C_e)$, $V'_t = V_r - V(C_e)$, and $V'_i = V_i$ for all $i \neq r, s$. Then $V'_1, V'_2, \ldots, V'_{t+3}$ form a representation of a $K_{3,t}$-minor.

We propose to show that $|(V(C_e) \cup V(e)) \cap \{x, y, z\}| \geq 2$. Otherwise, $|(V(C_e) \cup V(e)) \cap \{x, y, z\}| \leq 1$. Choose minimal $(f, S_f, C_f) \in \mathcal{N}(G)$ such that $S_f \cup V(C_f) \subseteq S_c \cup V(C_e)$ and $V(C_f) \subseteq V(C_e)$. Let $f = uv$ and $S_f = \{u, v, w\}$. Suppose $a \neq w$ (so $w \in V(C - e) \cup V(e)$). By Lemma 3.1, any edge $ww'$ with $w' \in V(C_f)$ is contractible in $G$. Since $G/ww'$ has a $K_{3,t}$-minor, we have $w, w' \in \{x, y, z\}$, for otherwise the present lemma follows from induction. Thus we may assume $a = w$. Then $u, v \in V'_r$. If $|V(C_f)| \geq 2$ then again by Lemma 3.1 any edge $g$ in $C_f$ is contractible, and so we may assume $V(g) \subset \{x, y, z\}$ as before. Hence there is only one vertex $b$ in $C_f$. It follows that $d(b) = 3$ and $bu, bv \in E(G)$. If $bu$ or $be \in E_s(G)$, then again we can either apply induction to show that $\tau(G; x, y, z) \geq t/3$ or conclude that $b, u \in \{x, y, z\}$ or $b, v \in \{x, y, z\}$. Therefore we may assume that $bu, bv \in E_s(G)$. By Lemma 3.2 and the remark on (ii) of Lemma 3.2, $d(u) = d(v) = 3$ and the triangle $T = buv$ is contractible. Note that $b, u, v \in V'_r$. If $|\{b, u, v\} \cap \{x, y, z\}| \leq 1$ then the lemma follows from induction. So we may assume the contrary, which implies that $|(V(C_e) \cup V(e)) \cap \{x, y, z\}| \geq 2$, as desired.

From the assumption on $V_s$ we deduce that $r \neq s$. Next, let us show that $|V'_j| \geq 2$ for some $j \neq s$. Then $|V'_j \cap \{x, y, z\}| \leq 1$ as $|V'_s \cap \{x, y, z\}| \geq 2$. Using the same argument with respect to $V'_1, V'_2, \ldots, V'_{t+3}$ (in place of $V_1, V_2, \ldots, V_{t+3}$) and $j$ (in place of $s$), we deduce that no edge $e'$ in $G[V'_j]$ is contractible, so there exists $(e', S'_e, C'_e) \in \mathcal{N}(G)$ such that $V(C'_e) \subseteq V'_j \cup V'_k$ for some $k$ and $G - (V'_j \cup V'_k)$ is connected. As before, $|(V(C'_e) \cup V(e')) \cap \{x, y, z\}| \geq 2$. This implies that $V(C'_e) \cap V(C_e) \cap \{x, y, z\} \neq \emptyset$. Since $V'(e') \cap V(e) = \emptyset$, $V(e')$ can have at most one vertex in common with $S_e \cup C_e$. By 3-connectedness of $G$, we thus have $S'_e \subseteq G - V(C_e)$. It follows that $C_e$ is properly contained in $C'_e$. So $V'_1, V'_2, \ldots, V'_{t+3}$ and $(e', S'_e, C'_e)$ contradict the choices of $V_1, V_2, \ldots, V_{t+3}$ and $(e, S_c, C_e)$, completing the proof of (1).

By (1), there exists a unique subscript $s$ such that $|V_s| \geq 2$. Renaming vertices if necessary, we assume that $x, y \in V_s$. Let $v_i$ be the only vertex in $V'_i$ for all $i \neq s$. Since $G$ is 3-connected, there exist three disjoint paths $P_x, P_y, P_z$ from $x, y, z$ to some distinct vertices $v_i, v_j, v_k$, respectively, which are disjoint from $v_t$ for all $t \notin \{i, j, k, s\}$, where $s \notin \{i, j, k\}$. Now let us consider two possible cases.

**Case 1.** $z \in V_s$.

Since $V_s$ is connected, it can be partitioned into three connected sets $Q_x, Q_y, Q_z$ such that $V(P_x) \subseteq Q_x$, $V(P_y) \subseteq Q_y$, and $V(P_z) \subseteq Q_z$.

**Subcase 1.1.** $s \geq 4$.

Without loss of generality, we assume that $s = 4$. If $\max\{i, j, k\} = 3$ then $Q_x \cup V_1, Q_y \cup V_2, Q_z \cup V_4, V_5, \ldots, V_{t+3}$ form a representation of $K_{3,t-1}$-minor in $G$ rooted at $\{x, y, z\}$, so $\tau(G; x, y, z) \geq t - 1 \geq t/3$ as $t \geq 4$.
Suppose $\min\{i,j,k\} \geq 5$. Then we may assume $i = 5, j = 6, k = 7$. Since $Q_x \cup V_5, Q_y \cup V_6, Q_z \cup V_7, V_1, V_2, V_3$ form a representation of a $K_{3,3}$-minor in $G$ rooted at $\{x,y,z\}$, we may assume that $t \geq 10$. Thus $Q_x \cup V_5 \cup V_1, Q_y \cup V_6 \cup V_2, Q_z \cup V_7 \cup V_3, V_8, \ldots, V_{t+3}$ form a representation of a $K_{3,t-4}$-minor in $G$ rooted at $\{x,y,z\}$. Hence $\tau(G;x,y,z) \geq t-4 \geq t/3$ as $t \geq 10$.

So we suppose $\min\{i,j,k\} \leq 3$. Remaining subscripts if necessary, we assume $i = 1$. If $\min\{j,k\} \leq 3$ then we may assume $j = 2$ and $k = 5$; in this case, $Q_x \cup V_1, Q_y \cup V_2, Q_z \cup V_5 \cup V_3, V_6, \ldots, V_{t+3}$ form a representation of a $K_{3,2}$-minor in $G$ rooted at $\{x,y,z\}$. So $\tau(G;x,y,z) \geq t-2 \geq t/3$ as $t \geq 4$. Hence we assume $j = 5$ and $k = 6$. Note that $Q_x \cup V_1 \cup V_7, Q_y \cup V_5, Q_z \cup V_6, V_2, V_3$ form a representation of a $K_{3,2}$-minor in $G$ rooted at $\{x,y,z\}$; so we may assume $t \geq 7$. Since $Q_x \cup V_1, Q_y \cup V_5 \cup V_2, Q_z \cup V_6 \cup V_3, V_7, \ldots, V_{t+3}$ form a representation of a $K_{3,t-3}$-minor in $G$ rooted at $\{x,y,z\}$, we have $\tau(G;x,y,z) \geq t-3 \geq t/3$ as $t \geq 7$.

Subcase 1.2. $s \leq 3$.

Clearly we may assume that $s = 1$. By the pigeonhole principle, there exists $\{i_1,i_2,\ldots,i_p\} \subseteq \{4,5,\ldots,t+3\}$, with $p = \lceil t/3 \rceil$, such that one of the following (a), (b), and (c) holds:

(a) $j, k \notin \{i_1, i_2, \ldots, i_p\}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in $Q_x$;
(b) $i, k \notin \{i_1, i_2, \ldots, i_p\}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in $Q_y$; and
(c) $i, j \notin \{i_1, i_2, \ldots, i_p\}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in $Q_z$.

Since $t \geq 4$, we have $t-p \geq 2$.

If $\{j, k\} \subseteq \{v_4, v_5, \ldots, v_{t+3}\}$, then symmetry allows us to assume that (a) occurs. Thus $Q_x, Q_y \cup V_2, V_7, Q_2, V_8 \cup V_9, V_1, \ldots, V_p$ form a representation of a $K_{3,p}$-minor in $G$ rooted at $\{x,y,z\}$. Now suppose $\{j, k\} = \{2, 3\}$. If (a) occurs then $Q_x, Q_y \cup V_2, Q_z \cup V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p}$-minor in $G$ rooted at $\{x,y,z\}$. If (b) occurs then $Q_x \cup V_1 \cup V_7, Q_y, Q_z \cup V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p}$-minor in $G$ rooted at $\{x,y,z\}$. The proof goes along the same line if (c) occurs.

It remains to consider that $j \in \{2, 3\}$ and $k \geq 4$. If (b) occurs then $Q_x \cup V_1 \cup V_2, Q_y, Q_z \cup V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p}$-minor in $G$ rooted at $\{x,y,z\}$. By symmetry we may assume that (a) occurs and $j = 2$. Then $Q_x, Q_y \cup V_2, Q_z \cup V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p}$-minor in $G$ rooted at $\{x,y,z\}$.

Case 2. $z = v_r$ for some $r \neq s$.

In this case we may assume that $P_z = z = v_k$. So $r = k$. Let us partition $V_s$ into two connected sets $Q_x, Q_y$ such that $V(P_z) \subseteq Q_x$ and $V(P_y) \subseteq Q_y$.

Subcase 2.1. $s \leq 3$.

Without loss of generality, we may assume $s = 1$. By the pigeonhole principle and by symmetry, we may assume that $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in $Q_x$, such that $j, k \notin \{4, 5, \ldots, p\}$, $p-3 \geq (t-2)/2$, and $p-3 \geq (t-1)/2$ if $\min\{j,k\} \leq 3$.

If $k \leq 3$, say $k = 3$, then $Q_x, Q_y \cup V_2, V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p-3}$-minor in $G$ rooted at $\{x,y,z\}$. Hence $\tau(G;x,y,z) \geq t-1/2 \geq t/3$ (since $t \geq 4$). Similarly, if $j \leq 3$, say $j = 3$, then $Q_x, Q_y \cup V_2, V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p-3}$-minor in $G$ rooted at $\{x,y,z\}$; and hence $\tau(G;x,y,z) \geq t/3$.

So we may assume $\min\{j,k\} > p$. Then $Q_x \cup V_1, Q_y \cup V_2, V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,2}$-minor in $G$ rooted at $\{x,y,z\}$; and hence we may assume $t \geq 7$. Since $Q_x, Q_y \cup V_2, V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p-3}$-minor in $G$ rooted at $\{x,y,z\}$, we have $\tau(G;x,y,z) \geq (t-2)/2 \geq t/3$.
and any integer $k$.

**Corollary 3.6.** Suppose $m, n_1, \ldots, n_k$ are positive numbers such that $m \geq \frac{n_i}{a}$ for $1 \leq i \leq k$. Then, for any integer $b \geq 4$, 

$$m^{\log_a 2} + \sum_{i=1}^{k} n_i^{\log_a 2} \geq (m + \frac{b - 1}{a} \sum_{i=1}^{k} n_i)^{\log_a 2}.$$
4 Cycles in Weighted Graphs

In our proof we shall use weights to keep track of the lengths of paths generated in 3-blocks output by Algorithm 2.2 and some of their unions, so we study the longest cycle problem on weighted graphs (with parallel edges allowed) in this section.

Let $H$ be a 2-connected graph, let $S \subseteq E(H)$, and let $(e, f)$ be an ordered pair of distinct edges in $H$. For each ladder $L = (A, B, e, f)$ with top $e$ and bottom $f$ in $H$, the edges in $S \cap [A, B] - \{e\}$ are called the $S$-rungs of $L$. Note that the bottom $f$ is counted as an $S$-rung whenever $f \in S$ while the top $e$ will never be counted. Moreover, $S$ may contain parallel edges.

Let $f = xy \in E(H)$ and let $P$ be an $x$-$y$ path in $H$. For any $e = uv \in E(P)$ with $x, u, v, y$ on $P$ in this order, a ladder generated by $P$ with top $e$ is a ladder $(A, B, e, f)$ with $P[x, u] \subseteq A$ and $P[v, y] \subseteq B$. Let $\sigma_{H,S}(P,e)$, or $\sigma(P,e)$ (if there is no confusion), denote the maximum number of $S$-rungs of a ladder generated by $P$ with top $e$. In the extreme case $E(P) = \{f\}$, we define $\sigma(P,f)$ as 1 if $f \in S$ and $|S| \geq 2$ and as 0 otherwise. (The theorem and its corollary established in this section will only be used in Section 5, where we always have $f \notin S$.)

The following is a strengthening of Theorem 3.1 in [4].

**Theorem 4.1.** Let $H$ be a 2-connected graph, let $\omega : E(H) \mapsto \mathbb{R}^+$, and let $S = \{e \in E(H) : \omega(e) > 0\}$. Then for any $xy \in E(H)$, there exists an $x$-$y$ path $P$ in $H$ such that

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \omega(H),$$

where $\omega(H) = \sum_{e \in E(H)} \omega(e)$.

**Proof.** Note that $\omega(H) = \omega(S) = \sum_{e \in S} w(e)$. We proceed by induction on $|E(H)| + |S|$. If $|S| = 0$, then $\omega(H) = 0$. Hence any $x$-$y$ path $P$ in $G$ is as desired. If $|S| = 1$, then $H$ has an $x$-$y$ path $P$ containing the edge in $S$ for $H$ is 2-connected. Clearly, $\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \omega(H)$. So we may assume $|S| \geq 2$.

Suppose $|E(H)| = 3$. Then $H$ is a triangle. Let $P$ and $Q$ be the two $x$-$y$ paths in $H$, with $\omega(P) \geq \omega(Q)$. If $S \subseteq E(P)$, then

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \sum_{e \in E(P)} \omega(e) = \omega(H).$$

If $S \cap E(Q) \neq \emptyset$, then $\sigma(P,e) \geq 1$ for any $e \in E(P)$. It follows that

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq 2\omega(P) \geq \omega(H).$$

So the desired statement holds in either case. Therefore we assume hereafter that $|E(H)| \geq 4$. The remainder of the proof is divided into two cases.

**Case 1.** $\{x, y\}$ is a cutset of $H$ or $S$ contains an edge incident with both $x$ and $y$.

In this case there exist subgraphs $H_1$ and $H_2$ of $H$ such that $H_1 \cup H_2 = H$, $V(H_1) \cap V(H_2) = \{x, y\}$, $E(H_1) \cap E(H_2) = \emptyset$, and for each $i$ either $|H_i| \geq 3$ or $E(H_i) \cap S \neq \emptyset$. Renaming subscripts if necessary, we assume $\omega(H_1) \geq \omega(H_2)$, which implies $2\omega(H_1) \geq \omega(H)$.
Suppose \( H_1 \) is induced by an edge \( f \in S \). Then \( f \) is incident with both \( x \) and \( y \). Let \( P = H_1 \). As \( |S| \geq 2 \), by definition \( \sigma(P, f) = 1 \). Thus \( \sum_{e \in E(P)} 2^{\sigma(P, e)} \omega(e) \geq 2\omega(f) = 2\omega(H_1) \geq \omega(H) \).

So we assume that \( |H_1| \geq 3 \). Let \( H^* := H_1 + xy \) if \( H_1 \) contains an edge between \( xy \); otherwise let \( H^* := H_1 + xy \). Let \( S^* := S \cap E(H_1) \), and let \( \omega^* : E(H^*) \to \mathbb{R}^+ \) be such that \( \omega^*(e) = \omega(e) \) if \( e \in E(H_1) \) and \( \omega^*(xy) = 0 \) if \( xy \notin E(H_1) \). Then \( S^* := \{e \in E(H^*) : \omega^*(e) > 0\} \). Note that \( H^* \) is 2-connected and \( |E(H^*)| + |S^*| < |E(H)| + |S| \). So the induction hypothesis on \((H^*, \omega^*)\) guarantees the existence of an \( x-y \) path \( P \) in \( H^* \) such that
\[
\sum_{e \in E(P)} 2^{\sigma^*(P, e)} \omega^*(e) \geq \omega^*(H^*) = \omega(S^*) = \omega(H_1),
\]
where \( \sigma^*(P, e) \) is defined for \( P \) in \( H^* \).

If \( S \cap E(H_2) = \emptyset \), then \( \omega(H_1) = \omega(H) \) and \( \sigma(P, e) = \sigma^*(P, e) \) for all \( e \in E(P) \). Thus
\[
\sum_{e \in E(P)} 2^{\sigma(P, e)} \omega(e) = \sum_{e \in E(P)} 2^{\sigma^*(P, e)} \omega^*(e) \geq \omega(H_1) = \omega(H).
\]
So we may assume that \( S \cap E(H_2) \neq \emptyset \). Using 2-connectedness of \( H \), we have \( \sigma(P, e) \geq \sigma^*(P, e) + 1 \) for all \( e \in E(P) \). Hence
\[
\sum_{e \in E(P)} 2^{\sigma(P, e)} \omega(e) \geq \sum_{e \in E(P)} 2^{\sigma^*(P, e)+1} \omega^*(e) \geq 2\omega(H_1) \geq \omega(H).
\]

**Case 2.** \( \{x, y\} \) is not a cutset of \( H \) and no edge in \( S \) is incident with both \( x \) and \( y \).

In this case \( y \) is contained in a unique block of \( H - x \), denoted by \( Y \). Let \( X \) be an \( (x, Y) \)-bridge of \( H \) with \( \omega(X) \) maximum, and let \( u \) be the unique vertex in \( V(X) \cap V(Y) \). If \( X \) is a nontrivial \( (x, Y) \)-bridge of \( H \), then \( u \neq y \) because \( \{x, u\} \) is a cutset of \( H \) while \( \{x, y\} \) is not. Otherwise, we may choose \( X \) so that \( u \neq y \), since no edge in \( S \) is between \( x \) and \( y \). Thus we can assume that \( u \neq y \).

Let \( S_X = S \cap E(X) \) and \( S_Y = S \cap E(Y) \). Clearly, both \( |E(X)| + |S_X| \) and \( |E(Y)| + |S_Y| \) are less than \( |E(H)| + |S| \). Let \( \omega_X \) and \( \omega_Y \) be the restrictions of \( \omega \) on \( X \) and \( Y \), respectively. Without loss of generality, we may assume that \( xu \) is an edge in \( X \), for otherwise we add such a dummy edge to \( X \) and define \( \omega(xu) = 0 \). Similarly, we assume that \( yu \) is an edge in \( Y \).

If \( |X| = 2 \), set \( P_x := X \). If \( |X| \geq 3 \), applying the induction hypothesis on \((X, \omega_X)\), we find an \( x-u \) path \( P_x \) (excluding the dummy edge, if any) in \( X \) such that \( \sum_{e \in E(P_x)} 2^{\sigma(X, e)} \omega_X(e) \geq \omega_X(X) = \omega(X) \), where \( \sigma_X(P_x, e) \) is the maximum number of \( S_X \)-rungs in a ladder induced by \( P_x \) in \( X \) with top \( e \).

If \( |Y| = 2 \), set \( P_y := Y \). If \( |Y| \geq 3 \), applying the induction hypothesis on \((Y, \omega_Y)\), we find a \( u-y \) path \( P_y \) (excluding the dummy edge, if any) in \( Y \) such that \( \sum_{e \in E(P_y)} 2^{\sigma(Y, e)} \omega_Y(e) \geq \omega_Y(Y) = \omega(Y) \), where \( \sigma_Y(P_y, e) \) is the maximum number of \( S_Y \)-rungs in a ladder induced by \( P_y \) in \( Y \) with top \( e \).

Let \( P := P_x \cup P_y \). Clearly, \( \sigma(P, e) \geq \sigma_Y(P_y, e) \) for any \( e \in E(P_y) \). Let \( k \) be the number of \( (x, Y) \)-bridges other than \( X \) containing an edge of \( S \). From the definition of \( \sigma(P, e) \), we deduce that \( \sigma(P, e) \geq \)
\[ \sigma_X(P_x, e) + k \text{ for any } e \in E(P_x). \] So

\[
\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \sum_{e \in E(P_y)} 2^{\sigma(P_y,e)} \omega(e) + \sum_{e \in E(P_x)} 2^{\sigma(P,x,e)} \omega(e) \\
\geq \omega(Y) + \sum_{e \in E(P_x)} 2^{k+2\sigma_x(P,x,e)} \omega(e) \\
\geq \omega(Y) + 2^k \omega(X) \\
\geq \omega(H),
\]

where the last inequality holds since \( 2^k \omega(X) \geq (k+1) \omega(X) \geq \omega(H - Y). \) 

For each ordered edge pair \((e,f)\) of \(H\) and \(S \subseteq E(H),\) let \(r(e,f;H)\) denote the maximum number of \(S\)-rungs of a ladder with top \(e\) and bottom \(f.\) Clearly, \(r(e,f;H) \geq \sigma(P,e)\) for any \(x-y\) path \(P\) passing through \(e,\) where \(f = xy.\)

**Corollary 4.2.** Let \(H\) be a 2-connected graph, let \(f = xy \in E(H),\) let \(\omega : E(H) \rightarrow \mathbb{R}^+,\) and let \(S = \{e \in E(H) : \omega(e) > 0\}.\) Suppose \(r(e,f;H) = 0\) for some \(e \in E(H).\) Then there exists an \(x-y\) path \(P\) passing through \(e\) in \(H\) such that

\[
\sum_{g \in E(P)} 2^{\sigma(P,g)} \omega(g) \geq \omega(H).
\]

**Proof.** Let \(P\) be the \(x-y\) path as exhibited in Theorem 4.1. If \(e \in E(P),\) then we are done. So we assume \(e \notin E(P).\) Since \(H\) is 2-connected, it contains two vertex-disjoint paths \(Q_1, Q_2\) from the ends of \(e\) to \(P.\) Let \(v_1\) and \(v_2\) be the ends of \(Q_1\) and \(Q_2\) on \(P,\) respectively, and let \(R\) be the path obtained from \(P \cup Q_1 \cup Q_2\) by deleting all vertices on \(P[v_1,v_2].\) Since \(r(e,f;H) = 0,\) we have \(P[v_1,v_2] \cap S = \emptyset;\) that is, \(w(g) = 0\) for all edges \(g\) on \(P[v_1,v_2].\) So \(\sum_{g \in E(R)} 2^{\sigma(R,g)} \omega(g) \geq \sum_{g \in E(P)} 2^{\sigma(P,g)} \omega(g) \geq \omega(H).\)

5 Proof of Theorem 2.5(a)

The following lemma serves as the induction step in the proof of Theorem 2.5(a). Recall that \(b = 1729\) and \(\beta = \log_{2} b.\)

**Lemma 5.1.** Suppose \(n > b^{t(t-1)},\) \(t \geq 2,\) and Theorem 2.5 holds for graphs with at most \(n - 1\) vertices and for graphs containing no \(K_{3,4}\)-minors. Then Theorem 2.5(a) holds for graphs with \(n\) vertices.

**Proof.** Let \(G\) be a 3-connected \(n\)-vertex graph with \(\tau(G) \leq t,\) let \(x, y, z\) be three distinct vertices of \(G\) with \(xz, yz \in E(G),\) and let \(H = (G - z) + xy\) (so \(|H| = n - 1\)). Suppose Algorithm 2.2 has been applied to \((H;x,y).\) Our objective is to prove that there exists an \(x-y\) path in \(G - z\) of length at least \(\alpha(t)(\delta(t,H)(n - 1))^{\beta},\) where \(\delta(t,H)\) is as defined in (2.2) with \(e_0 = xy\) in place of \(f.\)

In our proof we shall frequently use the following identities:

\[ \alpha(t - 1) = \alpha(t)4^{t-1} \text{ and } 4^{(t-1)/\beta} = 2^{(t-1)\log_2 b} = b^{(t-1)}. \]  

(5.1)
Claim 5.1. We may assume \( \tau(G) = t \).

Otherwise, \( \tau(G) \leq t - 1 \), so \( G \) contains no \( K_{3,1} \)-minors. Hence the induction hypothesis of Theorem 2.5(a) guarantees the existence of an \( x-y \) path \( P \) in \( G - z \) such that

\[
\ell(P) \geq \alpha(t - 1) (\delta(t - 1, H)|H|)^\beta
= \alpha(t) \left( \delta^2(t-1)\delta(t-1, H)|H| \right)^\beta \quad \text{(by (5.1))}
\geq \alpha(t) (\delta(t, H)|H|)^\beta \quad \text{(by Lemma 2.4(iii) and since } b = 1729). \]

Claim 5.2. We may assume that \( H \) is not 3-connected.

Suppose on the contrary that \( H \) is 3-connected. Then \( \theta(H) = \phi(H) = 0 \) (see the comment above (2.2)), so \( \delta(t, H) = 1 \). Hence, by the induction hypothesis of Theorem 2.5(c), there exists an \( x-y \) path \( P \) in \( H \) (hence in \( G - z \)) with \( \ell(P) \geq \alpha(t)|H|^\beta = \alpha(t) \delta(t, H)|H|\beta \).

For each \( f = uv \in \Psi(H) \), let \( H_f \) be as defined in Algorithm 2.2 and let \( G_f = G[V(H_f) \cup z] + \{uz, vz\} \). Recall that in Algorithm 2.2 we set \( e_0 = xy \).

Claim 5.3. If \( f \neq e_0 \), then \( H_f \) contains a \( u-v \) path of length at least \( \alpha(t) (\delta(t, H)|H_f|)^\beta \).

Since \( f \neq e_0 \), we have \( |f| < |G| \). By Lemma 2.3(iv), \( G_f \) is a 3-connected minor of \( G \). Let \( s \) be any integer such that \( \tau(G_f) \leq s \). Then the induction hypothesis of Theorem 2.5(a) implies the existence of a \( u-v \) path \( P \) in \( H_f \) with \( \ell(P) \geq \alpha(s) \delta(s, H_f)|H_f|\beta \).

Suppose \( \tau(G_f) = \tau(G) \). Set \( s = t \). By Lemma 2.4(iv), we get \( \delta(t, H_f) \geq \delta(t, H) \). Thus \( \ell(P) \geq \alpha(t) (\delta(t, H)|H_f|)^\beta \), as desired. So we may assume that \( \tau(G_f) < \tau(G) = t \). Set \( s = t - 1 \). Then the same argument used in the proof of Claim 5.1 implies

\[
\ell(P) \geq \alpha(t - 1) (\delta(t - 1, H_f)|H_f|)^\beta \geq \alpha(t) (\delta(t, H)|H_f|)^\beta.
\]

Suppose Case 2 or Case 3 of Algorithm 2.2 occurs; see the descriptions. Set \( \hat{H}_i = H_i + a_i b_i \), where \( a_i b_i = x_i x_{i+1} \) in Case 2 and \( a_i b_i = u_i v_i \) in Case 3, and set \( G_i = G[\hat{H}_i \cup \{z\}] + a_i z + b_i z \). By the hypotheses of Cases 2 and Case 3, \( \{x, y\} \) is not a cutset of \( H \), so \( \{a_i, b_i\} \neq \{x, y\} \). Using exactly the same proof as that of Lemma 2.3(iv), we see that \( G_i \) is a 3-connected minor of \( G \). In particular, \( \tau(G_i) \leq \tau(G) \). By applying Algorithm 2.2 directly to the input \( (\hat{H}_i; a_i b_i) \) (that is, with \( (\hat{H}_i; a_i b_i) \) in place of \((H;e_0))\), we can define \( \theta(\hat{H}_i), \phi(\hat{H}_i), \) and \( \delta(t, \hat{H}_i) \) accordingly. Now the same argument of Lemma 2.4(iv) implies that if \( \tau(G_i) = \tau(G) \), then \( \delta(t, H) \leq \delta(t, \hat{H}_i) \) for any \( t \geq \tau(G) \). Finally, imitating the proof of Claim 5.3, we get the following statement.

Claim 5.4. There exists an \( a_i - b_i \) path in \( H_i \) of length at least \( \alpha(t) (\delta(t, H)|H_i|)^\beta \).

Claim 5.5. For any \( f = uv \in \Psi(H) \) with \( \tau(G_f) \leq t - 1 \), we may assume \( |H_f| < \frac{|H|}{8t^2} \).
Suppose $|H_f| \geq \frac{|H|}{816}$. By Lemma 2.3(iv), $G_f$ is a 3-connected minor of $G$. In view of Claim 5.1, $G_f \neq G$, so $|G_f| < |G|$. Thus the induction hypothesis of Theorem 2.5(a) yields a $u$-$v$ path $P$ in $H_f$ such that

$$
\ell(P) \geq \alpha(t - 1) \left( \frac{\delta(t, H_f)|H_f|}{27} \right) \geq \alpha(t - 1) \left( \frac{|H_f|}{27} \right)^3 \geq \alpha(t) \left( \frac{|H_f|}{216t^2} \right)^3 \geq \alpha(t) \left( \frac{\delta(t, H)|H|}{216t^2} \right)^3 > \alpha(t) \left( \delta(t, H)|H| \right)^3.
$$

Since $H$ is 2-connected, it contains two vertex-disjoint paths $Q_1$ and $Q_2$ from $x, y$ such that $Q_1 \cup Q_2$ is a 2-connected minor of $H$. Then, by (2.1), we have

$$
\delta(H) \leq \frac{1}{3\delta(H_t) + k} \left( 1 - \phi(H_t) + \frac{m - 1 - k}{3t} \right). \quad (5.2)
$$

If $\delta(H) = \delta(H_t) + k$ then, by (2.1), we have $\phi(H) \geq \phi(H_t) + (m - 1)$. Thus (5.2) follows from the definition of $\delta(t, H)$. So we may assume $\delta(H) \geq \delta(H_t) + k + 1$. Then

$$
\delta(t, H) \leq \frac{1}{3\delta(H_t) + k + 1} \leq \frac{1}{3\delta(H_t) + k} \left( 1 - \phi(H_t) + \frac{m - 1}{3t} \right) \leq \text{RHS of (5.2)},
$$

where the third inequality holds since $m \leq t$ and $\phi(H_t) \leq \tau(G_{e_i}) \leq t$ by Lemma 2.4(i). Observe that the RHS of (5.2) is at most

$$
\frac{1}{3\delta(H_t) + k} \left( 1 - \phi(H_t) - \theta(H_t) + \frac{m - 1 - k}{3t} \right) = \delta(t, H) \left( \frac{1}{3\theta(H_t) + k - m - 1 - k} \right). \quad (5.3)
$$

Hence

$$
\delta(t, H) \leq \delta(t, H_t) \left( \frac{1}{3k} \left( 1 - \frac{m - 1 - k}{3t} \right) \right). \quad (5.3)
$$

By Claim 5.5, $|H_{e_i}| \leq \frac{|H|}{816}$ for any $j$ with $k + 2 \leq j \leq m$. It follows from the maximality of $|H_{e_i}|$ that

$$
|H_{e_i}| \geq \frac{m - 1 - k}{816} |H| \geq \frac{1}{3k} \left( 1 - \frac{m - 1 - k}{3t} \right) |H|. \quad (5.4)
$$
By the induction hypothesis of Theorem 2.5(a), there exists an \( x \)-\( y \) path \( P \) in \( H_{e_1} \) such that \( \ell(P) \geq \alpha(t) (\delta(t, H_{e_1})|H_{e_1}|)^{\beta} \). From (5.3) and (5.4), we conclude that

\[
\ell(P) \geq \alpha(t) \left( \delta(t, H_{e_1}) \frac{1}{3k} \left( 1 - \frac{m - 1 - k}{3t} \right) |H| \right)^{\beta} \geq \alpha(t) (\delta(t, H)|H|)^{\beta}.
\]

Recall that \( H^* \) is the leading block \( H^* \) output by Algorithm 2.2.

**Claim 5.7.** We may assume that \( H^* \) is not a multicycle; so Case 2 of Algorithm 2.2 cannot occur and \( H^* \) is 3-connected by Lemma 2.3(iii).

Suppose to the contrary that \( H^* \) is a multicycle. By Claim 5.6, \( \{x, y\} \) is not a cutset of \( H \). So \( xy \) is the unique edge in \( H^* \) with ends \( x \) and \( y \). Therefore, \( H^* - xy \) can be obtained from a simple path \( a_0a_1 \ldots a_k \) by adding parallel edges, where \( a_0 = x \) and \( a_k = y \). For each \( i \) with \( 0 \leq i \leq k - 1 \), let \( \bar{H}_i \) be the graph as defined right above Claim 5.4, where \( b_i = a_{i+1} \). By Claim 5.4, there exists an \( a_i-a_{i+1} \) path \( P_i \) in \( H_i \) such that \( \ell(P_i) \geq \alpha(t) (\delta(t, H)|H_i|)^{\beta} \). Concatenating all these \( P_i \), we obtain an \( x-y \) path \( P \) with

\[
\ell(P) \geq \sum_{i=0}^{k-1} \alpha(t) (\delta(t, H)|H_i|)^{\beta} \geq \alpha(t) \left( \delta(t, H) \sum_{i=0}^{k-1} |H_i| \right)^{\beta} \geq \alpha(t) (\delta(t, H)|H|)^{\beta},
\]

where the second inequality follows from Corollary 3.6.

For \( i = 1, 2 \), set \( \Psi_i := \Psi_i(H) \cap E(H^*) - \{e_0\} \), where \( e_0 = xy \), and define a weight function \( \omega_i : E(H^*) \mapsto \mathbb{R}^+ \) as follows:

\[
\omega_i(f) = \begin{cases} 
|H_f|, & \text{if } f \in \Psi_i \\
0, & \text{otherwise}.
\end{cases}
\]

In addition, set \( \omega_i(H^*) := \sum_{f \in \Psi_i} |H_f| \).

**Claim 5.8.** We may assume that \( \omega_2(H^*) < \frac{|H|}{9t^2} \).

Otherwise, \( \omega_2(H^*) \geq \frac{|H|}{9t^2} \). By Theorem 4.1 (with respect to \( \omega_2 \)), there exists an \( x-y \) path \( Q \) in \( H^* \) such that \( \sum_{e \in E(Q)} 2^{\sigma(Q, e)} \omega_2(e) \geq \omega_2(H^*) \), where \( \sigma(Q, e) \) is the maximum number of \( \Psi_2 \)-rungs of a ladder in \( H^* \) generated by \( Q \) with top \( e \) and bottom \( xy \). Since \( \tau(G) = t \), we have \( \sigma(Q, e) \leq t \), which implies \( \sum_{e \in E(Q)} \omega_2(e) \geq \omega_2(H^*)/2^t \). So \( \sum_{e \in E(Q) \cap \Psi_2} |H_e| \geq \omega_2(H^*)/2^t \geq |H|/(9t^2) \).

For each \( e \in E(Q) \cap \Psi_2 \), there holds \( \tau(G_e) \leq t - 1 \). So the induction hypothesis of Theorem 2.5(a) guarantees the existence of a path \( P_e \) in \( H_e \) between the ends of \( e \) such that

\[
\ell(P_e) \geq \alpha(t - 1) (\delta(t - 1, H_e)|H_e|)^{\beta} \geq \alpha(t - 1) \left( \frac{|H_e|}{27} \right)^{\beta} = \alpha(t) \left( \frac{b^{2(t-1)}}{27} |H_e| \right)^{\beta},
\]

where the second inequality follows from Lemma 2.4(iii) and the equality from (5.1).
Claim 5.10. \( \ell(P) \geq \sum_{e \in E(Q) \cap \Psi_2} \alpha(t) \left( \frac{b^2(t-1)}{27} |H_e| \right)^\beta \)
\[ \geq \alpha(t) \left( \frac{b^2(t-1)}{27} \sum_{e \in E(Q) \cap \Psi_2} |H_e| \right)^\beta \quad \text{(by Corollary 3.6)} \]
\[ \geq \alpha(t) \left( \frac{b^2(t-1)}{27 \cdot 9t \cdot 2^t} |H| \right)^\beta \]
\[ \geq \alpha(t) |H|^\beta \quad \text{(since } b^2(t-1) \geq 27 \cdot 9t \cdot 2^t) \]
\[ \geq \alpha(t) (\delta(t, H)|H|)^\beta \quad \text{(since } \delta(t, H) \leq 1). \]

Claim 5.9. We may assume \( \theta(H) \geq 1; \) so \( \delta(t, H) \leq 1/3 \) by Lemma 2.4(ii).
Suppose \( \theta(H) = 0. \) By Claim 5.2 and Claim 5.6, \( \Psi(H) - \{f\} \neq \emptyset; \) so \( \phi(H) \geq 1 \) (by definition). Thus
\[ \delta(t, H) = \frac{1}{3\theta(H)} \left( 1 - \frac{\phi(H) - \theta(H)}{3t} \right) = 1 - \frac{\phi(H)}{3t} \leq \frac{3t - 1}{3t}. \]
Since \( \Psi_1(H) = \emptyset \) (as \( \theta(H) = 0 \)), by Claim 5.8 we have
\[ |H^*| \geq |H| - \omega_2(H^*) \geq |H| - \frac{|H|}{9t} \geq \frac{3t - 1}{3t} (n - 1) \geq \delta(t, H)(n - 1). \]
As \( H^* \) is a 3-connected minor of \( G \) (by Lemma 2.3(ii) and Claim 5.7), the induction hypothesis of Theorem 2.5(c) gives an \( x \)-\( y \) path \( Q \) in \( H \) with \( \ell(Q) \geq \ell(P) \geq \alpha(t)|H^*|^\beta \geq \alpha(t) (\delta(t, H)(n - 1))^\beta. \)

Claim 5.10. We may assume that \( |H^*| < \frac{|H|}{3}. \)
Assume \( |H^*| \geq |H|/3. \) Since \( H^* \) is a 3-connected minor of \( G \) (by Lemma 2.3(ii) and Claim 5.7), the induction hypothesis of Theorem 2.5(c) gives an \( x \)-\( y \) path \( P \) in \( H^* \), which clearly leads to an \( x \)-\( y \) path \( Q \) in \( H \), such that \( \ell(Q) \geq \ell(P) \geq \alpha(t)|H^*|^\beta \geq \alpha(t) (\frac{1}{3}|H|)^\beta \geq \alpha(t) (\delta(t, H)|H|)^\beta \) (by Claim 5.9).

Since \( t \geq 2 \), combining Claims 5.10 and 5.8 we obtain
\[ \omega_1(H^*) = \sum_{e \in \Psi_1} |H_e| \geq |H| - |H^*| - \omega_2(H^*) \geq \left( 1 - \frac{1}{3} - \frac{1}{9t} \right) |H| > |H|/2. \quad (5.5) \]
By Theorem 4.1, there exists an \( x \)-\( y \) path \( Q \) in \( H^* \) such that
\[ \sum_{e \in E(Q)} 2^{\sigma(Q,e)} \omega_1(e) \geq \omega_1(H^*), \quad (5.6) \]
where \( \sigma(Q,e) \) is the maximum number of \( \Psi_1 \)-rungs of a ladder in \( H^* \) generated by \( Q \) with top \( e \) and bottom \( e_0 = xy. \) We shall use \( Q \) to produce a desired path in \( H \), by comparing \( |H^*|, w_1(H^*) \) and \( w_2(H^*). \)
Claim 5.11. For each $e \in E(Q) \cap \Psi_1$, there holds $\delta(t, H_e) \geq \delta(t, H) \cdot 3^{\sigma(Q, e)}$.

Since $e \in \Psi_1(H)$, we have $\tau(G_e) = \tau(G)$. It is then a routine matter to check that $\theta(H) \geq \theta(H_e) + \sigma(Q, e)$. To justify the claim, we distinguish between two cases. If $\theta(H) \geq \theta(H_e) + \sigma(Q, e) + 1$, then

$$\delta(t, H) \leq \frac{1}{3^{\sigma(Q, e)} \eta(H_e)} \leq \frac{1}{3^{\sigma(Q, e)} \eta(H_e)} \left(1 - \frac{\phi(H_e) - \theta(H_e)}{3^l}\right) = \frac{\delta(t, H_e)}{3^{\sigma(Q, e)}},$$

where the last inequality holds since $\phi(H_e) \leq t$ by Lemma 2.4(i). So we assume $\theta(H) = \theta(H_e) + \sigma(Q, e)$. Now from (2.1) we deduce that $\theta(H) \geq \phi(H_e) + \sigma(Q, e)$. Thus $\phi(H) - \theta(H) \geq (\phi(H_e) + \sigma(Q, e)) - (\theta(H_e) + \sigma(Q, e)) \geq \phi(H_e) - \theta(H_e)$. Hence the desired inequality follows instantly as in the previous case, completing the proof of Claim 5.11.

Let $g$ be an edge on $Q$ such that $3^{\sigma(Q, g)} \omega_1(g) = \max_{e \in E(Q)} \{3^{\sigma(Q, e)} \omega_1(e)\}$, and set

$$\lambda := \sum_{e \in E(Q) \cap (g)} 3^{\sigma(Q, e)} \omega_1(e).$$

For each $e = uv \in E(Q) \cap \Psi_1$, let $G_e = G[V(H_e) \cup \{z\}] + \{zu, vz, uv\}$. By Lemma 2.3(iv), $G_e$ is a 3-connected minor of $G$. So by the induction hypothesis of Theorem 2.5(a), there exists a path $P_e$ in $H_e$ between the ends of $e$ such that

$$\ell(P_e) \geq \alpha(t) (\delta(t, H_e))^{\beta} \geq \alpha(t) (\delta(t, H) 3^{\sigma(Q, e)} \omega_1(e))^{\beta}, \quad (5.7)$$

where the second inequality follows from Claim 5.11.

Claim 5.12. We may assume that $\lambda < \frac{1}{b - 2}(|H^*| + \omega_2(H^*))$.

Otherwise, $(b - 2) \lambda \geq |H^*| + \omega_2(H^*)$; so $w_1(H^*) + (b - 2) \lambda \geq |H^*| + \omega_2(H^*)$. Concatenating all $P_e$, with $e \in E(Q) \cap \Psi_1$, and paths in $H_e$ corresponding to all edges $e \in E(Q) \cap \Psi_1$, we obtain an $x-y$ path $P$ in $G - z$ such that

$$\ell(P) \geq \sum_{e \in E(Q) \cap \Psi_1} \ell(P_e)$$

$$\geq \alpha(t) \sum_{e \in E(Q) \cap \Psi_1} \left(\delta(t, H) 3^{\sigma(Q, e)} \omega_1(e)\right)^{\beta} \quad \text{by (5.7)}$$

$$\geq \alpha(t) \left(\delta(t, H) \left(3^{\sigma(Q, g)} \omega_1(g) + (b - 1) \sum_{e \in E(Q) \cap \Psi_1} 3^{\sigma(Q, e)} \omega_1(e)\right)\right)^{\beta} \quad \text{(by Corollary 3.6)}$$

$$= \alpha(t) \left(\delta(t, H) \left(3^{\sigma(Q, g)} \omega_1(g) + (b - 1) \lambda\right)\right)^{\beta}$$

$$\geq \alpha(t) \delta(t, H) (\omega_1(H^*) + (b - 2) \lambda)^{\beta} \quad \text{(by (5.6))}$$

$$\geq \alpha(t) \delta(t, H) |H|^{\beta}.$$
Claim 5.13. We may assume \(|H^*| < \omega_2(H^*)\).

Suppose \(|H^*| \geq \omega_2(H^*)\). Then by Claim 5.12 and Claim 5.10, we have \(\lambda \leq \frac{2|H^*|}{3t-2} \leq \frac{2|H|}{3t-2}\). From the definition of \(\lambda\), (5.6) and (5.5), it follows that \(3^\sigma(Q,g)\omega_1(g) \geq \omega_1(H^*) - \lambda \geq \frac{|H|}{3t-2} - \frac{2|H|}{3t-2} \geq \frac{|H|}{3t-2}\). So by Claim 5.10, we have

\[
3^\sigma(Q,g)\omega_1(g) \geq |H^*|. \tag{5.8}
\]

Let \(P_g\) be the path as specified in (5.7) with \(e = g\). By Lemma 2.3(ii) and Claim 5.7, \(H^*\) is a 3-connected minor of \(G\). So the induction hypothesis of Theorem 2.5(b) gives an \(x\)-\(y\) path \(Q_g\) passing \(g\) in \(H^*\) such that \(\ell(Q_g) \geq \alpha(t)(|H^*/28|^\beta + 1)\). Let \(P\) be the \(x\)-\(y\) path obtained from \(Q_g\) by replacing \(g\) with \(P_g\). Then

\[
\ell(P) = \ell(P_g) + \ell(Q_g) - 1 \\
\geq \alpha(t) \left( \delta(t, H) 3^\sigma(Q,g)\omega_1(g) \right)^\beta + \alpha(t) \left( |H^*|/28 \right)^\beta \\
\geq \alpha(t) \left( \delta(t, H) 3^\sigma(Q,g)\omega_1(g) \right)^\beta + \alpha(t) \left( \delta(t, H)|H^*/28| \right)^\beta \quad \text{(because \(\delta(t, H) \leq 1\))} \\
\geq \alpha(t) \left( \delta(t, H) \left( 3^\sigma(Q,g)\omega_1(g) + (b-1)|H^*/28| \right) \right)^\beta \quad \text{(by (5.8) and Corollary 3.6)} \\
\geq \alpha(t) \left( \delta(t, H) (\omega_1(H^*) + |H^*| + \omega_2(H^*)) \right)^\beta \quad \text{(by Claim 5.12 ans since \(b = 1729\))} \\
\geq \alpha(t) \left( \delta(t, H) |H^*/28|^\beta \right)^\beta \text{.}
\]

Path \(P\) obviously leads to an \(x\)-\(y\) path \(R\) in \(H\) with \(\ell(R) \geq \ell(P) \geq \alpha(t) \left( \delta(t, H)|H| \right)^\beta\).

Claim 5.14. We may assume that \(\sigma(Q,g) = 0\), \(\omega_1(g) \geq \left( 1 - \frac{1}{3t} \right)|H|\), and \(\delta(t, H_g) < \delta(t, H)/(1 - \frac{1}{3t})\).

By (5.6), \(\sum_{e \in E(Q)} 2^\sigma(Q,e)\omega_1(e) \geq \omega_1(H^*) \geq |H| - |H^*| - \omega_2(H^*)\). So by Claims 5.12 and 5.13,

\[
2^\sigma(Q,g)\omega_1(g) \geq |H| - |H^*| - \omega_2(H^*) - \lambda \geq |H| - \frac{1}{b-2}(|H^*| + \omega_2(H^*)) \geq |H| - \frac{2(b-1)}{b-2}\omega_2(H^*). 
\]

Since \(\omega_2(H^*) < \frac{|H|}{3t}\) (by Claim 5.8), \(2^\sigma(Q,g)\omega_1(g) \geq \left( 1 - \frac{2(b-1)}{3t(b-2)} \right)|H|\). Hence

\[
2^\sigma(Q,g)\omega_1(g) \geq \left( 1 - \frac{1}{3t} \right)|H|. \tag{5.9}
\]

Let \(P_g\) be the path as exhibited in (5.7) with \(e = g\). Then by (5.7) and (5.9),

\[
\ell(P_g) \geq \alpha(t) \left( \delta(t, H) 3^\sigma(Q,g)\omega_1(g) \right)^\beta \geq \alpha(t) \left( \delta(t, H) \left( \frac{3}{2} \right)^\sigma(Q,g)(1 - \frac{1}{3t})|H| \right)^\beta .
\]

Suppose \(\sigma(Q,g) \geq 1\). Then \(\ell(P_g) \geq \alpha(t) \left( \delta(t, H)|H| \right)^\beta\). Let \(R_1\) and \(R_2\) be two vertex-disjoint paths in \(H\) from \(\{x, y\}\) to the two ends of \(g\) (and internally disjoint from \(H_g\)). Clearly \(R_1 \cup P_g \cup R_2\) leads to
an \(x-y\) path in \(H\) with length at least \(\alpha(t) \left( \delta(t, H) |H| \right)^\beta\). So we may assume \(\sigma(Q,g) = 0\). Then by (5.9), 

\[
|H_g| = \omega_1(g) \geq (1 - \frac{1}{3t}) |H|.
\]

If \(\delta(t, H_g) \geq \delta(t, H) / (1 - \frac{1}{3t})\), then 

\[
\ell(P_g) \geq \alpha(t) (\delta(t, H_g) |H_g|)^\beta \geq \alpha(t) \left( \frac{\delta(t, H)}{1 - \frac{1}{3t}} \cdot \left(1 - \frac{1}{3t} \right) |H| \right)^\beta = \alpha(t) \left( \delta(t, H) |H| \right)^\beta.
\]

Clearly \(P_g\) leads to a desired path for the lemma. So we may assume \(\delta(t, H_g) < \delta(t, H) / (1 - \frac{1}{3t})\).

**Claim 5.15.** \(\theta(H_g) = \theta(H)\) and \(\phi(H_g) = \phi(H)\).

By Claim 5.14, we have \(\omega_1(g) \neq 0\). So \(g \in \Psi_1(H)\) and hence \(\tau(G_g) = \tau(G)\). It follows that \(\theta(H_g) \leq \theta(H)\). If \(\theta(H) \geq \theta(H_g) + 1\) then, by Lemma 2.4(i), 

\[
\delta(t, H_g) = \frac{1}{3t} (1 - \frac{\phi(H_g) - \theta(H_g)}{3t}) \geq \frac{1}{3t} (1 - \frac{t}{3t}) \geq \frac{2}{3t} \geq 2 \delta(t, H) > \delta(t, H) / (1 - \frac{1}{3t}),
\]

contradicting Claim 5.14. So \(\theta(H_g) = \theta(H)\).

From \(\theta(H_g) = \theta(H)\) and (2.1), we deduce that \(\phi(H_g) \leq \phi(H)\). If \(\phi(H) \geq \phi(H_g) + 1\), then \(\phi(H) - \theta(H) \geq \phi(H_g) - \theta(H_g) + 1\). It follows that 

\[
\delta(t, H) \leq \frac{1}{3t} (1 - \frac{\phi(H_g) - \theta(H_g) + 1}{3t}) \leq \frac{1}{3t} (1 - \frac{\phi(H_g) - \theta(H_g)}{3t})(1 - \frac{1}{3t}) = \delta(t, H_g)(1 - \frac{1}{3t}).
\]

This contradicts Claim 5.14, and so \(\phi(H_g) = \phi(H)\), proving Claim 5.15.

Finally, we define the third weight function \(\omega_3\) as follows:

\[
\omega_3(f) = \begin{cases} |H_f|, & \text{if } f \in \Psi_2 \cup \{g\} \\ 0, & \text{otherwise.} \end{cases}
\]

Notice that \(\omega_2\) and \(\omega_3\) are identical except that \(\omega_2(g) = 0\) while \(\omega_3(g) = |H_g|\). From Claim 5.15 (\(\phi(H_g) = \phi(H)\)), we deduce that no \(\Psi_2\)-rungs exist for any ladder in \(H^*\) with top \(g\) and bottom \(e_0 = xy\). So, using the notation introduced right above Corollary 4.2, we obtain \(r(g,e_0;H^*) = 0\). By this corollary, there exists an \(x-y\) path \(R\) passing through \(g\) in \(H^*\) such that 

\[
\sum_{e \in E(R)} 2^{\sigma(R,e)} \omega_3(e) \geq \omega_3(H^*) = \omega_3(g) + \omega_2(H^*),
\]

where \(\sigma(R,e)\) is the maximum number of \(\Psi_2\)-rungs of a ladder in \(H^*\) generated by \(R\) with top \(e\) and bottom \(e_0 = xy\). Since \(r(g,e_0;H^*) = 0\), we have \(\sigma(R,g) = 0\). So 

\[
\sum_{e \in E(R)} 2^{\sigma(R,e)} \omega_2(e) \geq \omega_2(H^*),
\]

because \(\omega_2(g) = 0\). As \(\tau(G) \leq t\), it is easy to see that \(\sigma(R,e) \leq t - 1\) for each \(e \in E(R) \cap \Psi_2\) (recall that \(\sigma(R,e)\) does not count \(e\)). Hence 

\[
\sum_{e \in E(R)} \omega_2(e) \geq \omega_2(H^*)/2^{t-1}. \tag{5.10}
\]

For each \(e \in E(R) \cap \Psi_2\), we have \(\tau(G_e) \leq \tau(G) - 1 \leq t - 1\). So the induction hypothesis of Theorem 2.5(a) gives a path \(R_e\) in \(H_e\) between the ends of \(e\) such that 

\[
\ell(R_e) \geq \alpha(t-1) \left( \delta(t-1, H_e) |H_e| \right)^\beta \geq \alpha(t-1) \left( \frac{|H_e|}{2} \right)^\beta = \alpha(t) 4^{t-1} \left( \frac{\omega_2(e)}{27} \right)^\beta,
\]

as desired.
where the second inequality follows from Lemma 2.4(iii) and the equality follows from (5.1). Let $P_g$ be the path as specified in (5.7) with $e = g$. Concatenating $P_g$, all these $R_c$, and paths in $H_c$ corresponding to all edges $e \in E(R) - (\Psi_2 \cup \{g\})$, we obtain an $x$-$y$ path $T$ in $H$ such that

$$\ell(T) \geq \alpha(t) \left( (\delta(t,H)\omega_1(g)) \right) + \sum_{e \in E(R)} 4^{t-1} \left( \frac{\omega_2(e)}{27} \right)^\beta.$$  

(5.11)

By Lemma 2.4(iii) and Claim 5.14, $\delta(t,H)\omega_1(g) \geq \frac{\omega_2(g)}{27} \geq \frac{1}{27}(1 - \frac{1}{3})|H| = \frac{2}{9t}|H|$. By Claim 5.8, $\omega_2(H^*) < \frac{|H|}{9t}$. So

$$\delta(t,H)\omega_1(g) \geq \frac{3t-1}{9} \omega_2(H^*) \geq \frac{1}{2} \omega_2(H^*) \geq \frac{\omega_2(e)}{27}$$

(5.12)

for any $e \in E(R)$. Let us view $4^{t-1} \left( \frac{\omega_2(c)}{27} \right)^\beta$ as the sum of $4^{t-1}$ terms, each being $\left( \frac{\omega_2(c)}{27} \right)^\beta$. Applying Corollary 3.6 to the RHS of (5.11) and using (5.12), we have

$$\ell(T) \geq \alpha(t) \left( \delta(t,H)\omega_1(g) + 4^{t-1}(b-1) \sum_{e \in E(R)} \frac{\omega_2(e)}{27} \right)^\beta.$$  

Plugging in (5.10) and using the inequality $(b-1)4^{t-1}/(27 \cdot 2^{t-1}) \geq 3$, we obtain

$$\ell(T) \geq \alpha(t) \left( \delta(t,H)\omega_1(g) + 4^{t-1}(b-1) \sum_{e \in E(R)} \frac{\omega_2(e)}{27} \right)^\beta$$

$$\geq \alpha(t) \left( \delta(t,H)\omega_1(g) + 4^{t-1}(b-1)\omega_2(H^*)/2^{t-1} \right)^\beta \quad \text{(by (5.10))}$$

$$> \alpha(t) (\delta(t,H)(\omega_1(g) + 3\omega_2(H^*)))^\beta$$

$$\geq \alpha(t) \delta(t,H) (|H_g| + |H^*| + \omega_2(H^*) + \lambda)^\beta \quad \text{(by Claims 5.12 and 5.13)}$$

$$\geq \alpha(t) \delta(t,H) (|H^*| + \omega_1(H^*) + \omega_2(H^*))^\beta$$

$$\geq \alpha(t) \delta(t,H)|H|^\beta,$$

where the second last inequality holds because $\sigma(Q, g) = 0$ (by Claim 5.14). From the definition of $\lambda$ and (5.6) it follows that $\lambda + |H_g| \geq \omega_2(H^*)$. This completes the proof of Lemma 5.1.

6 Proof of Theorem 2.5(b)

Let us establish the following lemma, which serves as the induction step for proving Theorem 2.5(b).

**Lemma 6.1.** Suppose $n > b^{t(t-1)}$, $t \geq 2$, and Theorem 2.5 holds for graphs with at most $n - 1$ vertices and for graphs containing no $K_{3,1}$-minors. Then Theorem 2.5(b) holds for graphs with $n$ vertices.

**Proof.** We may assume that $e$ and $f$ are nonadjacent, for otherwise, symmetry allows us to assume that $y$ is the common end of both $e$ and $f$. Let $w$ be the other end of $e$ and let $H := G - y$. Then, by
Lemma 5.1, $H$ contains an $x$-$w$ path $P$ with length $\ell(P) \geq \alpha(t)(\delta(t, H)|H|)^\alpha \geq \alpha(t)(\frac{n-1}{27})^3 \geq \alpha(t)(\frac{n}{28})^3$ for $n \geq 28$, where the second inequality follows from Lemma 2.4(iii). Let $Q$ be the path obtained from $P$ by appending the edge $e$. Clearly, $Q$ is an $x$-$y$ path through $e$ with length at least $\alpha(t)(n/28)^3 + 1$ in $G$.

As $G$ is 3-connected, it contains an $x$-$y$ path $Q$ through $e$. Let $Q_x$ and $Q_y$ be the components of $Q - e$ containing $x$ and $y$, respectively. Let $x_0x_1x_2\ldots x_pX_p x_{p+1}$ denote the chain of blocks in $G - V(Q_y)$ from $x$ to $x_{p+1}$, where $x_0 = x$ and $x_{p+1}$ is incident with $e$. Let $X = \bigcup_{i=1}^{p} X_i$. Clearly, $Q_x \subseteq X$.

Since $G$ is 3-connected, $U_i := G - V(X_i)$ is connected for each $i$ with $0 \leq i \leq p$. From Lemma 3.3, we deduce that $X_i^* := G/U_i$ is either a triangle or a 3-connected minor of $G$. Let $u_i$ denote the vertex of $X_i^*$ resulted from the contraction of $U_i$. Clearly, $u, x_i, u, x_{i+1} \in E(X_i^*)$. Since $|U_i| \geq 2$, we have $|X_i^*| < n$.

From 3-connectedness of $G$, we see that $Y := G - V(X)$ is a chain of blocks in $G - V(X)$. Suppose $Y$ is $y_0Y_0y_1Y_1\ldots y_qY_qy_{q+1}$, where $y_{q+1}$ is incident with $e$ and $y_0 = y$. Since $G$ is 3-connected, $W_j := G - V(Y_j)$ is connected for each $j$ with $0 \leq j \leq q$. By Lemma 3.3, $Y_j^* := G/W_j$ is either a triangle or a 3-connected minor of $G$. Let $w_j$ denote the vertex of $Y_j^*$ resulted from the contraction of $W_i$. Clearly, $w_jy_j, w_jy_{j+1} \in E(Y_j^*)$. Since $|W_j| \geq 2$, we have $|Y_j^*| < n$.

Let us now define an $x_i$-$x_{i+1}$ path $P_i$ in $X_i$ and an $y_j$-$y_{j+1}$ path $Q_j$ in $Y_j$ for all $i$ and $j$ as follows:

Set $P_i := X_i$ if $|X_i| = 2$. Clearly, $\ell(P_i) = 1 \geq \alpha(t)(\frac{|X_i|}{27})^3$. In the other case, $|X_i| \geq 3$. By Lemma 5.1 and Lemma 2.4(iii), there is an $x_i$-$x_{i+1}$ path $P_i$ in $X_i := X_i^* - u_i$ satisfying $\ell(P_i) \geq \alpha(t)(\frac{|X_i^* - 1|}{27})^3 = \alpha(t)(\frac{|X_i|}{27})^3$.

Set $Q_j := Y_j$ if $|Y_j| = 2$. Clearly, $\ell(Q_j) = 1 \geq \alpha(t)(\frac{|Y_j|}{27})^3$. In the other case, $|Y_j| \geq 3$. By Lemma 5.1 and Lemma 2.4(iii), there is a $y_j$-$y_{j+1}$ path $Q_j$ in $Y_j := Y_j^* - w_j$ satisfying $\ell(Q_j) \geq \alpha(t)(\frac{|Y_j^* - 1|}{27})^3 = \alpha(t)(\frac{|Y_j|}{27})^3$.

Finally, concatenating all these $P_i$, all these $Q_j$, and the edge $e$, we obtain an $x$-$y$ path $R$ through $e$ in $G$ such that

$$\ell(R) = \sum_{i=1}^{p} \ell(P_i) + \sum_{j=1}^{q} \ell(Q_j) + 1 \geq \sum_{i=1}^{p} \alpha(t)(\frac{|X_i|}{27})^3 + \sum_{j=1}^{q} \alpha(t)(\frac{|Y_j|}{27})^3 + 1 \geq \alpha(t) \left( \frac{1}{27} \left( \sum_{i=1}^{p} |X_i| + \sum_{j=1}^{q} |Y_j| \right) \right)^\beta + 1 \geq \alpha(t) \left( \frac{|G| - 1}{27} \right)^\beta + 1 \geq \alpha(t) \left( \frac{|G|}{28} \right)^\beta + 1.$$  

This completes the proof of Lemma 6.1.
7 Proof of Theorem 2.5(c)

In this section, we establish the induction step for proving Theorem 2.5(c).

**Lemma 7.1.** Suppose \( n > t^{(t-1)} \), \( t \geq 2 \), and Theorem 2.5 holds for graphs with at most \( n - 1 \) vertices and for graphs containing no \( K_{3,1} \)-minors. Then Theorem 2.5(c) holds for graphs with \( n \) vertices.

**Proof.** To show the existence of an \( x \)-\( y \) path of length at least \( \alpha(t)n^3 \) in \( G \), we search for it from \( x \) and proceed step by step to \( y \). At a certain point, the remaining graph may no longer be 3-connected. In this case, we are forced to choose one out of several parts of this graph. While our choice may be “good” at some stage, it may become undesirable at certain later stage, thereby we have to come back to modify our choice. This process is very sophisticated, and the notion of “magic minor” was used in [4] to guide the direction of our search and to help us explain things in a precise and concise way. To prove the present lemma, we need a modified version of this concept.

Let \( H_0 \) be an induced subgraph of \( G \) and let \( x_0 \) and \( y_0 \) be two distinct vertices of \( H_0 \) such that \( H_0 + x_0y_0 \) is 2-connected. We say that \((H_0, x_0, y_0)\) is a **magic minor** of \((G, x, y)\) if the following conditions are satisfied:

(M1) \( G - (V(H_0) - \{x_0, y_0\}) \) contains two vertex-disjoint paths \( X_0, Y_0 \) from \( x \) to \( x_0, y_0 \), respectively; 
(M2) \( U_0 := G - V(H_0) \) is connected and \( H_0^* \) is 3-connected, where \( H_0^* := G/U_0 \) if \( H_0 \) is 2-connected and \( H_0^* := (G/U_0) + x_0y_0 \) otherwise; 
(M3) \( U_0 \) is the disjoint union of two connected vertex subsets \( \Lambda_0 \) and \( \Omega_0 \) such that \( V(X_0) \subseteq \Lambda_0 \cup \{x_0\} \), \( V(Y_0) \subseteq \Omega_0 \cup \{y_0\} \), and \( N(V(H_0) - \{y_0\}) \subseteq \Lambda_0 \cup \{y_0\} \); and 
(M4) \( |H_0| \geq n/2 \) and the inequality \( \alpha(t)a^3 + \ell(X_0) + \ell(Y_0) \geq (a + 4(n - |H_0|))^3 \) holds for any \( a \geq \frac{n}{11} \).

We also say that \((H_0, x_0, y_0)\) is a **near-magic minor** of \((G, x, y)\) if (M1), (M2) and (M3) hold.

**Claim 7.1.** Let \( \mathcal{M} \) denote the set of all magic minors of \((G, x, y)\). Then \( \mathcal{M} \neq \emptyset \).

To justify this, let \( H_0 := G - x \), let \( y_0 := y \), and let \( x_0 \) be a neighbor of \( x \) other than \( y \). Then \( G - (V(H_0) - \{x_0, y_0\}) \) contains two vertex-disjoint paths \( X_0 := xx_0 \) and \( Y_0 := y_0 \). So (M1) holds. Clearly, \( U_0 := G - V(H_0) = \{x\} \) is connected. From the 3-connectivity of \( G \), we see that \( H_0 \) is 3-connected and \( H_0^* := G/U_0 = G \) is 3-connected. So (M2) holds. Setting \( \Lambda_0 = \{x\} \) and \( \Omega_0 = \emptyset \) yields (M3). Obviously, \(|H_0| = n - 1 \geq n/2 \). Moreover, for any \( a \geq \frac{n}{11} \), we have

\[
\alpha(t)a^3 + \ell(X_0) + \ell(Y_0) = \alpha(t)a^3 + 1 \geq \alpha(t)(a^3 + 1) \geq \alpha(t)(a + (b - 1))^3 \geq \alpha(t)(a + 4)^3,
\]

where the second inequality follows from Lemma 3.4. Note that \((a + 4(n - |H_0|))^3 = (a + 4)^3 \) for \(|H_0| = n - 1 \), so (M4) also holds. Therefore \((H_0, x_0, y_0) \in \mathcal{M} \), as claimed.

We reserve the triple \((H_0, x_0, y_0)\) for a magic minor in \( \mathcal{M} \) with smallest \(|H_0| \) hereafter. Now let us recursively define a sequence of near-magic minors of \((G, x, y)\) starting from \((H_0, x_0, y_0)\). (The construction of this sequence is quite complex. However, once it is understood, the remaining arguments are mostly easy consequences of this construction and previous claims.)

At a general step, suppose we have already had a near-magic minor \((H_i, x_i, y_i)\) of \((G, x, y, z)\) for some \( i \geq 0 \); that is,
(m0) $H_i$ is an induced subgraph of $G$ and $H_i + x_iy_i$ is 2-connected;

(m1) $G - (V(H_i) - \{x_i, y_i\})$ contains two vertex-disjoint paths $X_i, Y_i$ from $x_i, y_i$ to $x_i, y_i$, respectively;

(m2) $U_i := G - V(H_i)$ is connected and $H_i^*$ is 3-connected, where $H_i^* := G/U_i$ if $H_i$ is 2-connected and $H_i^* := (G/U_i) + x_iy_i$ otherwise;

(m3) $U_i$ is the disjoint union of two connected sets $\Lambda_i$ and $\Omega_i$ such that $V(X_i) \subseteq \Lambda_i \cup \{x_i\}$, $V(Y_i) \subseteq \Omega_i \cup \{y_i\}$, and $N(V(H_i) - \{y_i\}) \subseteq \Lambda_i \cup \{y_i\}$;

(m4) $|H_i| \geq n/2$.

Depending on whether or not $\{x_i, y_i\}$ is a cutset of $H_i$, we construct the following objects according to two rules (R1) and (R2):

- $(H_{i+1}, x_{i+1}, y_{i+1}), \Lambda_{i+1}$, and $\Omega_{i+1}$, where $H_{i+1}$ is a subgraph of $H_i$ and $x_{i+1}, y_{i+1} \in V(H_{i+1})$. Let $U_{i+1} := G - V(H_{i+1})$, let $H^*_{i+1} := G/U_{i+1}$ if $H_{i+1}$ is 2-connected and $H^*_{i+1} := (G/U_{i+1}) + x_{i+1}y_{i+1}$ otherwise, and let $u_{i+1}$ be the vertex of $H^*_{i+1}$ resulted from the contraction of $U_{i+1}$. $(U_{i+1}, u_{i+1})$, and $H^*_{i+1}$;

- $(H_{i+1,j}, x_{i+1,j}, y_{i+1,j})$ for $j = 1, 2, \ldots, s_{i+1}$, where $H_{i+1,j}$ is a subgraph of $H_i$ and $x_{i+1,j}, y_{i+1,j} \in V(H_{i+1,j})$. Let $U_{i+1,j} := G - V(H_{i+1,j})$, let $H^*_{i+1,j} := G/U_{i+1,j}$ if $H_{i+1,j}$ is 2-connected and $H^*_{i+1,j} := (G/U_{i+1,j}) + x_{i+1,j}y_{i+1,j}$ otherwise, and let $u_{i+1,j}$ be the vertex of $H^*_{i+1,j}$ resulted from the contraction of $U_{i+1,j}$; and

- $(F_{i+1,j}, x_{i+1,j}', y_{i+1,j}')$ for $j = 0, 1, \ldots, t_{i+1}$, where $F_{i+1,j}$ is a subgraph of $H_i$ and $x_{i+1,j}', y_{i+1,j}' \in V(F_{i+1,j})$. Let $W_{i+1,j} := G - F_{i+1,j}$, let $F^*_{i+1,j} := G/W_{i+1,j}$ if $F_{i+1,j}$ is 2-connected and $F^*_{i+1,j} := (G/W_{i+1,j}) + x_{i+1,j}'y_{i+1,j}'$ otherwise, and let $w_{i+1,j}$ be the vertex of $F^*_{i+1,j}$ resulted from the contraction of $W_{i+1,j}$.

In what follows, we set $\overline{\tau}(D) := \tau(G/(G - D))$ for each subgraph $D$ of $G$.

(R1) Suppose $\{x_i, y_i\}$ is a cutset of $H_i$. Let $B_i$ be an $\{x_i, y_i\}$-bridge of $H_i$ with largest size and set $H_{i+1} := G[B_i]$. Let $H_{i+1,j}, j = 1, 2, \ldots, s_{i+1}$, be all the nontrivial $\{x_i, y_i\}$-bridges of $H_i$ different from $B_i$, and let $x_{i+1,j} = x_i, y_{i+1,j} = y_i$, and $y_{i+1,j} = y_i$ for $1 \leq j \leq s_{i+1}$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$. In this case $(F_{i+1,j}, x_{i+1,j}', y_{i+1,j}')$, $(W_{i+1,j}, w_{i+1,j})$, and $F^*_{i+1,j}$ for $j = 0, 1, \ldots, t_{i+1}$ are all set to $\emptyset$.

(R2) Suppose $\{x_i, y_i\}$ is not a cutset of $H_i$. Let $B_i$ be the unique block of $H_i - x_i$ containing $y_i$, and let $\hat{B}_i$ be the union of all nontrivial $(x_i, B_i)$-bridges of $H_i$, if any, and be a trivial such bridge otherwise.

If there exists some $(x_i, B_i)$-bridge $B_{i,x}$ in $\hat{B}_i$ with $|B_{i,x}| \geq |\hat{B}_i|/4$, we choose such $B_{i,x}$ with largest size; otherwise, there exists some $(x_i, B_i)$-bridge $B_{i,x}$ in $\hat{B}_i$ with $\overline{\tau}(B_{i,x}) < t$ (see Claim 7.10); we choose such $B_{i,x}$ with largest size. Let $\{z_i\} = V(B_i) \cap V(B_{i,x})$.

If there exists some $(y_i, z_i)$-bridge $B_{i,y}$ of $B_i$ with $|B_{i,y}| \geq |B_i|/4$, we choose such $B_{i,y}$ with largest size; otherwise, there exists some $(z_i, y_i)$-bridge $B_{i,y}$ of $B_i$ with $\overline{\tau}(B_{i,y}) < t$ (see Claim 7.10); we choose such $B_{i,y}$ with largest size.

Depending on the sizes of $B_{i,x}$ and $B_{i,y}$, we distinguish between two cases:
(1) $|B_{i,x}| \geq |B_{i,y}|$. In this case, let $H_{i+1} := G[B_{i,x}]$ and set $x_{i+1} := x_i$ and $y_{i+1} := z_i$. Let $H_{i+1,j}, j = 1, 2, \ldots, s_{i+1}$, be all the nontrivial $(x_i, B_i)$-bridges of $H_i$ different from $H_{i+1}$, and let $\{y_{i+1,j}\} := V(H_{i+1}) \cap V(B_i)$. Let $F_{i+1,0} = G[B_{i,y}]$ and set $x'_{i+1} := z_i$ and $y'_{i+1} := y_i$. Let $F_{i+1,j}, j = 1, 2, \ldots, t_{i+1}$, be all the nontrivial $\{y_i, z_i\}$-bridges of $B_i$ different from $F_{i+1,0}$ (the only trivial $\{y_i, z_i\}$-bridge is the edge $y_iz_i$, if any), and set $x'_{i+1,j} := z_i$ and $y'_{i+1,j} := y_i$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - (V(H_{i+1}) \cup V(F_{i+1,0})))$ and $\Omega_{i+1} := \Omega_i \cup V(F_{i+1,0} - y_{i+1})$.

(2) $|B_{i,x}| < |B_{i,y}|$. In this case, let $H_{i+1} := G[B_{i,y}]$ and set $x_{i+1} := z_i$ and $y_{i+1} := y_i$. Let $H_{i+1,j}, j = 1, 2, \ldots, s_{i+1}$, be all the nontrivial $\{z_i, y_i\}$-bridges of $B_i$ different from $H_{i+1}$, and set $y_{i+1,j} := y_i$. Let $F_{i+1,0} := G[B_{i,x}]$, let $x_{i+1} := x_i$, and let $y'_{i+1} := z_i$. Let $F_{i+1,j}, j = 1, 2, \ldots, t_{i+1}$, be all the nontrivial $(x_i, B_i)$-bridges of $H_i$ different from $F_{i+1,0}$, and let $\{y'_{i+1,j}\} = V(F_{i+1}) \cap V(B_i)$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$.

We shall verify that $(H_{i+1}, x_{i+1}, y_{i+1})$ is a near-magic minor of $(H, x, y)$. We terminate this construction process when $|H_{i+1}| < n/2$ or when $\sum_{p=1}^i \sum_{j=1}^{s_{i+1}} |H_{p,j}| > (n - |H_i|)/2$.

From the construction process we see that

$$|H_i| \leq |H_{i+1}| + |\bigcup_{j=1}^{s_{i+1}} H_{i+1,j}| + |\bigcup_{j=0}^{t_{i+1}} F_{i+1,j}|.$$  \hspace{1cm} (7.1)

Let us exhibit some additional properties enjoyed by the objects constructed above.

**Claim 7.2.** The graphs $U_{i+1}, U_{i+1,j}$ and $W_{i+1,j}$ are all connected. Both $H_{i+1}$ and $F_{i+1,0}$ are induced subgraphs of $G$. The graph $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected. The graphs $H_{i+1}^*, H_{i+1,j}^*$ and $F_{i+1,j}^*$ are all 3-connected. \(\{u_{i+1}x_{i+1}, u_{i+1}y_{i+1}\} \subseteq E(H_{i+1}^*)\), \(\{u_{i+1,j}x_{i+1}, u_{i+1,j}y_{i+1,j}\} \subseteq E(H_{i+1,j}^*)\), and \(\{w_{i+1}x'_{i+1}, w_{i+1}y'_{i+1}\} \subseteq E(F_{i+1,0}^*)\). Moreover, $U_{i+1}$ is the disjoint union of $\Lambda_{i+1}$ and $\Omega_{i+1}$, both $G[\Lambda_{i+1}]$ and $G[\Omega_{i+1}]$ are connected, and $N(V(H_{i+1}) - \{y_{i+1}\}) \subseteq \Lambda_{i+1} \cup \{y_{i+1}\}$. In particular, $G - (V(H_{i+1}) - \{x_{i+1}, y_{i+1}\})$ contains two vertex-disjoint paths from $x, y$ to $x_{i+1}, y_{i+1}$, respectively.

Indeed, since $G$ is 3-connected and $U_i$ is connected (by (m2)), it follows from (R1) and (R2) that $U_{i+1}, U_{i+1,j}$, and $W_{i+1,j}$ are all connected. Since $H_i$ is an induced subgraph of $G$ (by (m0)), from (R1) and (R2) we deduce that both $H_{i+1}$ and $F_{i+1,0}$ are induced subgraphs of $G$. Since $|H_i| \geq n/2 \geq b^{(t-1)}/2$ (by (m4)) and $H_i + x_iy_i$ is 2-connected (by (m9)), it is easy to see that $|H_{i+1}| \geq 3$ and $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected. If $H_{i+1}$ is 2-connected, then $H_{i+1}^*$ is 3-connected by Lemma 3.3. If $H_{i+1}$ is not 2-connected then, since $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected, $H_{i+1}^* = (G[U_{i+1}] + x_{i+1}y_{i+1} = (G + x_{i+1}y_{i+1})/U_{i+1}$ is again 3-connected. Similarly, we can show that both $F_{i+1,0}^*$ (if nonempty) and $H_{i+1,j}^*$ are 3-connected. The properties enjoyed by $\Lambda_{i+1}$ and $\Omega_{i+1}$ follow instantly from (m3) and the construction of $\Lambda_{i+1}$ and $\Omega_{i+1}$. The rest of Claim 7.2 are implied by the definitions of $H_{i+1}^*, H_{i+1,j}^*$ and $F_{i+1,0}^*$ in (R1) or (R2).

The next two claims follow instantly from (R1) and (R2).

**Claim 7.3.** There exist two vertex-disjoint paths in $H_i - (V(H_{i+1}) - \{x_{i+1}, y_{i+1}\})$ from $x_{i+1}, y_{i+1}$ to $x_i, y_i$, respectively. For each $j$ with $1 \leq j \leq s_{i+1}$, there exist two vertex-disjoint paths in $H_i - (V(H_{i+1,j}) - \{x_{i+1}, y_{i+1,j}\})$ from $x_{i+1}, y_{i+1,j}$ to $x_i, y_i$, respectively. Moreover, for each $j$ with $1 \leq j \leq t_{i+1}$, if $F_{i+1,j}$ is defined, then there exist two vertex-disjoint paths in $H_i - (V(F_{i+1,j}) - \{x'_{i+1,j}, y'_{i+1,j}\})$ from $x'_{i+1,j}, y'_{i+1,j}$ to $x_i, y_i$, respectively. Hence, by Claim 7.2, $(H_{i+1}, x_{i+1}, y_{i+1})$ is a near-magic minor of $(G, x, y)$.
Claim 7.4. For (R2), either \(x_{i+1} = x_i, \ y_{i+1} = z_i = x'_{i+1}, \ y'_{i+1} = y_i\); or \(x_{i+1} = z_i = y'_{i+1}, \ y_{i+1} = y_i\), and \(x'_{i+1} = x_i\). Moreover, \(H_{i+1} \cap F_{i+1} = \{z_i\}, |V(H_{i+1,j}) \cap V(F_{i+1})| \leq 1, V(H_{i+1,j}) \cap V(F_{i+1}) \subseteq \{y'_{i+1}, y_{i+1,j}\}\), and \(H_{i+1} = \{x_{i+1}, y_{i+1}\}\) and \(H_{i+1,j} = \{x_{i+1}, y_{i+1,j}\}\), for all \(j = 1, 2, \ldots, s_{i+1}\), are pairwise vertex-disjoint.

Claim 7.5. Let \(D\) be an induced subgraph of \(G\) that is a chain of blocks \(v_0 D_1 v_1 D_2 v_2 \ldots v_m D_m v_{m+1}\). Suppose \(G - D\) and \(G - D_k\) are connected for all \(k = 1, 2, \ldots, m\). Then \(\tau(D) = \max_{1 \leq k \leq m} \tau(D_k)\).

It is obvious that any \(K_{3,p}\)-minor in \(G/(G - D_k)\) yields a \(K_{3,p}\)-minor in \(G/(G - D)\). So \(\tau(D) \geq \max_{1 \leq k \leq m} \tau(D_k)\).

Conversely, suppose \(V_1, V_2, \ldots, V_{p+3}\) form a representation of a \(K_{3,p}\)-minor in \(G/(G - D)\). Let \(u\) denote the vertex resulted from the contraction of \(G - D\).

If \(u \notin V_i\) for any \(i \in \{1, 2, \ldots, p + 3\}\) then, by 3-connectedness of \(K_{3,p}\), there exists \(k \in \{1, 2, \ldots, m\}\) such that \(V' := V_j \cap V(D_k)\) are all connected. Clearly, \(V_1, V_2, \ldots, V_{p+3}\) form a representation of a \(K_{3,p}\)-minor in \(G/(G - D_k)\). So \(\tau(D) \leq \max_{1 \leq k \leq m} \tau(D_k)\).

If \(u \in V_i\) for some \(i \in \{1, 2, \ldots, p + 3\}\) then, by 3-connectedness of \(K_{3,p}\), there exists \(k \in \{1, 2, \ldots, m\}\) such that \(V_i := V_j \cap V(D_k)\), with \(j \neq i\), are all connected. Let \(V_i := V(G) \setminus V(D_k)\). Clearly, \(V_i, V_2, \ldots, V_{p+3}\) form a representation of a \(K_{3,p}\)-minor in \(G/(G - D_k)\). So \(\tau(D) \leq \max_{1 \leq k \leq m} \tau(D_k)\).

Thus the claim is justified.

Claim 7.6. If \(H_{i+1}^*\) is not a minor of \(G\), then \(s_{i+1} = 0\) (which means that there is no \(H_{i+1,j}\)). All \(H_{i+1,j}^*\), and all \(F_{i+1,j}^*\) with \(j \geq 1\) are minors of \(G\). If \(F_{i+1,j}^*\) is not a minor of \(G\), then \(t_{i+1} = 0\). There exists a path \(P_{i+1}\) (resp. \(P_{i+1,j}\), and \(R_{i+1,j}\)) in \(H_{i+1}\) (resp. \(H_{i+1,j}\), and \(F_{i+1,j}\)) connecting \(x_{i+1}\) and \(y_{i+1}\) (resp. \(x_{i+1} + y_{i+1}\), and \(x'_{i+1} + y'_{i+1}\) ) such that

\[
\begin{align*}
\ell(P_{i+1}) & \geq \alpha(\tau(H_{i+1}))/|H_{i+1}/27|, \\
\ell(P_{i+1,j}) & \geq \alpha(\tau(H_{i+1,j}))/|H_{i+1,j}/27|, \text{ and} \\
\ell(R_{i+1,j}) & \geq \alpha(\tau(F_{i+1,j}))/|F_{i+1,j}/27|.
\end{align*}
\]

Clearly, \(H_{i+1}\) is a minor of \(G\) if \(H_{i+1} = G/U_{i+1}\). It remains to consider the case when \(H_{i+1} = G/U_{i+1} + x_{i+1} + y_{i+1}\). If \(s_{i+1} > 0\) then \(H_{i+1}\) exists; in this case, \(H_{i+1}\) can be obtained from \(G\) by contracting to \(y_{i+1}\) the graph \(H_{i+1} = \{x_{i+1}\}\), and by contracting \(U_{i+1} = V(H_{i+1}) - \{x_{i+1}, y_{i+1}\}\) (which is connected since it contains \(U_i\) and all \(H_{i+1,j}\) and \(F_{i+1,j}\) (if nonempty) have neighbors in \(U_i\)).

So \(H_{i+1}\) is again a minor of \(G\).

Similarly, if \(F_{i+1,j}^*\) is not a minor of \(G\), then \(t_{i+1} = 0\); and all \(H_{i+1,j}^*\) and all \(F_{i+1,j}^*\) with \(j \geq 1\) are minors of \(G\).

To show the existence of the desired path \(P_{i+1}\), note that \(H_{i+1}\) is a chain of blocks \(v_0 D_0 v_1 D_1 v_2 \ldots v_m D_m v_{m+1}\), with \(v_0 = x_{i+1}\) and \(v_{m+1} = y_{i+1}\). By Lemma 3.3, \(G/(G - D_k)\) is either 3-connected or a triangle for \(k = 1, 2, \ldots, m\). In the former case Lemma 5.1 guarantees the existence of a \(u_k\)-path \(R_k\) in \(D_k\) with \(\ell(R_k) \geq \alpha(\tau(H_{i+1}))/|D_k|/27| \geq \alpha(\tau(H_{i+1}))/|D_k|/27|\), where the first inequality follows from Lemma 7.5 and the second from Lemma 2.4(iii); in the latter case this statement holds trivially. Concatenating all these \(R_k\), we obtain an \(x_{i+1}y_{i+1}\) path \(P_{i+1}\) in \(H_{i+1}\) with

\[
\ell(P_{i+1}) \geq \alpha(\tau(H_{i+1}))/\sum_{k=0}^m |D_k|/27| \geq \alpha(\tau(H_{i+1}))/\sum_{k=0}^m |D_k|/27| \geq \alpha(\tau(H_{i+1}))/|H_{i+1}/27|\alpha(\tau(H_{i+1}))/|H_{i+1}/27|,
\]
where the second inequality follows from Corollary 3.6.

The existence of $P_{t+1,j}$ and $R_{t+1,j}$ can be justified likewise. This establishes the claim.

**Claim 7.7.** (i) $\tau(G) \geq \lceil \tau(H_{t+1})/3 \rceil + \sum_{j=0}^{t+1} \lceil \tau(F_{t+1,j})/3 \rceil$ and (ii) $\tau(G) \geq \sum_{j=0}^{t+1} \lceil \tau(F_{t+1,j})/3 \rceil$.

We only prove the first inequality as the second one can be established similarly.

Let us first show that $H_{t+1}'$ contains a $K_{3,\lceil \tau(H_{t+1})/3 \rceil}$-minor $\Sigma_{t+1}$ rooted at $\{x_{t+1}, y_{t+1}, u_{t+1}\}$. For this purpose, note that $H_{t+1}$ is a chain of blocks $v_0 D_0 v_1 D_1 v_2 \ldots v_m D_m v_{m+1}$, with $v_0 = x_{t+1}$ and $v_{m+1} = y_{t+1}$. Statement (i) holds trivially if $G[H_{t+1}]$ is a path (which implies $\tau(H_{t+1}) = 1$). Thus, by Claim 7.5, we may assume the existence of a nontrivial block $D_k$ of $H_{t+1}$ such that $\tau(H_{t+1}) = \tau(G/(G-D_k))$. In view of Lemma 2.1, $G/(G-D_k)$ contains a $K_{3,\lceil \tau(H_{t+1})/3 \rceil}$-minor rooted at $v_k, v_{k+1}$, and the vertex resulted from contracting $G - V(D_k)$. Clearly, this minor leads to a $K_{3,\lceil \tau(H_{t+1})/3 \rceil}$-minor $\Sigma_{t+1}$ of $H_{t+1}'$ rooted at $\{x_{t+1}, y_{t+1}, u_{t+1}\}$, as desired.

Similarly, $H_{t+1}'$ contains a $K_{3,\lceil \tau(H_{t+1})/3 \rceil}$-minor $\Sigma_{t+1,j}$ rooted at $\{x_{t+1}, y_{t+1,j}, u_{t+1,j}\}$. By (R1) and (R2), the neighbors of $H_{t+1} - \{x_{t+1}, y_{t+1}\}$ and those of $H_{t+1,j} - \{x_{t+1}, y_{t+1,j}\}$, for $1 \leq j \leq s_{t+1}$, are all in $U_i = G - V(H_i)$ if $H_{t+1}$ is defined in (R1) or $H_{t+1} = B_{i,x}$ in (R2), and all in $U_i = G - V(H_i)$ and $\bigcup_{j=1}^{s_{t+1}} F_{t+1,k} - z_i$ otherwise. By (m2), $U_i$ is connected. So we can merge the above $\Sigma_{t+1}$ and $\Sigma_{t+1,j}$ to form a $K_{3,\lceil \tau(H_{t+1})/3 \rceil}$-minor $G_{i} \geq \sum_{j=0}^{t+1} \lceil \tau(H_{t+1,j})/3 \rceil$. Hence (i) follows.

Clearly, $1 \leq \tau(H_{t+1,j}) \leq \tau(G) \leq t$ for $1 \leq j \leq s_{t+1}$. Similarly, $1 \leq \tau(F_{t+1,j}) \leq \tau(G) \leq t$ for $0 \leq j \leq t_{t+1}$. From Claim 7.7 we see that

**Claim 7.8.** $s_{t+1} \leq t - 1$ and $t_{t+1} \leq t - 1$.

Set $H_{t+1,0} = H_{t+1}$. Throughout the remainder of our proof, $s_{t+1}^*$ stands for the number of bridges $H_{t+1,j}$, with $0 \leq j \leq s_{t+1}$, satisfying $\tau(H_{t+1,j}) = t$, and $t_{t+1}^*$ stands for the number of bridges $F_{t+1,j}$, with $0 \leq j \leq t_{t+1}$, satisfying $\tau(F_{t+1,j}) = t$.

**Claim 7.9.** $s_{t+1}^* \leq 3$ and equality holds only if $s_{t+1} = 2$; $t_{t+1}^* \leq 3$ and equality holds only if $t_{t+1} = 2$.

From Claim 7.7(i) we deduce that $t \geq s_{t+1}^*/3 + (s_{t+1} + 1 - s_{t+1}^*)$, so the first part of our claim follows. The second part can be justified likewise.

**Claim 7.10.** $B_{i,x}$ and $B_{i,y}$ in (R2) are well defined. Moreover, $|B_{i,x}| \geq |B_i|/(2t)$ and $|B_{i,y}| \geq |B_i|/(2t)$.

We only prove the statements for $B_{i,x}$ as the proof for $B_{i,y}$ goes along the same line.

We may assume that all $(x_i, B_i)$-bridges $B$ in $B_i$ satisfy $|B| < |B_i|/4$, for otherwise, according to (R2) we choose $B_i$ to be one with $|B_{i,x}|$ maximum. This implies $|B_{i,x}| \geq |B_i|/4 \geq |B_i|/(2t)$ (as $t \geq 2$).

So there is an $(x_i, B_i)$-bridges $B$ in $B_i$ with $\tau(B) < t$, for otherwise, all such $B$ satisfy $\tau(B) = t$. Since $s_{t+1}^* \leq 3$ (by Claim 7.9), there exists an $(x_i, B_i)$-bridge $B$ in $B_i$ such that $|B| \geq |B_i|/3 \geq |B_i|/4$, a contradiction.

By Claim 7.9, $s_{t+1}^* \leq 2$; that is, the number of $(x_i, B_i)$-bridges $B$ in $B_i$ with $\tau(B) = t$ is at most two. For such $B$, the definition of $B_{i,x}$ (see (R2)) implies that $|B| < |B_i|/4$. Hence, using Claim 7.8, we get $|B_{i,x}| \geq (|B_i| - 2|B_i|/4)/t = |B_i|/(2t)$, as desired.
Claim 7.11. We may assume that the following three statements hold:

(i) if $\bar{\tau}(H_{i+1}) < t$, then $|H_{i+1}| < |H_i|/(8t^2)$,

(ii) if $\bar{\tau}(H_{i+1,j}) < t$, then $|H_{i+1,j}| < |H_i|/(8t^2)$, and

(iii) if $\bar{\tau}(F_{i+1,j}) < t$, then $|F_{i+1,j}| < |H_i|/(8t^2)$.

We may assume that the following three statements hold:

(i) $|H_{i+1}| \geq n/(16t^2)$ by (m4). Hence the $x_{i+1}-y_{i+1}$ path exhibited in Claim 7.6 has length at least

$$\alpha(t-1) \left( \frac{|H_{i+1}|}{27} \right)^{\beta} \geq \alpha(t-1) \left( \frac{n}{432t^2} \right)^{\beta} = \alpha(t) \left( \frac{2^{t-1}n}{432t^2} \right)^{\beta} \geq \alpha(t) n^\beta,$$

where the equality follows from (5.1). (Observe that in the last inequality, we need $t \geq 2$; and when $t = 2$ we need $b = 1729$ as $432t^2 = 1728 = b - 1$.) Clearly, $P_{i+1}$ can be extended to an $x-y$ path in $G$ with length at least $\alpha(t) n^\beta$.

Claim 7.12. (i) $\bar{\tau}(H_{i+1}) = t$, and (ii) $|H_{i+1}| \geq |H_{i+1,j}|$ for $j = 1, 2, \ldots, s_{i+1}$.

If (R1) applies, then $|H_{i+1}| \geq |H_i|/t$. If (R2) applies, then $|H_{i+1}| = \max\{|B_{i,x}|, |B_{i,y}|\} \geq (|B_{i,x}| + |B_{i,y}|)/2 \geq |H_i|/(2t)$ by Claim 7.10. It follows from Claim 7.11(i) that $\bar{\tau}(H_{i+1}) = t$. By (i) and (R2), we get (ii) immediately.

Claim 7.13. $|H_{i+1}| \geq |F_{i+1,j}|$ for $j = 0, 1, \ldots, t_{i+1}$.

To justify this, recall (R2); we only prove the statement for the case when $|B_{i,x}| \geq |B_{i,y}|$ as the proof for the other case goes along the same line.

If $|B_{i,j}| \geq |B_{i}|/4$ then according to (R2), $|F_{i+1,0}| \geq |F_{i+1,j}|$ for $j = 1, 2, \ldots, t_{i+1}$, and hence $|H_{i+1}| = |B_{i,x}| \geq |B_{i,y}| = |F_{i+1,0}|$. So we assume that $|B_{i,y}| < |B_{i}|/4$. Then, by (R2), $\bar{\tau}(F_{i+1,0}) < t$ and $|F_{i+1,j}| < |B_{i}|/4$ for $0 \leq j \leq t_{i+1}$. By Claim 7.9, we have $t_{i+1} \leq 3$. From Claim 7.8 and Claim 7.11(iii), it follows that $|B_{i}| \leq \sum_{j=0}^{t_{i+1}} |F_{i+1,j}|/4 + t|H_i|/(8t^2) \leq 3|B_{i}|/4 + t|H_i|/(8t^2)$, implying $|B_{i}| \leq |H_i|/(2t)$. Hence, by Claim 7.8 and Claim 7.12(ii), $|H_{i+1}| \geq |B_{i}|/t \geq |H_i - B_{i}|/t \geq (1 - 1/(2t))|H_i|/t \geq |H_i|/(2t) \geq |B_{i}|$, which yields the statement as desired.

Claim 7.14. (i) $|H_{i+1}| \geq |H_i|/8 \geq n/16$, (ii) $|H_{i+1}| \geq \bigcup_{j=0}^{t_{i+1}} F_{i+1,j}/4$, and (iii) $|F_{i+1,0}| \geq \bigcup_{j=0}^{t_{i+1}} F_{i+1,j}/4$ if $\bar{\tau}(F_{i+1,0}) = t$.

To justify (i), we appeal to inequality (7.1). In view of Claim 7.8, Claim 7.11, Claim 7.12(ii), and Claim 7.13, we obtain

$$|H_{i+1}| \leq (s_{i+1} + t_{i+1}^*)|H_{i+1}| + 2t|H_i|/8t^2 \leq 6|H_{i+1}| + |H_i|/4t,$$

where the second inequality follows from Claim 7.9. So $|H_{i+1}| \geq (4t - 1)|H_i|/(24t) \geq |H_i|/8$ as $t \geq 2$. By (m4), $|H_i| \geq n/2$. Hence inequality (i) is established.

By Claim 7.9, $t_{i+1}^* \leq 3$. From Claim 7.11(iii) we see that $\bigcup_{j=0}^{t_{i+1}^*} F_{i+1,j} \leq 3|H_{i+1}| + t|H_i|/(8t^2) \leq 3|H_{i+1}| + |H_{i+1}|/t \leq 4|H_{i+1}|$, where the second inequality follows from (i). So inequality (ii) is also proved.

Inequality (iii) follows instantly from the construction rule (R2).
Claim 7.15. Let $P_{i+1}$ and $R_{i+1} := R_{i+1,0}$ be the paths as described in Claim 7.6. Then

(i) $\ell(P_{i+1}) \geq \alpha(t)(|H_i|/216)^\beta$ and

(ii) $\ell(R_{i+1}) \geq \alpha(t)(|\bigcup_{j=0}^{k+1} F_{i+1,j}|/108)^\beta$.

As in Claim 7.6, $\ell(P_{i+1}) \geq \alpha(t)(|H_{i+1}|)/27)^\beta$. By Claim 7.12 and Claim 7.14(i), we have $\tau(H_{i+1}) = t$ and $|H_{i+1}| \geq |H_i|/8$. So (i) follows instantly.

By Claim 7.6, $\ell(R_{i+1}) \geq \alpha(t)(|F_{i+1,0}|)/27\beta$. Clearly, $\tau(F_{i+1,0}) \leq \tau(G) \leq t$. If $\tau(F_{i+1,0}) = t$, then (ii) follows from Claim 7.14(ii). So we assume that $\tau(F_{i+1,0}) \leq t - 1$. Thus

$$\ell(R_{i+1}) \geq \alpha(t - 1)(|F_{i+1,0}|)/27\beta = \alpha(t)\left(t^2(t-1)|F_{i+1,0}|/27\right)^\beta,$$

where the equality follows from (5.1). By Claim 7.10, $|F_{i+1,0}| \geq |\bigcup_{j=0}^{k+1} F_{i+1,j}|/(2t)$. Plugging this into the above inequality, we get (ii). So the claim is justified.

Suppose $\{(H_i, x_i, y_i) : i = 0, 1, \ldots, k\}$ is a maximal sequence of near-magic minors recursively constructed from $(H_0, x_0, y_0)$ according to (R1) and (R2), subject to the following two constraints:

(S1) $|H_k| \geq \frac{n}{2}$, and

(S2) For each $s$ with $1 \leq s \leq k$,

$$\sum_{i=1}^{s} \sum_{j=1}^{s_j} |H_{i,j}| \leq \frac{1}{2}(n - |H_s|).$$

Starting from $(H_k, x_k, y_k)$ and using (R1) and (R2), we can still construct

- $(H_{k+1,1}, x_{k+1,1}, y_{k+1,1})$ and $(H_{k+1,j}, x_{k+1,j}, y_{k+1,j})$ for $j = 1, 2, \ldots, s_{k+1},$
- $(F_{k+1,j}, x_{k+1,j}, y_{k+1,j})$ for $j = 1, 2, \ldots, t_{k+1},$ and
- $U_{k+1,1}, W_{k+1,1}, H_{k+1}, F_{k+1}, u_{k+1}, w_{k+1}, \Lambda_{k+1},$ and $\Omega_{k+1}.$

Since $(H_0, x_0, y_0)$ is a magic minor of $(G, x, y)$, from Claim 7.2 and Claim 7.3 we deduce that $(H_i, x_i, y_i)$ is a near-magic minor of $(G, x, y)$ for $1 \leq i \leq k+1$. Thereby, Claim 7.2 - Claim 7.15 hold for $i = 1, 2, \ldots, k+1$.

Note that, for $s = 1, 2, \ldots, k+1$, the vertices of $G$ outside $H_s$ is either outside $H_0$, or in $H_{i,j}$ for some pair $i, j$ with $1 \leq i \leq s$ and $1 \leq j \leq s_i$, or in $F_{i,j}$ for some pair $i, j$ with $1 \leq i \leq s$ and $0 \leq j \leq t_i$. Since $n - |H_s|$ is the number of vertices of $G$ outside $H_s$, and $n - |H_0|$ is the number of vertices of $G$ outside $H_0$, we have

Claim 7.16. For any $s$ with $1 \leq s \leq k+1$,

$$|\bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} H_{i,j}| + |\bigcup_{i=1}^{s} \bigcup_{j=0}^{t_i} F_{i,j}| + (n - |H_0|) \geq n - |H_k|.$$

By Claim 7.3, $(H_{k+1,1}, x_{k+1,1}, y_{k+1,1})$ is a near-magic minor of $(G, x, y)$. The maximality on $k$ implies

Claim 7.17. Either $|H_{k+1}| < n/2$, or $|H_{k+1}| \geq n/2$ and $\sum_{i=1}^{s_{k+1}} \sum_{j=1}^{s_i} |H_{i,j}| > \frac{1}{2}(n - |H_{k+1}|)$. 

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By (7.1), Claim 7.11 and Claim 7.9, we have \(|H_k| \leq 3|H_{k+1}| + t|H_k|/(8t^2) + |\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}|; so |
|H_{k+1}| + |\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}|/3 \geq (1 - \frac{1}{8t})|H_k|/3. Since \(t \geq 2\) and \(|H_k| \geq \frac{n}{2}\) by (S1), we obtain

Claim 7.18. \(|H_{k+1}| + |\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}|/3 \geq |H_k|/4 \geq n/8.

Since \(n \geq |\bigcup_{j=0}^{t_{k+1}} F_{i,j}|\) for all \(i\), from Claim 7.18 the following statement follows.

Claim 7.19. \(|H_{k+1}| + 4|\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}| \geq \frac{1}{276} \big| \bigcup_{j=0}^{t_{k+1}} F_{i,j}\big|\) for any \(i\) with \(1 \leq i \leq k\).

Let \(P_{k+1}\) be the \(x_{k+1}y_{k+1}\) path in \(H_{k+1}\) and let \(R_i := R_{i,0}\) be the \(x_i'\)-\(y_i'\) path in \(F_i,0\), for \(1 \leq i \leq k+1\), as exhibited in Claim 7.6 (with \(R_i = \emptyset\) when \(F_i,0\) is empty). Set \(Q_{k+1} := P_{k+1}\). Then

\[\ell(Q_{k+1}) \geq \alpha(t) \left(\frac{|H_{k+1}|}{27}\right)^\beta. \tag{7.2}\]

By Claim 7.15(ii) (or trivially when \(R_0 = \emptyset\)), we have

\[\ell(R_i) \geq \alpha(t) \left(\frac{\big| \bigcup_{j=0}^{t_{k+1}} F_{i,j}\big|}{108}\right)^\beta. \tag{7.3}\]

In view of Claim 7.4, there exists an \(x_ky_k\) path \(Q_k\) in \(H_k\) such that \(Q_k \supseteq Q_{k+1} \cup R_{k+1}\). Note that

\[\ell(Q_k) \geq \ell(Q_{k+1}) + \ell(R_{k+1}) \geq \alpha(t) \left(\frac{|H_{k+1}|}{27}\right)^\beta + \alpha(t) \left(\frac{\big| \bigcup_{j=0}^{t_{k+1}} F_{k+1,j}\big|}{108}\right)^\beta \quad \text{(by (7.2) and (7.3))}\]

\[\geq \alpha(t) \left(\frac{|H_{k+1}| + b - 1}{276} \big| \bigcup_{j=0}^{t_{k+1}} F_{k+1,j}\big|\right)^\beta \quad \text{(by Claim 7.14(ii) and Lemma 3.4)}\]

\[\geq \alpha(t) \left(\frac{|H_{k+1}| + 4| \bigcup_{j=0}^{t_{k+1}} F_{k+1,j}\big|}{108}\right)^\beta \quad \text{(for \(b - 1 = 1728\)).}\]

Similarly, let \(Q_{k-1}\) be an \(x_{k-1}y_{k-1}\) path in \(H_{k-1}\) such that \(Q_{k-1} \supseteq Q_k \cup R_k\). Then

\[\ell(Q_{k-1}) \geq \ell(Q_k) + \ell(R_k) \geq \alpha(t) \left(\frac{|H_{k+1}| + 4| \bigcup_{j=0}^{t_{k+1}} F_{k+1,j}\big|}{276}\right)^\beta \quad \text{(by (7.3))}\]

\[\geq \alpha(t) \left(\frac{|H_{k+1}| + 4| \bigcup_{j=0}^{t_{k+1}} F_{k+1,j}\big| + \frac{b - 1}{216} \big| \bigcup_{j=0}^{t_{k+1}} F_{k,j}\big|}{276}\right)^\beta \quad \text{(by Claim 7.19 and Lemma 3.4)}\]

\[\geq \alpha(t) \left(\frac{|H_{k+1}| + 4| \bigcup_{j=0}^{t_{k+1}} F_{i,j}\big|}{216}\right)^\beta \quad \text{(for \(b - 1 = 1728\)).}\]
Using Claim 7.19 and continuing in this fashion, we obtain an \(x_0-y_0\) path \(Q_0\) in \(H_0\) such that \(Q_0 \supseteq Q_{k+1} \cup (\bigcup_{i=1}^{k+1} R_i)\) and that
\[
\ell(Q_0) \geq \alpha(t) \left( \frac{|H_{k+1}|}{27} + 4\left| \bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j} \right| \right)^{\beta}.
\]

Let \(P := X_0 \cup Q_0 \cup Y_0\) (see (M1) for the definitions of \(X_0\) and \(Y_0\)). Then \(P\) is an \(x-y\) path in \(G\) with
\[
\ell(P) = \ell(Q_0) + \ell(X_0) + \ell(Y_0)
\]
\[
\geq \alpha(t) \left( \frac{|H_{k+1}|}{27} + 4\left| \bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j} \right| \right)^{\beta} + \ell(X_0) + \ell(Y_0)
\]
\[
\geq \alpha(t) \left( \frac{|H_{k+1}|}{27} + 4\left| \bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j} \right| + 4(n - |H_0|) \right)^{\beta},
\]
where the last inequality follows from (M4) because, by Claim 7.14(i), \(\frac{|H_{k+1}|}{27} \geq \frac{n}{272}\).

**Claim 7.20.** We may assume that \(\sum_{i=1}^{k+1} \sum_{j=1}^{r_i} |H_{i,j}| \geq \frac{1}{2} (n - |H_{k+1}|)\).

Suppose, on the contrary, that \(\sum_{i=1}^{k+1} \sum_{j=1}^{r_i} |H_{i,j}| \leq \frac{1}{2} (n - |H_{k+1}|)\). Then, by Claim 7.17, we have \(|H_{k+1}| < n/2\). From Claim 7.16 it can be seen that
\[
\left| \bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j} \right| + (n - |H_0|) \geq n - |H_{k+1}| - \sum_{i=1}^{k+1} \sum_{j=1}^{r_i} |H_{i,j}| \geq \frac{1}{2} (n - |H_{k+1}|).
\]

Hence
\[
\ell(P) \geq \alpha(t) \left( \frac{|H_{k+1}|}{27} + 4\left| \bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j} \right| + 4(n - |H_0|) \right)^{\beta} \geq \alpha(t)(2(n - |H_{k+1}|))^{\beta} \geq \alpha(t)n^{\beta}.
\]

In view of Claim 7.20, we use \(p\) to denote the smallest integer \(i\), with \(0 \leq i \leq k\), such that \(H_{i+1,1}\) exists. Set \(H := H_p\) if (1) of (R2), with \(i = p\), occurs; and set \(H := B_p\) if (R1) or (2) of (R2) occurs for \(i = p\). We call \(B_i\) light if (1) of (R2) occurs for \(i\). Let \(\tilde{H}\) be the graph obtained from \(G\) by first contracting each light \(B_i\) to a single vertex for \(i = p, p+1, \ldots, k\) (in that order), then contracting \(H_{k+1} - x_{k+1}\) to \(y_{k+1}\), and finally contracting \(V(G) - V(\tilde{H})\) to a single vertex \(\tilde{w}\).

**Claim 7.21.** The following statements hold for \(\tilde{H}\):

(i) All the vertices \(y_i\) and \(y_{i,j}\), for \(p+1 \leq i \leq k+1\) and \(1 \leq j \leq s_i\), are contracted into \(y_p\) in \(\tilde{H}\);

(ii) All \(H_{i,j}\) remain intact in \(\tilde{H}\), for \(p+1 \leq i \leq k+1\) and \(1 \leq j \leq s_i\);

(iii) \(\tilde{H}\) is 3-connected.
To justify the claim, for each $q$ with $p \leq q \leq k$, let $D_{q+1}$ be the graph obtained from $H$ by contracting light $B_i$ to a single vertex for $i = p, p+1, \ldots, q$ (in that order). Set $\tilde{x} := x_p$ if $H = H_p$, and $\tilde{x} := x_{p+1}$ if $H = B_p$; and set $\tilde{y} := y_p$. Since $G$ is 3-connected and $H_{p+1}$ exists, from the contraction process of light $B_i$’s, we deduce by induction on $q$ (using (R1) and (R2)) that, for $q = p, p+1, \ldots, k$,

1. all the vertices $y_i$ and $y_{i,j}$, for $p+1 \leq i \leq q+1$ and $1 \leq j \leq s_i$, are contracted into $\tilde{y}$ in $D_{q+1}$;
2. all $H_{i,j}$ remain intact in $D_{q+1}$, for $p+1 \leq i \leq q+1$ and $1 \leq j \leq s_i$;
3. $D_{q+1}$ is 2-connected;
4. if $\{u, v\}$ is a cutset of $D_{q+1}$ and $C$ is a component of $D_{q+1} - \{u, v\}$ containing neither $\tilde{x}$ nor $\tilde{y}$, then $C$ is adjacent to $V(G) - V(H)$;
5. if $A$ is a cutset of $D_{q+1}$ with $A$ contained in $H_{q+1} - x_{q+1}$, then all $A$-bridges in $D_{q+1}$ are induced subgraphs of $H_{q+1}$, except one which contains both $\tilde{x}$ and $\tilde{y}$; and
6. there is at least one edge between $V(D_{q+1}) - (V(H_{q+1}) \cup \{\tilde{x}, \tilde{y}\})$ and $V(G) - V(H)$. (Indeed, if $H = H_p$, then $H_{p+1} - \{x_{p+1}, y_{p+1,1}\}$ is adjacent to $V(G) - V(H)$; if $H = B_p$, with $i = p$, occurs, then $F_{p+1,0} - \{x_{p+1}, y_{p+1,1}\}$ is adjacent to $V(G) - V(H)$.)

Clearly, (i) follows instantly from (1) and (ii) from (2) with $q = k$. Let $\tilde{D}$ be the graph obtained from $D_{k+1}$ by contracting $H_{k+1} - x_{k+1}$ to $y_{k+1}$. Then $\tilde{D}$ is 2-connected by (3) and (5). Since $\tilde{H}$ is obtained from $\tilde{D}$ by adding $\tilde{w}$ (and edges from $\tilde{D}$ to $V(G) - V(H)$), from (4) and (6) we conclude that $\tilde{H}$ is 3-connected. So (iii) also holds.

Since $\tilde{H}$ is a 3-connected minor of $G$ and both $\tilde{y}w$ and $\tilde{y}x_{k+1}$ are edges in $\tilde{H}$, the induction hypothesis of Theorem 2.5(a) guarantees the existence of a $\tilde{w}$-$x_{k+1}$ path $\tilde{Q}$ in $\tilde{H} - \tilde{y}$ with

$$
\ell(\tilde{Q}) \geq \alpha(t) \left( \frac{|\tilde{H}|}{27} \right)^{\beta} \\
\geq \alpha(t) \left( \frac{\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j} - \{x_i, y_{i,j}\}|}{27} \right)^{\beta} \quad \text{(by Claim 7.21(ii))} \\
\geq \alpha(t) \left( \frac{1}{3} \frac{\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}|}{27} \right)^{\beta} \\
\geq \alpha(t) \left( \frac{1}{162} (n - |H_{k+1}|) \right)^{\beta} \quad \text{(by Claim 7.20)}. $$

By Claim 7.2, $U_p$ is the disjoint union of $\Lambda_p$ and $\Omega_p$, both $G[\Lambda_p]$ and $G[\Omega_p]$ are connected, and $N(V(H_p) - \{y_p\}) \subseteq \Lambda_p \cup \{y_p\}$. Therefore, using Claim 7.21(i), $G - (V(H_{k+1}) - \{x_{k+1}, y_{k+1}\})$ contains two vertex-disjoint paths $X_{k+1}$ and $Y_{k+1}$ from $x$, $y$ to $x_{k+1}$, $y_{k+1}$, respectively, such that

- $\tilde{Q} - \tilde{w} \subseteq X_{k+1}$, and $V(X_{k+1}) - V(\tilde{Q} - \tilde{w})$ is contained in $\Lambda_p$ if $H = H_p$ and in $\Lambda_{p+1}$ if $H = B_p$;
- $V(Y_{k+1}) \cap U_p$ is contained in $\Omega_p$, and $V(Y_{k+1}) - U_p$ is contained in the union of $F_{i+1,0}$ for all $i$ with $B_i$ light.
Observe that $\ell(X_{k+1}) \geq \ell(\tilde{Q}) \geq \alpha(t) \left( \frac{1}{162} (n - |H_{k+1}|) \right)^\beta$.

Let us consider the triple $(H_{k+1}, x_{k+1}, y_{k+1})$. Since $X_{k+1}$ and $Y_{k+1}$ are vertex-disjoint paths in $G - (V(H_{k+1}) - \{x_{k+1}, y_{k+1}\})$ from $x$ to $x_{k+1}, y_{k+1}$, respectively, (M1) holds for $(H_{k+1}, x_{k+1}, y_{k+1})$.

By Claims 7.2 and 7.3, (M2) and (M3) also hold for $(H_{k+1}, x_{k+1}, y_{k+1})$. From the maximality on $k$, we deduce that $(H_{k+1}, x_{k+1}, y_{k+1})$ is not a magic minor of $(G, x, y)$ and hence it does not satisfy (M4).

Note that for any $a \geq \frac{n}{432}$, we have $a \geq \frac{1}{2} \cdot \frac{n - |H_{k+1}|}{216}$. So

$$
\alpha(t)a^\beta + \ell(X_{k+1}) + \ell(Y_{k+1}) \\
\geq \alpha(t) \left( a^\beta + \left( \frac{1}{162} (n - |H_{k+1}|) \right)^\beta \right) \\
\geq \alpha(t) \left( a + \frac{b - 1}{432} (n - |H_{k+1}|) \right)^\beta \quad \text{(by Lemma 3.4)} \\
= \alpha(t) (a + 4(n - |H_{k+1}|))^\beta \quad \text{(for $b - 1 = 1728$)}.
$$

It follows that

**Claim 7.22.** $|H_{k+1}| < \frac{n}{2}$.

Now we are ready to present the last part of our proof. Let $Q_{k+1}$ be the $x_{k+1}$-$y_{k+1}$ path in $H_{k+1}$ as exhibited in (7.2). Set $Q := X_{k+1} \cup Q_{k+1} \cup Y_{k+1}$. Then $Q$ is an $x$-$y$ path in $G$ with

$$
\ell(Q) \geq \ell(Q_{k+1}) + \ell(X_{k+1}) \geq \alpha(t) \left( \left( \frac{|H_{k+1}|}{27} \right)^\beta + \left( \frac{1}{162} (n - |H_{k+1}|) \right)^\beta \right).
$$

If $\frac{|H_{k+1}|}{27} \geq \frac{1}{162} (n - |H_{k+1}|)$, then

$$
\ell(Q) \geq \alpha(t) \left( \frac{|H_{k+1}|}{27} + \frac{b - 1}{162} (n - |H_{k+1}|) \right)^\beta \geq \alpha(t)n^\beta
$$

for $(b - 1)/162 \geq 2$ and $n - |H_{k+1}| \geq n/2$. If $\frac{|H_{k+1}|}{27} < \frac{1}{162} (n - |H_{k+1}|)$, then

$$
\ell(Q) \geq \alpha(t) \left( \frac{b - 1}{27} |H_{k+1}| + \frac{1}{162} (n - |H_{k+1}|) \right)^\beta \geq \alpha(t)n^\beta
$$

for $|H_{k+1}| \geq n/16$ by Claim 7.14 and $b = 1729$. This completes the proof of Lemma 7.1 and hence of Theorem 2.5. □

### References


