

# Hamilton Cycles in Plane Triangulations

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April 10, 2002

## Abstract

We extend Whitney's Theorem that every plane triangulation without separating triangles is hamiltonian by allowing some separating triangles. More precisely we define a decomposition of a plane triangulation  $G$  into 4-connected 'pieces' and show that if each piece shares a triangle with at most three other pieces then  $G$  is hamiltonian. We provide an example to show that our hypothesis that 'each piece shares a triangle with at most three other pieces' cannot be weakened to 'four other pieces'. As part of our proof we also obtain new results on Tutte cycles through specified vertices in planar graphs.

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\*Partially supported by NSF grants DMS-9531824 and 9970527

# 1 Introduction

We consider finite simple graphs. Let  $G$  be a graph,  $X$  a set of vertices and edges of  $G$ . Then  $G - X$  denotes the graph obtained from  $G$  by deleting  $X$  and all edges incident to some vertex in  $X$ . When  $X = \{x\}$ , we write  $G - x$  instead of  $G - \{x\}$ . For a subgraph  $H$  of  $G$ , we use  $H \cap X$  to denote the set of elements of  $X$  which are vertices or edges of  $H$ . For convenience, we use  $A := B$  to define  $A$  as  $B$  or to re-name  $B$  as  $A$ . Let  $x, y$  be two vertices in  $G$ . Then  $G + xy := G$  if  $xy \in E(G)$ , otherwise  $G + xy$  denotes the graph obtained from  $G$  by adding the edge  $xy$ . A *circuit graph* is an ordered pair  $(G, C)$  consisting of a 2-connected plane graph  $G$  and a facial cycle  $C$  of  $G$  such that for any 2-cut  $U$  of  $G$ , each component of  $G - U$  contains a vertex of  $C$ . A *plane chain of blocks* is a plane graph with blocks  $B_1, B_2, \dots, B_k$  such that for  $i, j \in \{1, 2, \dots, k\}$ ,  $B_i \cap B_j$  consists of exactly one vertex if  $|i - j| = 1$ ,  $B_i \cap B_j = \emptyset$  for  $|i - j| \geq 2$ , and  $(\bigcup_{i=1}^k B_i) - B_j$  is in the infinite face of  $B_j$ . We also say that  $B_1$  and  $B_k$  are the *endblocks*. Given a path  $P$  in a plane graph  $G$  we define the *plane chain of blocks in  $G$  along  $P$*  to be the union of all blocks of  $G$  which contain an edge of  $P$ . (It can easily be seen that this union is indeed a plane chain of blocks.) For a cycle  $C$  in a plane graph and  $x, y \in V(C)$ ,  $xCy := x$  if  $x = y$ , and otherwise,  $xCy$  denotes the subpath of  $C$  from  $x$  to  $y$  in the clockwise direction. The *outerwalk*  $D$  in a plane graph is the facial walk bounding its infinite face. Given  $x, y \in V(D)$ , we shall use  $xDy$  to denote a subwalk of  $D$  from  $x$  to  $y$  in the clockwise direction. An outerwalk which is a cycle is an *outercycle*.

Obviously, if  $C$  is a facial cycle of a 3-connected plane graph  $G$ , then  $(G, C)$  is a circuit graph. Circuit graphs have the following nice inductive properties. Let  $(G, C)$  be a circuit graph such that  $C$  is the outercycle. Let  $C'$  be a cycle in  $G$  and let  $H$  be the subgraph of  $G$  contained in the closed disc bounded by  $C'$ . Then  $(H, C')$  is a circuit graph (because if  $T$  is a 2-cut of  $H$  such that  $H - T$  has a component  $D$  containing no vertex of  $C'$ , then by planarity,  $D$  is also a component of  $G - T$  containing no vertex of  $C$ ). Also for  $v \in V(C)$ ,  $G - v$  is a plane chain of blocks  $B_1, \dots, B_k$  such that  $v$  has a neighbor in  $B_1$  and a neighbor in  $B_k$  neither of which is a cutvertex of  $G - v$ . This can be seen as follows. Let  $x, y$  be distinct neighbors of  $v$  on  $C$  such that  $xCy = xvy$ . Since  $(G, C)$  is a circuit graph, every block of  $G - v$  contains an edge of  $yCx$ . Hence,  $G - v$  is a plain chain of blocks along  $yCx$ , and each endblock contains  $x$  or  $y$ .

Let  $P$  be a subgraph of a graph  $G$ . A  $P$ -bridge of  $G$  is either a single edge of  $G - E(P)$  with both ends on  $P$  (which is *trivial*), or a component of  $G - V(P)$  together with the edges joining the component to  $P$  (and all incident vertices). For any  $P$ -bridge  $B$  of  $G$ , the set of *attachments of  $B$  on  $P$*  is  $V(B) \cap V(P)$ . We say that  $P$  is a *Tutte* subgraph of  $G$  if every  $P$ -bridge of  $G$  has at most three attachments on  $P$ . Given a subgraph  $C$  of  $G$ , we say that  $P$  is a  *$C$ -Tutte* subgraph of  $G$  if  $P$  is a Tutte subgraph of  $G$ , and every  $P$ -bridge of  $G$  containing an edge of  $C$  has at most two attachments. A *Tutte path* (or *Tutte cycle*) is a path (or cycle) which is a Tutte subgraph.

Note that if  $(G, C)$  is a circuit graph and  $x, y \in V(C)$  are distinct, then  $G$  has at most three  $\{x, y\}$ -bridges and at most two non-trivial  $\{x, y\}$ -bridges. Moreover,  $G$  has exactly two  $\{x, y\}$ -bridges if and only if  $xy \in E(C)$ , or  $\{x, y\}$  is a 2-cut of  $G$  and  $xy \notin E(G)$ .

Tutte [5] proved that every 2-connected planar graph contains a Tutte cycle through two given edges on a facial cycle, and deduced that every 4-connected planar graph contains a Hamilton cycle. Thomas and Yu [3] extended these results by finding a Tutte cycle through any three given edges on a facial cycle and deducing that every 4-connected projective planar graph is hamiltonian. We shall show that every circuit graph  $(G, C)$  has a Tutte cycle through a given edge of  $C$  and two other given vertices. We then use this result to extend the theorem of Whitney [6] that every plane triangulation without separating triangles is hamiltonian. To do this, we define a decomposition of a plane triangulation  $G$  into 4-connected ‘pieces’. We show that if each piece in the decomposition shares a triangle with at most three other pieces, then  $G$  is hamiltonian. A different generalization of Whitney’s result was obtained by Dillencourt [2].

## 2 Preliminary Results

We shall need several lemmas. The first is due to Thomassen [4].

**Lemma 1** *Let  $G$  be a connected plane graph with outerwalk  $C$ . Let  $x \in V(C)$ ,  $e \in E(C)$ , and  $y \in V(G) - \{x\}$  such that  $G$  contains a path from  $x$  to  $y$  through  $e$ . Then  $G$  contains a  $C$ -Tutte path  $P$  from  $x$  to  $y$  through  $e$ .*

Lemma 1 is proved in [4] for 2-connected graphs, but it easily generalizes to connected graphs. Note that Lemma 1 implies that every 4-connected

planar graph is Hamilton connected.

The following result finds a Tutte path between two vertices through two specified edges.

**Lemma 2** ([3], (2.6)) *Let  $G$  be a 2-connected plane graph with outercycle  $C$ , let  $u, v \in V(C)$  be distinct, let  $e, f \in E(C)$ , and assume that  $u, v, e, f$  occur on  $C$  in this clockwise order. Then  $G$  has a  $vCu$ -Tutte path  $P$  from  $u$  to  $v$  using  $e$  and  $f$ .*

We also need two lemmas concerning the existence of paths avoiding specified vertices.

**Lemma 3** ([7], (2.5)) *Let  $G$  be a connected plane graph with outerwalk  $C$ , and let  $x, y, z \in V(C)$  be distinct and occur on  $C$  in this order such that  $G - z$  has a path from  $x$  to  $y$ . Then  $G - z$  has a path  $P$  from  $x$  to  $y$  such that  $(P \cup \{z\})$  is an  $x Cz$ -Tutte subgraph of  $G$ .*

**Lemma 4** ([3], (2.4)) *Let  $G$  be a connected plane graph with outerwalk  $C$ , and let  $x, y, u, v$  be four distinct vertices on  $C$  in this clockwise order such that  $u, v \notin xCy$ . Then  $G - \{u, v\}$  contains a path  $P$  from  $x$  to  $y$  such that  $(P \cup \{u, v\})$  is an  $x Cy$ -Tutte subgraph of  $G$ .*

### 3 Tutte Paths Through Specified Vertices

**Lemma 5** *Let  $(G, C)$  be a circuit graph with  $C$  as its outercycle, let  $x$  and  $y$  be distinct vertices of  $C$ , let  $e = uv \in E(C)$  such that  $x, u, v, y$  occur in this clockwise order around  $C$ , and let  $z \in V(G)$ . Then the following hold.*

- (a)  $G$  contains a  $C$ -Tutte path  $T$  from  $x$  to  $y$  through  $z$ .
- (b)  $G$  contains an  $x Cy$ -Tutte path  $T$  from  $x$  to  $y$  through  $e$  and  $z$ , unless either (i) there is a facial cycle  $D$  of  $G$  other than  $C$  and there are distinct vertices  $a, b, c, d \in V(C \cap D)$  such that  $x, a, u, v, b, y, c, d$  occur on  $C$  in this clockwise order,  $z \notin \{c, d\}$ , and  $z$  is contained in the  $\{c, d\}$ -bridge of  $G$  containing  $cCd$ ; or (ii) there exist vertices  $w_1 \in yCx - x$  and  $w_2 \in vCy$  such that for both  $i \in \{1, 2\}$ , no  $\{x, w_i\}$ -bridge of  $G$  contains both  $z$  and  $e$ ; or (iii) there exist vertices  $w_1 \in yCx - y$  and  $w_2 \in xCu$

such that for both  $i \in \{1, 2\}$ , no  $\{y, w_i\}$ -bridge of  $G$  contains both  $z$  and  $e$ . In particular, if  $xy \in E(C) - \{e\}$  then the exceptional cases (i), (ii) and (iii) cannot occur, if  $x = u$  then the exceptional case (i) falls under the exceptional case (ii), and if  $y = v$  then the exceptional case (i) falls under the exceptional case (iii).

**Proof.** We proceed by way of contradiction. Suppose the lemma is false and let  $G$  be a counterexample such that  $|V(G)|$  is as small as possible.

We first derive a claim, which will be used heavily in the proof of this lemma to modify paths.

**Claim 1** *Let  $(G', C')$  be a circuit graph with  $C'$  as its outercycle. Let  $\alpha, \beta, \nu, \tau$  be vertices occurring in this order on  $C'$ ,  $\beta\nu \in E(C')$  and put  $L = \alpha C' \beta$ . Let  $H$  be the plane chain of blocks in  $G' - V(L)$  along  $\nu C' \tau$ . Let  $D$  denote the path in the outercycle of  $H$  from  $\tau$  to  $\nu$  which contains no edge of  $\nu C' \tau$ , and let  $\mu$  be the first vertex on  $D$  from  $\tau$  which is an attachment of some  $(H \cup L)$ -bridge of  $G'$ .*

*Let  $z \in V(G')$ . Define  $z_1 \in V(H)$  as follows: if  $z \in V(H)$  put  $z_1 = z$ ; if there exist vertices  $w \in V(H)$  and  $p_1, p_2 \in V(L)$  such that, for  $1 \leq i \leq 2$ ,  $w$  and  $p_i$  are both attachments of some  $(H \cup L)$ -bridge of  $G'$ , and  $z$  belongs to the union of  $p_1 L p_2$  with those  $(H \cup L)$ -bridges of  $G'$  whose attachments are all contained in  $V(p_1 L p_2) \cup \{w\}$ , then let  $z_1 = w$ ; otherwise let  $z_1 = \nu$ .*

*Suppose that  $Q$  is a  $\mu D \nu$ -Tutte path of  $H$  starting at  $\nu$  and containing  $z_1$ , and suppose that  $|V(G') - V(Q)| \leq |V(G)| - 2$ .*

- (c) *Suppose  $\mu \in V(Q)$ . Let  $L_\alpha = \alpha L \alpha'$  be the maximal subpath of  $L$  such that  $\alpha'$  belongs to some  $(H \cup L)$ -bridge of  $G'$  with  $\mu$  as an attachment. Let  $F_\alpha$  be the union of  $L_\alpha$  and those  $(H \cup L)$ -bridges of  $G'$  whose attachments are all contained in  $V(L_\alpha) \cup \{\mu\}$ . Suppose further that, if  $z \in F_\alpha$  then, for any vertices  $w_1 \in \mu C' \alpha - \mu$  and  $w_2 \in \alpha C' \alpha'$  both  $z$  and  $\alpha' \mu$  are contained in a  $\{\mu, w_i\}$ -bridge of  $F_\alpha + \alpha' \mu$  for some  $i \in \{1, 2\}$ . Then there is a path  $P$  in  $G' - V(Q)$  from  $\alpha$  to  $\beta$  such that  $z \in V(P) \cup V(Q)$ ,  $P \cup Q$  is an  $\alpha C' \beta$ -Tutte subgraph of  $G'$ , and every  $Q$ -bridge of  $H$  containing no edge of  $\mu D \nu$  is also a  $(P \cup Q)$ -bridge of  $G'$ . Furthermore, if  $z \notin F_\alpha - \mu$  then  $P$  can be chosen such that  $P \cup Q$  is a  $\mu C' \beta$ -Tutte subgraph of  $G'$ .*

(d) Suppose  $\mu \notin V(Q)$ . Let  $X$  be the  $Q$ -bridge of  $H$  which contains  $\mu$  and  $L_\alpha = \alpha L \alpha'$  be the maximal subpath of  $L$  such that  $\alpha'$  belongs to some  $(H \cup L)$ -bridge of  $G'$  with an attachment in  $V(X) - V(Q)$ . Let  $F_\alpha$  be the union of  $L_\alpha$ ,  $X$ , and those  $(H \cup L)$ -bridges of  $G'$  whose attachments are all contained in  $V(L_\alpha) \cup (V(X) - V(Q))$ . Suppose that  $z \notin F_\alpha - V(Q)$ . Then there is a path  $P$  in  $G' - V(Q)$  from  $\alpha$  to  $\beta$  such that  $z \in V(P) \cup V(Q)$ ,  $P \cup Q$  is an  $\alpha C' \beta$ -Tutte subgraph of  $G'$ , and every  $Q$ -bridge of  $H$  containing no edge of  $\mu D \nu$  is also a  $(P \cup Q)$ -bridge of  $G'$ .

**Proof of (c).** Since  $G'$  is a plane graph, we can divide  $L$  into non-trivial edge-disjoint subpaths  $pLq$  of three types according to the  $Q$ -bridges of  $H$ : (type 1) for some  $Q$ -bridge  $B$  of  $H$ ,  $pLq$  is a maximal subpath of  $L$  such that  $p$  and  $q$  belong to some  $(H \cup L)$ -bridges of  $G'$  each with an attachment contained in  $V(B) - V(Q)$ ; (type 2) for some vertex  $v$  of  $Q$ ,  $pLq$  is the maximal subpath of  $L$  such that both  $p$  and  $q$  belong to some  $(H \cup L)$ -bridges of  $G'$  with  $v$  as an attachment; (type 3) the remaining non-trivial edge maximal subpaths of  $L$ .

Obviously, each edge of  $L$  belongs to exactly one of these subpaths. We label these subpaths (from  $\alpha$  to  $\beta$ ) as  $L_i$ ,  $i = 1, \dots, m$ . Let the endvertices of  $L_i$  be  $x_i$  and  $x_{i+1}$  such that  $x_1, x_2, \dots, x_{m+1}$  occur on  $C'$  in this order, where  $x_1 = \alpha$  and  $x_{m+1} = \beta$ .

Suppose  $L_i$  is of type 1. Let  $F_i$  be the union of  $L_i$ ,  $B$ , and those  $(H \cup L)$ -bridges of  $G'$  whose attachments are all contained in  $V(L_i) \cup (V(B) - V(Q))$ . By Lemma 4,  $F_i - (V(B) \cap V(Q))$  has a path  $P_i$  from  $x_i$  to  $x_{i+1}$  such that  $P_i \cup (Q \cap B)$  is an  $x_i C' x_{i+1}$ -Tutte subgraph of  $F_i$ . Because  $z_1 \in V(Q)$ ,  $z \notin F_i - (V(Q) \cap V(B))$ .

Suppose  $L_i$  is of type 2. Let  $F_i$  be the union of  $L_i$  and those  $(H \cup L)$ -bridges of  $G'$  whose attachments are all contained in  $V(L_i) \cup \{v\}$ . Let  $C_i$  be the outerwalk of  $F_i$ . Applying (b) inductively to  $(F_i + x_{i+1}v, v C_i x_{i+1}v)$  for  $1 \leq i < m$ , we find an  $x_i C_i x_{i+1}v$ -Tutte path  $P_i$  from  $x_i$  to  $v$  through  $x_{i+1}v$ , and also through  $z$  if  $z \in V(F_i)$ . Similarly, if  $i = m$  we apply (b) inductively to  $(F_m + \nu x_m, \nu x_m C_m \nu)$  we find a  $\nu x_m C_m \beta$ -Tutte path  $P_m$  from  $\nu$  to  $\beta$  through  $\nu x_m$ , and also through  $z$  if  $z \in V(F_m)$ . Note that the exceptional cases of (b) cannot occur by the internal 3-connectivity of  $G'$  for  $1 < i < m$ , by the hypothesis on  $w_1, w_2$  in (c) and the fact that  $x_{i+1}\mu \in E(F_i + x_{i+1}\mu)$  for  $i = 1$ , and by the fact that  $\nu\beta \in E(C_m) - \{\nu x_m\}$  for  $i = m$ .

Suppose  $L_i$  is of type 3. Let  $F_i$  be the union of  $L_i$  and those  $(H \cup L)$ -bridges of  $G'$  whose attachments are all contained in  $V(L_i)$ . If  $F_i = x_i x_{i+1}$  let  $P_i = x_i x_{i+1}$ . Otherwise chose  $z_i \in V(F_i) - \{x_i, x_{i+1}\}$  with  $z_i = z$  if  $z \in V(F_i)$ . Applying (a) inductively to  $(F_i + x_i x_{i+1}, x_i L_i x_{i+1} x_i)$  we find an  $x_i L_i x_{i+1} x_i$ -Tutte path  $P_i$  from  $x_i$  to  $x_{i+1}$  through  $z_i$ .

It is now easy to see that  $P := \bigcup_{i=1}^m P_i$  is the desired path in  $G' - V(Q)$ .

To prove the second assertion of (c) we proceed similarly, but instead of using (b) inductively to deal with  $L_1 = L_\alpha$ , we use Lemma 3 to find a path  $P_1$  in  $F_1 - \mu$  from  $\alpha$  to  $x_2$  such that  $P_1 \cup \{\mu\}$  is a  $\mu C' x_2$ -Tutte subgraph of  $F_1$ .

**Proof of (d).** We proceed as in (c), with the exception that we use Lemma 4 to construct a path  $P_1$  in  $F_1 - (V(Q) \cap V(X)) := F_\alpha - (V(Q) \cap V(X))$  such that  $P_1 \cup (Q \cap X)$  is an  $\alpha C \alpha'$ -Tutte subgraph of  $F_1$ . ■

**Claim 2** (a) holds for  $G$ .

Suppose the claim is false. If  $z \in V(C)$  then (a) holds for  $G$  by Lemma 1. Hence  $z \notin V(C)$ .

**Claim 2.1**  $G$  is 3-connected.

**Proof.** Suppose  $\{s, t\}$  is a 2-cut of  $G$ . We may suppose that  $\{s, t\}$  has been chosen such that the  $\{s, t\}$ -bridge  $B$  which contains  $z$  is minimal. Let the outerwalk of  $B$  be denoted by  $t C s' s F t$ . Let  $J := G - (V(B) - \{s, t\})$ .

We first consider the case when  $x, y \in V(B)$ . Applying (a) inductively to the circuit graph  $(B + st, t C s t)$  we deduce that  $B + st$  has a  $t C s$ -Tutte path  $P$  from  $x$  to  $y$  through  $z$ . If  $st \notin E(P)$  then  $T := P$  satisfies (a) for  $G$ . Hence  $st \in E(P)$ . By Lemma 1 we find an  $s C t$ -Tutte path  $P'$  from  $s$  to  $t$  in  $J$ . Then  $T := (P - st) \cup P'$  satisfies (a) for  $G$ .

We next consider the case when  $x, y \in V(J)$ . Applying (a) inductively to the circuit graph  $(B + st, t C s t)$  we deduce that  $B + st$  has a  $t C s$ -Tutte path  $P$  from  $s$  to  $t$  through  $z$ . By Lemma 1 we find an  $s C t$ -Tutte path  $P'$  in  $J + st$  from  $x$  to  $y$  through  $st$ . Then  $T := (P' - st) \cup P$  satisfies (a) for  $G$ .

Finally we consider the case when  $\{s, t\}$  separates  $x$  and  $y$ . By symmetry we may assume that  $y \in V(B)$  and  $x \in V(J)$ . By Lemma 1 we find an  $s C t$ -Tutte path  $P'$  in  $J + st$  from  $x$  to  $s$  through  $ts$ . Let  $L := y C s'$ , let  $H$  be the plane chain of blocks in  $B - V(L)$  along  $s F t$ , and let  $W$  denote the path in the outerwalk of  $H$  from  $t$  to  $s$  which contains no edge of  $F$ . Let  $\mu$  be

the first vertex on  $W$  from  $t$  which is an attachment of some  $(H \cup L)$ -bridge of  $B$ . Define  $z_1 \in V(H)$  as follows: if  $z \in V(H)$  put  $z_1 = z$ ; if there exist vertices  $w \in V(H)$  and  $p_1, p_2 \in V(L)$  such that, for  $1 \leq i \leq 2$ ,  $w$  and  $p_i$  are both attachments of some  $(H \cup L)$ -bridge of  $G$  and  $z$  belongs to the union of  $p_1 C p_2$  with those  $(H \cup L)$ -bridges of  $G$  whose attachments are all contained in  $V(p_1 C p_2) \cup \{w\}$ , then let  $z_1 = w$ ; otherwise let  $z_1 = s$ . Let  $f$  be an edge of  $W$  incident with  $\mu$ . Applying (b) inductively to the circuit graph  $(H + st, tWst)$  and because  $st \in E(tWst) - \{f\}$ , we deduce that  $H + st$  has a  $tWst$ -Tutte path  $Q$  from  $s$  to  $t$  through  $z_1$  and  $f$  (and hence  $\mu$ ). Let  $L_1 = yLy'$  be the maximal subpath of  $L$  such that  $y'$  belongs to some  $(H \cup L)$ -bridge of  $B$  with  $\mu$  as an attachment. Then  $z$  does not belong to the union of those  $(H \cup L)$ -bridges of  $B$  whose attachments are all contained in  $V(L_1) \cup \{\mu\}$ , otherwise (because  $z \notin V(C)$ )  $\{\mu, y'\}$  would contradict the choice of  $\{s, t\}$  to minimize the  $\{s, t\}$ -bridge  $B$  containing  $z$ . Applying Claim 1(c) (with  $B, tCsFt, y, y', s', s, t$  as  $G', C', \alpha, \alpha', \beta, \nu, \tau$ , respectively), we obtain a path  $P$  from  $s'$  to  $y$  in  $B - V(Q)$  such that  $P \cup Q$  is a  $\mu C s'$ -Tutte subgraph of  $B$  and every  $Q$ -bridge of  $H$  containing no edge of  $tWst$  is also a  $(P \cup Q)$ -bridge of  $B$ . Then  $T := xP'tQs's'Py$  satisfies (a) for  $G$ . ■

We can now complete the proof of Claim 2. We may assume by symmetry that  $xCy$  has at least three vertices. Let  $C = xCy''y'xCx'$  and  $J := G - V(xCy'')$ , where  $y' \neq y$  but we may have  $x = y''$  or  $x' = y$  or both. Let  $H$  be the plane chain of blocks in  $J$  along  $y'Cx'$ . Since  $G$  is 3-connected,  $H$  is a block. Let the outerwalk of  $H$  be  $y'Cx'Fy'$ . Hence either  $(H, y'Cx'Fy')$  is a circuit graph, or else  $x' = y$  and  $H = y'x'$ . Define  $z_1 \in V(H)$  as follows: if  $z \in V(H)$  put  $z_1 = z$ ; if there exist vertices  $w \in V(H)$  and  $p_1, p_2 \in V(xCy'')$  such that, for  $1 \leq i \leq 2$ ,  $w$  and  $p_i$  are both attachments of some  $(H \cup xCy'')$ -bridge of  $G$ , and  $z$  belongs to the union of  $p_1 C p_2$  with those  $(H \cup xCy'')$ -bridges of  $G$  whose attachments are all contained in  $V(p_1 C p_2) \cup \{w\}$ , then let  $z_1 = w$ ; otherwise let  $z_1 = y'$ . If  $H = y'x'$  then let  $Q = H$ . Otherwise we may apply (b) inductively to  $(H, y'Cx'Fy')$  to deduce that  $H$  has a  $y'Cx'Fy'$ -Tutte path  $Q$  from  $y$  to  $y'$  through  $z_1$  and an edge  $f$  of  $yCx'$  incident to  $x'$ . Note that the exceptional cases of (b) do not occur by the fact that  $yy' \in E(y'Cx'Fy') - \{f\}$ . Applying Claim 1(c) (with  $G, C, x, y'', y', x'$  as  $G', C', \alpha, \beta, \nu, \tau = \mu$ , respectively) and using Claim 2.1 to show that the hypothesis in (c) on  $w_1, w_2$  must hold, we obtain a path  $P$  in  $G - V(Q)$  from  $y''$  to  $x$  such that  $z \in V(P) \cup V(Q)$ ,  $P \cup Q$  is an  $xCy''$ -Tutte subgraph of  $G$ ,

and every  $Q$ -bridge of  $H$  containing no edge of  $x'Fy'$  is also a  $(P \cup Q)$ -bridge of  $G$ . Then  $T := yQy'y''Px$  satisfies (a) for  $G$ . ■

**Claim 3** (b) holds for  $G$ .

**Proof.** Suppose  $z \in V(xCy)$ . Then by Lemma 2,  $G$  has an  $xCy$ -Tutte path  $T$  from  $x$  to  $y$  through  $e$  and  $z$ , and so, (b) holds for  $G$ . So assume  $z \notin V(xCy)$ .

We claim that either (1)  $y \neq v$  and  $z$  belongs to a block of  $G - V(xCu)$  which contains an edge of  $vCy$  or (2)  $x \neq u$  and  $z$  belongs to a block of  $G - V(vCy)$  which contains an edge of  $xCu$ . Suppose neither (1) nor (2) holds. If  $u = x$  and  $v = y$  then  $G$  belongs to the exceptional cases (ii) and (iii) of (b). Assume  $x = u$  and  $y \neq v$ . Then there is a vertex  $w_1 \in yCx - x$  such that  $\{x, w_1\}$  is a 2-cut of  $G$  separating  $z$  from  $e$ , or else (1) holds. Let  $w_2 = v$ . Then for each  $i \in \{1, 2\}$  no  $\{x, w_i\}$ -bridge of  $G$  contains both  $z$  and  $e$ , and we have exceptional case (ii) of (b). Similarly, if  $x \neq u$  and  $y = v$ , then exceptional case (iii) of (b) occurs. So assume that  $x \neq u$  and  $y \neq v$ . Let  $K$  denote the plane chain of blocks in  $G - xCu$  along  $vCy$ . Since (1) does not hold, there exists some  $(K \cup xCu)$ -bridge  $B$  of  $G$  such that  $z \in B - V(xCu \cup K)$ . Let  $c \in V(B) \cap V(K)$ . Then  $c \in yCx - x$ ; otherwise, (2) holds because  $(G, C)$  is a circuit graph. Let  $a \in V(B) \cap V(xCu)$  such that  $aCu$  is minimal. Then  $\{a, c\}$  is a 2-cut of  $G$  separating  $e$  from  $z$ . By a symmetric argument, we may assume that  $G$  has a 2-cut  $\{b, d\}$  separating  $e$  from  $z$ , where  $b \in vCy$  and  $d \in yCx - y$ . Clearly,  $a \neq b$ ,  $z \notin \{c, d\}$ , and  $c \in yCd$ . If  $a = d$  then exceptional case (ii) occurs, and if  $c = b$  the exceptional case (iii) occurs. So  $a \neq d$  and  $c \neq b$ . By planarity, we have exceptional case (i) of (b).

Hence we may assume by symmetry that (1) holds. Let  $L = xCu$  and  $H$  be the plane chain of blocks in  $G - V(L)$  along  $vCy$ . By (1),  $z \in V(H)$ . Define  $z_1 = z$ . Let  $F$  be the path in the outerwalk of  $H$  from  $y$  to  $v$  which contains no edge of  $vCy$ . Applying (a) inductively to each block of  $H$  we obtain a  $vCyFv$ -Tutte path  $Q$  in  $H$  from  $v$  to  $y$  through  $z_1$ . Let  $y'$  be the first vertex of  $F$  from  $y$  which is an attachment of some  $(H \cup L)$ -bridge of  $G$ . If  $y' \in V(Q)$ , let  $L_x = xLx'$  be the maximal subpath of  $L$  such that  $x'$  belongs to some  $(H \cup L)$ -bridge of  $G$  with  $y'$  as an attachment, and  $F_x$  be the union of  $L_x$  with those  $(H \cup L)$ -bridges of  $G$  whose attachments are all contained in  $V(L_x) \cup \{y'\}$ . If  $y' \notin V(Q)$  let  $X$  be the  $Q$ -bridge of  $H$  which contains  $y'$ ,  $L_x = xLx'$  be the maximal subpath of  $L$  such that  $x'$  belongs to some

$(H \cup L)$ -bridge of  $G$  with an attachment in  $V(X) - V(Q)$ , and  $F_x$  be the union of  $L_x$ ,  $X$  and those  $(H \cup L)$ -bridges of  $G$  whose attachments are all contained in  $V(L_x) \cup (V(X) - V(Q))$ . Since  $z \in V(Q)$ ,  $z \notin F_x - y'$ . Then by Claim 1 (c) and (d) (with  $G, C, x, x', u, v, y, y'$  as  $G', C', \alpha, \alpha', \beta, \nu, \tau, \mu$ , respectively), we can find a path  $P$  from  $u$  to  $x$  in  $G - V(Q)$  such that  $z \in V(P \cup Q)$ ,  $P \cup Q$  is a  $y'Cu$ -Tutte subgraph of  $G$ , and every  $Q$ -bridge of  $H$  containing no edge of  $y'Fv$  is also a  $(P \cup Q)$ -bridge of  $G$ . Then  $T := xPuvQy$  satisfies (b) for  $G$ . ■

**Lemma 6** *Let  $(G, C)$  be a circuit graph,  $r, z$  be vertices of  $G$  and  $e \in E(C)$ . Then  $G$  contains a  $C$ -Tutte cycle  $X$  through  $e, r$ , and  $z$ .*

**Proof.** Suppose the lemma is false and let  $G$  be a counterexample chosen such that  $|V(G)|$  is as small as possible. If either  $r$  or  $z$  lies on  $C$  then the lemma holds by Lemma 5(b) (because  $xy \in E(C) - \{f\}$ , where  $f \in E(C) - \{xy\}$  and  $f$  is incident to  $\{r, z\} \cap V(C)$ ). Hence  $\{r, z\} \cap V(C) = \emptyset$ . Without loss of generality, assume that  $C$  is the outercycle of  $G$ .

**Claim 4**  *$G$  is 3-connected.*

**Proof.** Let  $\{s, t\}$  be a 2-cut of  $G$  and  $B_1$  and  $B_2$  be the non-trivial  $\{s, t\}$ -bridges of  $G$  such that  $sCt \subset B_1$ . We first suppose that  $r$  is in  $B_1$  and  $e, z$  are in  $B_2$ . By Lemma 5(a) (with  $B_1 + st, sCts, s, t$  as  $G, C, x, y$ , respectively),  $B_1 + st$  has an  $sCts$ -Tutte path  $P_1$  from  $s$  to  $t$  through  $r$ . By Lemma 5(b) (with  $B_2 + st, tCst, s, t$  as  $G, C, y, x$ , respectively) and because  $st \in E(tCst) - \{e\}$ ,  $B_2 + st$  has a  $tCst$ -Tutte path  $P_2$  from  $s$  to  $t$  through  $e$  and  $z$ . Then  $X := P_1 \cup P_2$  is the required Tutte cycle in  $G$ . Thus we may assume that  $\{s, t\}$  does not separate  $r$  and  $z$ .

We next suppose that  $e$  is in  $B_1$  and  $r, z$  are in  $B_2$ . By Lemma 1,  $B_1 + st$  has an  $sCts$ -Tutte path  $P_3$  from  $s$  to  $t$  through  $e$ . By induction,  $B_2 + st$  has a  $tCst$ -Tutte cycle  $T_1$  through  $st, r$  and  $z$ . Then  $X := P_3 \cup (T_1 - st)$  is the required Tutte cycle in  $G$ .

Finally we suppose that  $e, r, z$  are in  $B_2$ . By induction,  $B_2 + st$  has a  $tCst$ -Tutte cycle  $T_2$  through  $e, r, z$ . If  $st \notin E(T_2)$  then  $T_2$  is the required Tutte cycle in  $G$ . Hence  $st \in E(T_2)$ . By Lemma 1,  $B_1$  has an  $sCts$ -Tutte path  $P_4$  from  $s$  to  $t$ . Then  $X := (T_2 - st) \cup P_4$  is the required Tutte cycle in  $G$ . ■

**Claim 5**  $G - V(C)$  is connected.

**Proof.** This follows immediately from Claim 4 and planarity. ■

Let  $H$  be a block of  $G - V(C)$  and  $W$  be the outerwalk of  $H$ . By Claim 5, every  $(H \cup C)$ -bridge of  $G$  has exactly one attachment on  $H$ . Let  $w_1, w_2, \dots, w_m$  be the attachments on  $H$  of the  $(H \cup C)$ -bridges of  $G$  in clockwise order around  $W$ . Let  $B_{i,1}, B_{i,2}, \dots, B_{i,n_i}$  be the  $(H \cup C)$ -bridges of  $G$  which contain  $w_i$  for  $1 \leq i \leq m$ . Let  $s_{i,j}Ct_{i,j}$  be the minimal segment of  $C$  such that  $B_{i,j} \cap C \subseteq s_{i,j}Ct_{i,j}$  and  $B \cap s_{i,j}Ct_{i,j} \subseteq \{s_{i,j}, t_{i,j}\}$  for all  $(H \cup C)$ -bridges  $B \neq B_{i,j}$  of  $G$ . Let  $F_{i,j} := B_{i,j} \cup s_{i,j}Ct_{i,j}$ .

**Claim 6** We may choose  $H$  such that  $|F_{i,j} \cap \{e, r, z\}| \leq 1$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n_i$ .

**Proof.** Choose  $H$  such that either  $|F_{i,j} \cap \{e, r, z\}| \leq 1$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n_i$ , or, if this is not possible, such that if  $|F_{i,j} \cap \{e, r, z\}| \geq 2$  then  $F_{i,j}$  is minimal. Suppose  $|F_{i,j} \cap \{e, r, z\}| \geq 2$ . Let  $H'$  be the block of  $G - V(C)$  with  $w_i \in H' \subseteq F_{i,j}$ . Then, in an obvious notation, we have that if  $|F'_{k,\ell} \cap \{e, r, z\}| \geq 2$  then  $F'_{k,\ell}$  is a proper subgraph of  $F_{i,j}$ . This would contradict the choice of  $H$  and hence Claim 6 holds. ■

We now continue with the proof of the lemma. Let  $t_i = t_{i,n_i}$  for  $1 \leq i \leq m$ . We may suppose that the labeling has been chosen such that  $e \in t_m C t_1$ . Choose  $(k, \ell)$  to be as small as possible in the lexicographic ordering such that  $2 \leq k \leq m$ ,  $1 \leq \ell \leq n_k$ , and  $t_{k,\ell} \neq t_1$ . Note that  $(k, \ell)$  exists since otherwise  $\{t_1, w_1\}$  would be a 2-cut in  $G$ , contradicting Claim 4. Note also that Claim 4 implies that if  $(k, \ell) > (2, 1)$  then  $V(F_{i,j}) = \{t_1, w_i\}$  for  $(2, 1) \leq (i, j) < (k, \ell)$ . For  $u \in \{r, z\}$ , define  $u_1 \in H$  as follows: if  $u \in H$  put  $u_1 = u$ ; if  $u \in F_{i,j}$  for some  $1 \leq i \leq m$  and  $1 \leq j \leq n_j$ , put  $u_1 = w_i$ . Let  $H'$  be the graph obtained from  $H$  by adding the vertex  $t_1$  and edges  $t_1 w_i$ ,  $i = 1, \dots, k$  and let  $C' := w_k W w_1 t_1 w_k$  be the outercycle of  $H'$ . By induction there exists a  $C'$ -Tutte cycle  $Q'$  in  $H'$  through  $t_1 w_k, r_1, z_1$ . Note that  $w_1 \in Q'$  since otherwise the  $Q'$ -bridge of  $H'$  containing  $w_1$  would contain an edge of  $C'$ , and would have at least three attachments on  $Q'$  (the vertex  $t_1$  and at least two attachments on  $H$  since  $H$  is a block). Let  $Q := Q' - t_1$ . Then  $w_1 \in Q$  and  $Q$  is a path from  $w_j$  to  $w_k$  in  $H$  for some  $j \in \{1, \dots, k-1\}$ .

Since  $G$  is a plane graph, we can divide  $C$  into non-trivial edge-disjoint subpaths  $sCt$  of three types according to the  $Q$ -bridges of  $H$ : (i) for some

$Q$ -bridge  $B$  of  $H$ ,  $sCt$  is a maximal subpath of  $C$  such that  $s$  and  $t$  belong to some  $(H \cup C)$ -bridges of  $G$  each with an attachment in  $V(B) - V(Q)$ ; (ii) for some vertex  $v$  of  $Q$  and some  $(H \cup C)$ -bridge  $B$  of  $G$  with  $v$  as an attachment,  $sCt$  is the maximal subpath of  $C$  such that both  $s$  and  $t$  belong to  $B$  (in this case  $sCt = s_{p,q}Ct_{p,q}$  for some  $s_{p,q} \neq t_{p,q}$ ); (iii) the remaining edges of  $C$ .

Each edge of  $C$  belongs to exactly one of these subpaths. We label these subpaths as  $C_i$ ,  $i = 1, \dots, b$ . Let the endvertices of  $C_i$  be  $x_i$  and  $x_{i+1}$  such that  $x_1, \dots, x_b$  are on  $C$  in this clockwise order, and subscripts are read modulo  $b$ . Since  $w_1, w_k \in Q$  we may choose the labeling such that  $x_1 = s_{1,n_1}$  and  $x_{h+1} = t_{k,\ell}$  for some  $h \geq 1$ .

We first consider intervals  $C_i$  of type (i). Let  $J_i$  be the union of  $C_i$ ,  $B$ , and those  $(H \cup C)$ -bridges of  $G$  whose attachments are all contained in  $V(C_i) \cup (V(B) - V(Q))$ . By Lemma 4,  $J_i - (V(B) \cap V(Q))$  has a path  $P_i$  from  $x_i$  to  $x_{i+1}$  such that  $P_i \cup (B \cap Q)$  is a  $C_i$ -Tutte subgraph of  $J_i$ . Note that  $e, r, z$  do not belong to  $J_i - (V(B) \cap V(Q))$  by the facts that  $e \in t_m C t_1$  and  $w_1, r_1, z_1 \in V(Q)$ .

We next consider intervals  $C_i$  of type (ii). Then  $C_i = s_{p,q}Ct_{p,q}$  for some  $1 \leq p \leq m$  and  $1 \leq q \leq n_p$ . Let  $J_i := F_{p,q}$ , and let  $W_i$  be the outerwalk of  $J_i$ . First assume that  $h+1 \leq i \leq b$  or  $i = 1$  and  $w_j \neq w_1$ . Since  $|J_i \cap \{e, r, z\}| \leq 1$  by Claim 6 and because  $x_i w_p \in E(w_p x_i W_i x_{i+1} w_p) - \{x_{i+1} w_p\}$ , we may apply Lemma 2 (when  $e \in E(J_i)$ ) or Lemma 5(b) (when  $\{r, z\} \cap V(J_i) \neq \emptyset$ ) to  $(J_i + \{x_{i+1} w_p, x_i w_p\}, w_p x_i W_i x_{i+1} w_p)$  to find an  $x_i W_i x_{i+1} w_p$ -Tutte path  $P'_i$  from  $x_i$  to  $w_p$  through  $x_{i+1} w_p$  and  $J_i \cap \{e, r, z\}$ . Let  $P_i = P'_i - w_p$  if  $i \neq 1$ , and otherwise, let  $P_1 = x_1 P'_1 x_2 w_j$  (when  $i = 1$  and  $w_j \neq w_1$ , we have  $x_1 = s_{1,n_1}$  and  $x_2 = t_{1,n_1} = t_1$ , and so  $x_2 w_j \in E(G)$ ). Now assume that  $i = 1$  and  $w_j = w_1$  (in this case,  $t_1 = x_2$ , and so,  $h \geq 2$  because  $x_{h+1} = t_{k,\ell} \neq t_1$ ). Because  $w_1 x_1 \in E(x_1 W_1 w_1 x_1) - \{f\}$ , where  $f \in E(W_1)$  and  $f$  is incident to  $x_2$ , we apply Lemma 2 (when  $e \in E(J_1)$ ) or Lemma 5(b) (when  $\{r, z\} \cap V(J_1) \neq \emptyset$ ) to  $(J_1 + w_1 x_1, x_1 W_1 w_1 x_1)$  to find an  $x_1 W_1 w_1$ -Tutte path  $P_1$  from  $x_1$  to  $w_1$  through  $f$  and  $J_1 \cap \{e, r, z\}$ . Finally assume that  $i = h$ . Because  $C_i$  is of type (ii),  $s_{k,\ell} \neq t_{k,\ell}$ . Hence  $s_{k,\ell} = x_h$ . Note that  $e \notin J_h$  because  $e \in t_m C t_1$  and  $C_h \subset t_1 C t_k$ . Suppose  $x_h = t_1$ . Let  $J'_h := (J_h - x_h) + x_{h+1} w_k$ . Assume that the edge  $x_{h+1} w_k$  is added so that  $w_k W_h x_{h+1} w_k$  is the outercycle of  $J_h$ . Let  $F_h$  denote the outercycle of  $J'_h$ , and let  $f \in E(F_h) - \{x_{h+1} w_k\}$  such that  $f$  is incident to the neighbor of  $x_h$  on  $C \cap J'_h$ . Because  $G$  is 3-connected,  $(J'_h, F_h)$  is a circuit graph. Applying Lemma 5(b) to  $(J'_h, F_h)$  and because

$x_{h+1}w_k \in E(F_h) - \{f\}$ , we find a  $w_k F_h x_{h+1}$ -Tutte path  $P_h$  from  $w_k$  to  $x_{h+1}$  through  $f$  and  $J_h \cap \{r, z\}$ . Now assume  $x_h \neq t_1$ . Let  $f \in E(W_h)$  such that  $f$  is incident to  $x_h$ . Because  $w_k x_{h+1} \in E(w_k W_h x_{h+1} w_k) - \{f\}$ , we may apply Lemma 5(b) to  $(J_h + x_{h+1} w_k, w_k W_h x_{h+1} w_k)$  to find a  $w_k W_h x_{h+1}$ -Tutte path  $P_h$  from  $w_k$  to  $x_{h+1}$  through  $f$  and  $J_h \cap \{r, z\}$ .

Finally, for intervals  $C_i$  of type (iii), let  $P_i := C_i$ .

Note that if  $s_{k,\ell} = t_{k,\ell}$  then  $V(F_{k,\ell}) = \{w_k, x_{h+1}\}$ , and if  $s_{1,n_1} = t_{1,n_1}$  then  $V(F_{1,n_1}) = \{w_1, x_1\}$ . We define  $X$  as follows. If  $s_{1,n_1} = t_{1,n_1}$  and  $s_{k,l} = x_{h+1}$ , then let  $X := w_j Q w_k x_{h+1} P_{h+1} \dots P_b x_1 w_j$ . If  $s_{1,n_1} = t_{1,n_1}$  and  $s_{k,l} \neq x_{h+1}$ , then let  $X := w_j Q w_k P_h x_{h+1} P_{h+1} \dots P_b x_1 w_j$ . If  $s_{1,n_1} \neq t_{1,n_1}$  and  $s_{k,l} = x_{h+1}$ , then let  $X := w_j Q w_k x_{h+1} P_{h+1} \dots P_b x_1 P_1 w_j$ . If  $s_{1,n_1} \neq t_{1,n_1}$  and  $s_{k,l} \neq x_{h+1}$ , then let  $X := w_j Q w_k P_h x_{h+1} P_{h+1} \dots P_b x_1 P_1 w_j$ .

It is then straightforward to verify that  $X$  is the required Tutte cycle in  $G$ . ■

## 4 Plane Triangulations

We first use Lemma 6 to show that we can find a Hamilton cycle through edges in specified triangles in a plane triangulation which has no separating triangles.

**Theorem 7** *Let  $G$  be a plane triangulation with no separating triangles. Let  $T, T_1, T_2$  be distinct triangles in  $G$ . Let  $V(T) = \{u, v, w\}$ . Then there exists a Hamilton cycle  $C$  of  $G$  and edges  $e_1 \in E(T_1)$ ,  $e_2 \in E(T_2)$  such that  $w, uw, e_1, e_2$  are distinct and contained in  $E(C)$ .*

**Proof.** Let  $H = G - u$  and  $F$  be the face of  $H$  which contained  $u$ . We may suppose that  $F$  is the infinite face of  $H$ . Let  $X_i = T_i \cap H$  for  $1 \leq i \leq 2$ . Then either  $X_i = T_i$  or  $X_i$  is an edge incident with  $F$ . Let  $H'$  be the graph obtained from  $H$  as follows: if  $X_i$  is an edge of  $H$  then subdivide  $X_i$  with a new vertex  $z_i$ ; if  $X_i = T_i$  then in the face of  $H$  bounded by  $T_i$  insert a new vertex  $z_i$  and join  $z_i$  to each vertex of  $T_i$ . Let  $F'$  be the outercycle of  $H'$ . Then  $(H', F')$  is a circuit graph. By Lemma 6,  $H'$  has an  $F'$ -Tutte cycle  $C'$  through  $vw, z_1$  and  $z_2$ . Using the facts that the only possible separating triangles in  $H'$  are  $T_1$  and  $T_2$  and that  $C'$  is an  $F'$ -Tutte cycle in  $H'$ , we deduce that  $C'$  is a Hamilton cycle in  $H'$ . Let  $C$  be the cycle obtained from

$C' - \{vw, z_1, z_2\}$  by adding  $uv$ ,  $uw$  and edges  $e_i$  between the neighbours of  $z_i$  on  $C'$  for  $1 \leq i \leq 2$ . Then  $C$  is the required Hamilton cycle in  $G$ . ■

We next apply Theorem 7 to general plane triangulations. To accomplish this we first need to describe how to decompose a plane triangulation into 4-connected ‘pieces’. Let  $G$  be a plane triangulation. Suppose that  $T$  is a separating triangle in  $G$ . Then we may separate  $G$  into two graphs  $G_1$  and  $G_2$  such that  $G = G_1 \cup G_2$ ,  $G_1 \cap G_2 = T$  and both  $G_1$  and  $G_2$  have at least four vertices. Then  $G_i$  is a plane triangulation and  $T$  is a facial triangle of  $G_i$  for  $1 \leq i \leq 2$ . We shall refer to  $T$  as a *marker triangle* in  $G_1$  and  $G_2$ . We now iterate this procedure for both  $G_1$  and  $G_2$ . We continue until we obtain a collection  $S$  of plane triangulations each of which has no separating triangles. We shall refer to the graphs in  $S$  as *pieces* of  $G$ . Note that each separating triangle of  $G$  will occur as a marker triangle in exactly two pieces of  $G$ . We define a new graph  $D$  whose vertices are the pieces in  $S$ , and in which two pieces are joined by an edge if they have a marker triangle in common. It follows from the decomposition theory developed by Cunningham and Edmonds [1] that  $D$  is a tree and also that the set  $S$  and the tree  $D$  are uniquely defined by  $G$ . We shall refer to  $D$  as *the decomposition tree* of  $G$ .

**Theorem 8** *Let  $G$  be a 3-connected plane triangulation. Suppose that  $G$  has a decomposition tree  $D$  of maximum degree at most three. Let  $H$  be a piece of  $G$  corresponding to a vertex of  $D$  of degree at most two,  $T$  be a facial cycle of both  $H$  and  $G$ , and  $V(T) = \{u, v, w\}$ . Then  $G$  has a Hamilton cycle through  $uv$  and  $uw$ .*

**Proof.** We proceed by induction on  $|V(D)|$ . If  $D$  has only one vertex then the result follows from Theorem 7. Hence we may assume that  $D$  has at least two vertices. To simplify notation we shall suppose that  $H$  has degree two in  $D$ . (A similar but simpler proof holds if  $H$  has degree one.) Let  $T_1$  and  $T_2$  be the marker triangles contained in  $H$ . By Theorem 7,  $H$  has a Hamilton cycle  $C$  which contains distinct edges  $uv$ ,  $uw$ ,  $f_1 \in E(T_1)$  and  $f_2 \in E(T_2)$ .

For  $1 \leq i \leq 2$ , let  $V(T_i) = \{u_i, v_i, w_i\}$  where  $f_i = v_i w_i$ . Let the pieces of  $G$  separated by  $T_i$  be  $G_i, G'_i$  where  $G_i \cap H = T_i = G_i \cap G'_i$ . Let  $D_1$  and  $D_2$  be the two components obtained by deleting the vertex  $H$  from the decomposition tree  $D$ , labeled in such a way that  $T_i$  is contained in a piece  $H_i$  of  $G$  which is a vertex of  $D_i$ . Then  $D_i$  is the decomposition tree for

$G_i$ , and  $H_i$  has degree at most two in  $D_i$ . Thus we may apply induction to  $G_i$  and find a Hamilton cycle  $C_i$  in  $G_i$  through  $u_i v_i$  and  $u_i w_i$ . Then  $(C - \{v_1 w_1, v_2 w_2\}) \cup (C_1 - u_1) \cup (C_2 - u_2)$  is the required Hamilton cycle in  $G$ . ■

The hypothesis of Theorem 8 that  $D$  has maximum degree at most three cannot be weakened to maximum degree at most four. To see this we define a sequence of graphs recursively by putting  $G_1 = K_4$  and, for  $i \geq 1$ , letting  $G_{i+1}$  be the graph obtained from  $G_i$  by inserting a new vertex into each face of  $G_i$  then joining the new vertex to every vertex incident with the face. Then  $G_i$  has a decomposition tree of maximum degree four for  $i \geq 2$ , and is non-hamiltonian for  $i \geq 3$ .

One may hope to extend Theorem 8 to an arbitrary 3-connected planar graph  $G$ . We can define a decomposition of  $G$  into 4-connected pieces in a similar way to the above decomposition for triangulations, the only difference being that, when we separate  $G$  along a 3-vertex cut, we add three new edges between the vertices of the cut on each side of the separation, to form the ‘marker triangle’. One drawback of this approach is that the decomposition need no longer be unique. The main drawback, however, is that the obvious extension of Theorem 8 is false. To see this consider the Herschel graph  $H$  of Figure 1. If we separate  $H$  along the three 3-cuts  $\{2, 4, 9\}$ ,  $\{1, 6, 11\}$ ,  $\{3, 8, 10\}$  we obtain a decomposition of  $H$  into four pieces,  $G_1, G_2, G_3, G_4$ , where  $G_1$  and  $G_4$  are isomorphic to  $K_4$ , and  $G_2$  and  $G_3$  are isomorphic to the octahedron. The corresponding decomposition tree is the path  $G_1 G_2 G_3 G_4$  of length three, which has maximum degree two. The graph  $H$  is not hamiltonian, however, since it is bipartite and has an odd number of vertices.

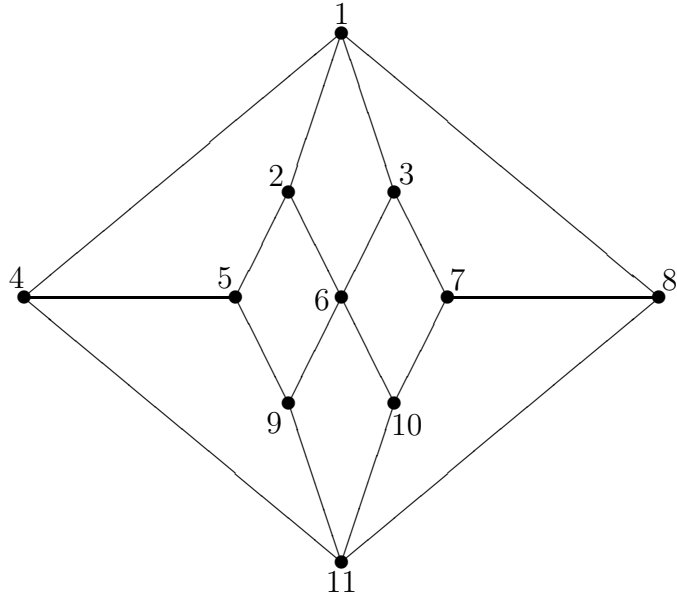


Figure 1

**Acknowledgment** This research was carried out during the workshop on paths and cycles held at Rutgers in July 1998. The authors thank DIMACS for its financial support during the workshop. We also thank the referee for very helpful suggestions and for pointing out the exceptional case (i) in Lemma 5(b).

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