Contractible Subgraphs in k-Connected Graphs*

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Abstract

For a graph G we define a graph $T(G)$ whose vertices are the triangles in G and two vertices of $T(G)$ are adjacent if their corresponding triangles in G share an edge. Kawarabayashi showed that if G is a k-connected graph and $T(G)$ contains no edge then G admits a k-contractible clique of size at most 3, generalizing an earlier result of Thomassen. In this paper, we further generalize Kawarabayashi's result by showing that if G is k-connected and the maximum degree of $T(G)$ is at most 1, then G admits a k-contractible clique of size at most 3 or there exist independent edges e and f of G such that e and f are contained in triangles sharing an edge and $G/e/f$ is k-connected.

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1 Introduction

Given a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively, and $\Delta(G)$ denotes the maximum degree of a vertex in G. An edge e (respectively, a subgraph K) in a k-connected graph G is *k-contractible* if the graph obtained from G by contracting e (respectively, K), denoted by G/e (respectively, G/K), is still k-connected. For two edges e, f of G, we use $G/e/f$ to denote the graph obtained from G by first contracting e and then contracting f.

Tutte [10] showed that K_4 is the only 3-connected graph which does not admit any 3-contractible edge. Fontet [4] and Martinov [8] proved independently that if a 4-connected graph contains no 4-contractible edge then it is the square of a cycle of length at least 5 or it is the line graph of a cyclically 4-edge-connected cubic graph.

For $k \geq 5$, it is seemingly difficult to characterize those k-connected graphs that do not contain any k-contractible edge. However, Thomassen [9] showed that if a k-connected graph is triangle-free then it contains a k-contractible edge. In fact, Egawa, Enomoto, and Saito [3] showed that every k-connected triangle-free graph G contains at least $\min\{|V(G)| + \frac{3}{2}k^2 - 3k, |E(G)|\}$ k-contractible edges.

Let K_4^- denote the graph obtained from K_4 by deleting an edge. Recently, Kawarabayashi [5] proved that when k is odd then every k-connected K_4^--free graph contains a k-contractible edge. Kawarabayashi also observed that the same result doesn't hold when k is even. However, he proved in [6] that every k-connected K_4^--free graph contains a k-contractible clique of size at most 3. Recall that a clique in a graph is a maximal complete subgraph.

It appears that the existence of a k-contractible subgraph is related to the number of triangles containing a common edge. Also several results are obtained in [1] concerning the existence of contractible edges by forbidding certain subgraphs involving triangles. Therefore, for a given graph G, we define a new graph $T(G)$ whose vertices are triangles in G, and two vertices of $T(G)$ are adjacent if their corresponding triangles in G share an edge. We are interested in those conditions on $T(G)$ which guarantee the existence of small k-contractible subgraphs in G.

The above mentioned result of Thomassen [9] is equivalent to the statement that if G is k-connected and $T(G) = \emptyset$ then G contains a k-contractible edge. Also, Kawarabayashi's result in [6] can be stated as follows: If G is a k-connected graph and $E(T(G)) = \emptyset$ then G contains a k-contractible clique of size at most 3. The squares of cycles of length at least 5 show that when $E(T(G)) \neq \emptyset$ it is possible that G contains neither a k-contractible edge nor a k-contractible triangle. The main result of this paper generalizes the results of Thomassen and Kawarabayashi.

(1.1) Theorem. *Let $k \geq 5$ be an integer, and let G be a k-connected graph such that $\Delta(T(G)) \leq 1$. Then one of the following holds.*

(i) G contains a k-contractible clique of size at most 3.

(ii) There exist two independent edges e and f of G such that $G[V(e) \cup V(f)]$ is isomorphic to K_4^- and $G/e/f$ is k-connected.

It would be interesting to know whether Theorem (1.1) holds when one relaxes the condition on $\Delta(T(G))$. Note that a result of Kawarabayashi [6] states that if a 4-connected graph contains neither a 4-contractible edge nor a 4-contractible triangle then it is the square of a cycle of length at least 5. We see that if G is the square of a cycle of length at least 7 then G has two independent edges e, f such that $G[V(e) \cup V(f)]$ is isomorphic to K_4^- and $G/e/f$ is 4-connected.

This means that 4-connected graphs can be reduced to K_5 or the square of a 6-cycle by a sequence of contractions of edges, triangles, and independent edges. In fact, Kriesell showed in [7] that 4-connected graphs can be reduced to K_5 or the square of a 6-cycle by a sequence of contractions of one edge or two edges.

Since loops and multiple edges do not affect connectivity, we only consider simple graphs. A component consisting of only one vertex is said to be *trivial*, and a *nontrivial* component is a component which is not trivial. Let G be a graph. For any $x \in V(G)$, $N_G(x)$ denotes the neighborhood of x in G, and let $d_G(x) = |N_G(x)|$. We use $\delta(G)$ to denote the minimum degree of a vertex in G. In addition, for a subset S of $V(G)$, we let $N_G(S) = (\bigcup_{v \in S} N_G(v)) - S$. For any subgraph H of G, we write $N_G(H)$ instead of $N_G(V(H))$.

An edge is said to be *lonely* if it is not contained in any triangle. A triangle is said to be *isolated* if it shares no edge with any other triangle. (Therefore, an isolated triangle corresponds to an isolated vertex of $T(G)$.) Two independent edges e, f of a graph G are said to be *cohesive* if e and f are contained in a subgraph of G isomorphic to K_4^-, and neither e nor f is contained in two triangles.

We organize this paper as follows. In Section 2, we define cuts associated with lonely edges, isolated triangles, and cohesive edges. We then prove a few results about these cuts and their associated components. In Section 3, we study how such cuts and their associated components can interact with each other. The results are then used in Section 4 to complete the proof of Theorem (1.1).

2 Cuts and components

In this section we define a collection of cuts, and investigate their associated components.

For an edge e, we use $V(e)$ to denote the set of vertices incident with e. Let G be a k-connected graph, and let e be an edge of G. Define $\mathcal{C}_e(G) := \{S : S \text{ is a } k\text{-cut in } G \text{ and } V(e) \subseteq S\}$. The following proposition is easy to verify.

(2.1) Proposition. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and let $e \in E(G)$. Then e is k-contractible if and only if $\mathcal{C}_e(G) = \emptyset$.*

Let G be a k-connected graph and let T be a triangle. Note that G/T is not k-connected if and only if there is a cut S' in G/T such that $|S'| < k$. Let t denote the vertex of G/T which corresponds to the contraction of T. Since G is k-connected, S' must contain t, and $S := (S' - \{t\}) \cup V(T)$ is a cut in G of size k or $k+1$. Thus T is k-contractible if and only if no cut in G containing $V(T)$ has size k or $k+1$. For convenience, we define $\mathcal{C}_T^k(G) := \{S : S \text{ is a } k\text{-cut in } G \text{ containing } V(T)\}$ and $\mathcal{C}_T^{k+1}(G) := \{S : S \text{ is a } (k+1)\text{-cut in } G \text{ containing } V(T)\}$. If $\mathcal{C}_T^k(G) \neq \emptyset$ then let $\mathcal{C}_T(G) = \mathcal{C}_T^k(G)$. If $\mathcal{C}_T^k(G) = \emptyset$ then let $\mathcal{C}_T(G) = \mathcal{C}_T^{k+1}(G)$. Note that the cuts in $\mathcal{C}_T^{k+1}(G)$ are not necessarily minimal, but the cuts in $\mathcal{C}_T(G)$ are minimal. We have the following observation.

(2.2) Proposition. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and let T be a triangle in G. Then T is k-contractible if and only if $\mathcal{C}_T(G) = \emptyset$.*

We now describe a collection of cuts associated with two independent edges. Let e and f be two independent edges of G. We wish to derive necessary and sufficient conditions for $G/e/f$ to be k-connected. Since G is k-connected, $G/e/f$ is $(k-2)$-connected. So $G/e/f$ is k-connected if and only if $G/e/f$ has neither $(k-2)$-cut nor $(k-1)$-cut. Therefore, we shall discuss when $G/e/f$ has a $(k-2)$-cut or $(k-1)$-cut. For convenience, denote by e^* and f^* the vertices of $G/e/f$ corresponding to the contractions of e and f, respectively.

Suppose $G/e/f$ has a $(k-2)$-cut S^*. Then since G is k-connected, $e^*, f^* \in S^*$. Hence $S := (S^* - \{e^*, f^*\}) \cup V(e) \cup V(f)$ is a k-cut in G. That is, G has a k-cut containing $V(e) \cup V(f)$. Let $\mathcal{C}^k_{e,f,1}(G) := \{S : S \text{ is a } k\text{-cut in } G \text{ containing } V(e) \cup V(f)\}$.

Suppose $G/e/f$ has a $(k-1)$-cut S^* such that $e^* \in S^*$ and $f^* \notin S^*$ or $f^* \in S^*$ and $e^* \notin S^*$. Then $S := (S^* - \{e^*\}) \cup V(e)$, or $S := (S^* - \{f^*\}) \cup V(f)$, is a k-cut in G. That is, G has a k-cut S such that $V(e) \subseteq S$ and $V(f) \cap S = \emptyset$, or $V(f) \subseteq S$ and $V(e) \cap S = \emptyset$. So we define $\mathcal{C}^k_{e,f,2}(G) := \{S : S \text{ is a } k\text{-cut in } G \text{ such that } V(e) \subseteq S \text{ and } V(f) \cap S = \emptyset, \text{ or } V(f) \subseteq S \text{ and } V(e) \cap S = \emptyset\}$.

Now assume $G/e/f$ has a minimal $(k-1)$-cut S^* such that $\{e^*, f^*\} \subseteq S^*$. Then $S' := (S^* - \{e^*, f^*\}) \cup V(e) \cup V(f)$ is a $(k+1)$-cut in G. We consider two cases: S' is not a minimal cut in G, and S' is a minimal cut in G.

Suppose S' is not a minimal cut in G. Then since G is k-connected and S^* is a minimal $(k-1)$-cut in $G/e/f$, there is a vertex $u \in V(e) \cup V(f)$ such that $S := S' - \{u\}$ is a k-cut in G. Clearly, if $\{u\}$ is a component of $G - S$ then $G - S$ has at least three components and u is adjacent to all vertices in $(V(e) \cup V(f)) - \{u\}$. Hence, we define $\mathcal{C}^k_{e,f,3}(G) := \{S : S$ is a k-cut in G, $|S \cap (V(e) \cup V(f))| = 3$, and either $G - S$ has at least three components or $(V(e) \cup V(f)) - S$ is not a component of $G - S\}$.

Now suppose that S' is a minimal cut in G. Then $S := S'$ is a minimal $(k+1)$-cut in G containing $V(e) \cup V(f)$. So we define $\mathcal{C}^{k+1}_{e,f}(G) := \{S : S \text{ is a minimal } (k+1)\text{-cut in } G \text{ such that } V(e) \cup V(f) \subseteq S\}$.

In view of the above discussions, we let $\mathcal{C}^k_{e,f}(G) := \mathcal{C}^k_{e,f,1}(G) \cup \mathcal{C}^k_{e,f,2}(G) \cup \mathcal{C}^k_{e,f,3}(G)$. We define $\mathcal{C}_{e,f}(G) = \mathcal{C}^k_{e,f}(G)$ if $\mathcal{C}^k_{e,f}(G) \neq \emptyset$; and $\mathcal{C}_{e,f}(G) = \mathcal{C}^{k+1}_{e,f}(G)$ otherwise. As a consequence of the above discussion, we have the following result.

(2.3) Proposition. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and let e, f be independent edges in G. Then $G/e/f$ is k-connected if and only if $\mathcal{C}_{e,f}(G) = \emptyset$.*

The following observation will be helpful.

(2.4) Proposition. *Let G be a k-connected graph. Then the following statements hold.*

(i) If $S \in \mathcal{C}_e(G)$ for some lonely edge e in G, then $|S - V(e)| = k - 2$.

(ii) If $S \in \mathcal{C}_T(G)$ for some isolated triangle T in G, then $|S - V(T)| = k - 3$ when $S \in \mathcal{C}^k_T(G)$, and $|S - V(T)| = k - 2$ when $S \in \mathcal{C}^{k+1}_T(G)$.

(iii) If $S \in \mathcal{C}_{e,f}(G)$ for some cohesive edges e and f in G, then $|S - (V(e) \cup V(f))| = k - 4$ when $S \in \mathcal{C}^k_{e,f,1}(G)$, $|S - (V(e) \cup V(f))| = k - 2$ when $S \in \mathcal{C}^k_{e,f,2}(G)$, and $|S - (V(e) \cup V(f))| = k - 3$ when $S \in \mathcal{C}^k_{e,f,3}(G) \cup \mathcal{C}^{k+1}_{e,f}(G)$.

Since we shall be dealing with lonely edges, isolated triangles, and cohesive edges, we define $\mathcal{C}(G) = (\bigcup_e \mathcal{C}_e(G)) \cup (\bigcup_T \mathcal{C}_T(G)) \cup (\bigcup_{e,f} \mathcal{C}_{e,f}(G))$, where the first union is taken over all lonely edges, the second union is taken over all isolated triangles, and the third union is taken over all cohesive edges. For $S \in \mathcal{C}(G)$, the following result shows when $G - S$ has a trivial component.

(2.5) Lemma. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and let $S \in \mathcal{C}(G)$. Then $G-S$ has at most one trivial component, and if $G-S$ has a trivial component then $S \in \mathcal{C}^k_{e,f,3}(G)$ for some cohesive edges e and f in G and $G-S$ has at least three components.*

Proof. Suppose that $G-S$ has a trivial component H, consisting of only one vertex, say x. Let $A = V(e)$ if $S \in \mathcal{C}_e(G)$ for some lonely edge e in G; $A = V(T)$ if $S \in \mathcal{C}_T(G)$ for some isolated triangle T in G; and $A = V(e) \cup V(f)$ if $S \in \mathcal{C}_{e,f}(G)$ for some cohesive edges e and f in G. Note that $|N_G(x) \cap A| \leq 1$ when $A = V(e)$ or $A = V(T)$, and $|N_G(x) \cap A| \leq 2$ when $A = V(e) \cup V(f)$.

First, assume $S \in \mathcal{C}_e(G)$ or $S \in \mathcal{C}_T(G)$. Then $N_G(x) \subseteq S$ and $|N_G(x) \cap A| \leq 1$. Therefore, it follows from (i) and (ii) of Proposition (2.4) that $|N_G(x)| \leq |S-A| + |N_G(x) \cap A| \leq (k-2)+1 = k-1$. This contradicts the assumption that G is k-connected.

Now assume $S \in \mathcal{C}^k_{e,f,1}(G)$ or $S \in \mathcal{C}^{k+1}_{e,f}(G)$. Then $|N_G(x) \cap A| \leq 2$ and it follows from (iii) of Proposition (2.4) that $|N_G(x)| \leq |S-A| + |N_G(x) \cap A| \leq (k-3)+2 = k-1$, contradicting the assumption that G is k-connected.

Suppose $S \in \mathcal{C}^k_{e,f,2}(G)$, and by symmetry assume that $V(e) \subseteq S$ and $V(f) \cap S = \emptyset$. Clearly, $x \notin V(f)$. Hence, since e is contained in only one triangle, $|N_G(x) \cap A| \leq 1$. Again it follows from (iii) of Proposition (2.4) that $|N_G(x)| \leq |S-A| + |N_G(x) \cap A| \leq (k-2)+1 = k-1$, a contradiction.

Finally, let $S \in \mathcal{C}^k_{e,f,3}(G)$. By definition, $|S \cap (V(e) \cup V(f))| = 3$. Therefore, $V(e) \subseteq S$ or $V(f) \subseteq S$. Without loss of generality, we may assume $V(e) \subseteq S$. Suppose $G-S$ has another trivial component consisting of only one vertex, say y. Since G is k-connected and $|S| = k$, we have $N_G(x) = S = N_G(y)$. However, this implies that e is contained in two triangles, a contradiction (since e and f are cohesive). Hence, $G-S$ has exactly one trivial component. It follows from the definition of $\mathcal{C}^k_{e,f,3}(G)$ that $G-S$ has at least three components. □

In later arguments, we need to decide when a certain cut in G belongs to $\mathcal{C}(G)$. Hence, we prove the the following lemma.

(2.6) Lemma. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and let $S \in \mathcal{C}(G)$. Let $A := V(e)$ when $S \in \mathcal{C}_e(G)$ for some lonely edge e in G; $A = V(T)$ when $S \in \mathcal{C}_T(G)$ for some isolated triangle T in G; and $A = V(e) \cup V(f)$ when $S \in \mathcal{C}_{e,f}(G)$ for some cohesive edges e and f in G. Suppose R is a cut in G such that $|R| \leq |S|$ and $A \cap S \subseteq R$. Then $R \in \mathcal{C}_e(G) \cup \mathcal{C}_T(G) \cup \mathcal{C}_{e,f}(G)$, unless $S \in \mathcal{C}^k_{e,f,2}(G)$ or $S \in \mathcal{C}^k_{e,f,3}(G)$, and $G-R$ has exactly two components one of which is trivial.*

Proof. Clearly, if $S \in \mathcal{C}_e(G)$ or $S \in \mathcal{C}_T(G)$ or $S \in \mathcal{C}^k_{e,f,1}(G)$ then $A \subseteq S$ and $|S| = k$, and therefore $A \subseteq R$ and $k \leq |R| \leq |S| = k$, which shows $R \in \mathcal{C}_e(G) \cup \mathcal{C}_T(G) \cup \mathcal{C}_{e,f}(G)$.

Now assume that $S \in \mathcal{C}^k_{e,f,2}(G)$ or $S \in \mathcal{C}^k_{e,f,3}(G)$. Then $|S| = k$, and so, $|R| = k$ (since $k \leq |R| \leq |S|$). Because $A \cap S \subseteq R$, we have $|A \cap R| \geq 2$. If $|A \cap R| = 2$ then $S \in \mathcal{C}^k_{e,f,2}(G)$, and so, $A \cap R = V(e)$ and $R \cap V(f) = \emptyset$, or $A \cap R = V(f)$ and $R \cap V(e) = \emptyset$; which shows that $R \in \mathcal{C}^k_{e,f,2}(G)$, and so, $R \in \mathcal{C}_{e,f}(G)$. We may therefore assume $|A \cap R| \geq 3$. If $A \subseteq R$ then we see that $R \in \mathcal{C}^k_{e,f,1}(G)$, and so, $R \in \mathcal{C}_{e,f}(G)$. So we may further assume $|A \cap R| = 3$. Since $|R| = |S| = k$ we see from the definition of $\mathcal{C}^k_{e,f,3}(G)$ that either $R \in \mathcal{C}^k_{e,f,3}(G) \subseteq \mathcal{C}_{e,f}(G)$ or $G-R$ has exactly two components one of which is trivial.

Finally, assume $S \in \mathcal{C}^{k+1}_{e,f}(G)$. Then $|S| = k+1$, $A = V(e) \cup V(f) \subseteq S$, and $\mathcal{C}^k_{e,f}(G) = \emptyset$ (because $S \in \mathcal{C}(G)$). Since $A \cap S \subseteq R$, we have $A \subseteq R$. Because $k \leq |R| \leq |S| = k+1$ and

$\mathcal{C}^k_{e,f}(G)=\emptyset$, $|R|=k+1$. If R is a minimal cut in G, then $R\in\mathcal{C}^{k+1}_{e,f}(G)\subseteq\mathcal{C}_{e,f}(G)$ (since $S\in\mathcal{C}(G)$ and $|S|=k+1$). So we may assume that R is not a minimal cut. Then there is a vertex $x\in R$ such that $R-\{x\}$ is a cut in G of size k. If $x\notin A$ then $R-\{x\}\in\mathcal{C}^k_{e,f,1}(G)\subseteq\mathcal{C}^k_{e,f}(G)$, a contradiction. So $x\in A$. If $\{x\}$ is a component of $G-(R-\{x\})$, then $G-(R-\{x\})$ has at least three components, and so, $R-\{x\}\in\mathcal{C}^k_{e,f,3}(G)\subseteq\mathcal{C}^k_{e,f}(G)$, a contradiction. Hence $\{x\}$ is contained in a nontrivial component of $G-(R-\{x\})$. If $G-(R-\{x\})$ has no trivial component, then again $R-\{x\}\in\mathcal{C}^k_{e,f,3}(G)\subseteq\mathcal{C}^k_{e,f}(G)$, a contradiction. So $G-(R-\{x\})$ has a trivial component consisting of only one vertex, say y. Without loss of generality, assume $x\in V(f)$. Then $V(e)\subseteq R-\{x\}$. Since $|R-\{x\}|=k$, we have $N_G(y)=R-\{x\}$, and this implies that e is contained in at least two triangles, a contradiction. □

Next, we prove a lemma concerning the size of components associated with cuts in $\mathcal{C}(G)$.

(2.7) Lemma. *Let $k\geq 5$ be an integer, and let G be a k-connected graph such that $\Delta(T(G))\leq 1$. Then for any $S\in\mathcal{C}(G)$ and any component H of $G-S$, $|V(H)|\geq k-2$ unless $S\in\mathcal{C}^k_{e,f,3}(G)$ and $V(H)=(V(e)\cup V(f))-S$ for some cohesive edges e and f in G.*

Proof. Let H be a component of $G-S$ and assume that if $S\in\mathcal{C}^k_{e,f,3}(G)$ for some cohesive edges e and f in G then $V(H)\neq(V(e)\cup V(f))-S$. Note that for each edge xy of G, since $\Delta(T(G))\leq 1$, $|N_G(x)\cap N_G(y)|\leq 2$. Therefore, because G is k-connected, $|N_G(x)\cup N_G(y)|\geq 2k-2$.

Case 1. $S\in\mathcal{C}_e(G)$ for some lonely edge e, or $S\in\mathcal{C}_T(G)$ for some isolated triangle T.

Let $A:=V(e)$ if $S\in\mathcal{C}_e(G)$ for some lonely edge e in G; and let $A:=V(T)$ if $S\in\mathcal{C}_T(G)$ for some isolated triangle T in G. Let $x\in V(H)$. Since e is lonely and T is isolated, $|N_G(x)\cap A|\leq 1$. Hence, $|N_G(x)\cap S|\leq|S-A|+|N_G(x)\cap A|\leq k-1$ (by (i) and (ii) of Proposition (2.4)). Therefore, because G is k-connected, x has a neighbor in H. Let xy be an edge of H. Then $|(N_G(x)\cup N_G(y))\cap A|\leq 2$, and so, $|(N_G(x)\cup N_G(y))\cap S|\leq|S-A|+|(N_G(x)\cup N_G(y))\cap A|\leq k$ (by (i) and (ii) of Proposition (2.4)). Hence $|V(H)|\geq|N_G(x)\cup N_G(y)|-|(N_G(x)\cup N_G(y))\cap S|\geq(2k-2)-k=k-2$.

Case 2. $S\in\mathcal{C}_{e,f}(G)$ for some cohesive edges e and f in G.

First, assume $S\in\mathcal{C}^k_{e,f,1}(G)\cup\mathcal{C}^{k+1}_{e,f}(G)$. Then $V(e)\cup V(f)\subseteq S$. Let x be a vertex of H. Since $G[V(e)\cup V(f)]$ is isomorphic to K_4^- and neither e nor f is contained in two triangles, $|N_G(x)\cap(V(e)\cup V(f))|\leq 2$. Hence $|N_G(x)\cap S|\leq(k-3)+2\leq k-1$ (by (iii) of Proposition (2.4)). Therefore, since G is k-connected, x must have a neighbor in H. Let xy be an edge of H. By a similar argument as for x, we have $|N_G(y)\cap(V(e)\cup V(f))|\leq 2$. If $|(N_G(x)\cup N_G(y))\cap(V(e)\cup V(f))|\leq 3$ then $|(N_G(x)\cup N_G(y))\cap S|\leq k$ (by (iii) of Proposition (2.4)), and hence, $|V(H)|\geq|N_G(x)\cup N_G(y)|-|(N_G(x)\cup N_G(y))\cap S|\geq(2k-2)-k=k-2$. So we may assume $|(N_G(x)\cup N_G(y))\cap(V(e)\cup V(f))|=4$. Then $|N_G(x)\cap(V(e)\cup V(f))|=2=|N_G(y)\cap(V(e)\cup V(f))|$ and $N_G(x)\cap N_G(y)\cap(V(e)\cup V(f))=\emptyset$. It is easy to check that the triangles in $G[\{x,y\}\cup V(e)\cup V(f)]$ show $\Delta(T(G))\geq 2$, a contradiction.

Now assume that $S\in\mathcal{C}^k_{e,f,2}(G)$. By symmetry, we may assume $V(e)\subseteq S$ and $V(f)\cap S=\emptyset$. Let x be a vertex of H. If $V(e)\subseteq N_G(x)$ then $x\in V(f)$ (since e is not in two triangles), and so, x must have a neighbor in H. If $V(e)\not\subseteq N_G(x)$, then $|N_G(x)\cap S|\leq k-1$, and hence, x has a neighbor in H. In any case, let xy be an edge of H. Then $|V(H)|\geq|N_G(x)\cup N_G(y)|-|S|\geq(2k-2)-k=k-2$.

Finally, assume that $S\in\mathcal{C}^k_{e,f,3}(G)$. Then $V(H)\neq(V(e)\cup V(f))-S$ (by assumption). Let u be the only vertex in $(V(e)\cup V(f))-S$. Suppose $u\in V(H)$. Then $|V(H)|\geq 2$. Let $xy\in E(H)$.

Then $|V(H)| \geq |N_G(x) \cup N_G(y)| - |S| \geq (2k-2) - k = k-2$. Now suppose $u \notin V(H)$. Let $x \in V(H)$. Since neither e nor f is contained in two triangles, $|N_G(x) \cap (V(e) \cup V(f))| \leq 2$. Therefore, $|N_G(x) \cap S| \leq k-1$ (by (iii) of Proposition (2.4)), and so, x must have a neighbor in H. Let xy be an edge of H. Again, $|V(H)| \geq |N_G(x) \cup N_G(y)| - |S| \geq (2k-2) - k = k-2$. □

Our final result in this section says that for $k \in \{5, 6\}$, the bound $k-2$ in Lemma (2.7) can be improved to $k-1$. Since the proof we have is similar to that of Lemma (2.7) and involves detailed case analysis, it is omitted.

(2.8) Lemma. *Let G be a k-connected graph, where $k \in \{5, 6\}$, such that $\Delta(T(G)) \leq 1$. Then for any $S \in \mathcal{C}(G)$ and any component H of $G - S$, $|V(H)| \geq k-1$, unless $S \in \mathcal{C}^k_{e,f,3}(G)$ and $V(H) = (V(e) \cup V(f)) - S$ for some cohesive edges e and f in G.*

3 Components of minimum size

In this section, we study the minimum size components associated with cuts in $\mathcal{C}(G)$ for a k-connected graph G. First, we make the following observation.

(3.1) Lemma. *Let $k \geq 5$ be an integer, let G be a k-connected graph such that $\Delta(T(G)) \leq 1$, and let uv be an edge of G. Then one of the following holds.*

(i) uv is lonely.

(ii) uv is contained in an isolated triangle.

(iii) There exist cohesive edges e, f of G such that $\{u, v\} \subseteq V(e) \cup V(f)$.

Proof. Assume (i) and (ii) fail. Then uv is contained in a triangle which is not isolated.

First, assume that uv is contained in only one triangle, say $uvwu$. Then $uvwu$ is not isolated. By symmetry, we may assume that vw is contained in another triangle, say $vwxv$. If xw is not contained in two triangles, then $e = uv$ and $f = xw$ show that (iii) holds. So assume xw is contained in another triangle, say $xwyx$. Since uv is contained in only one triangle, $y \neq u$. Hence the triangle $vwxv$ corresponds to a vertex of $T(G)$ with degree at least 2, a contradiction.

Now assume that uv is contained in two triangles: $uvwu$ and $uvxu$. If neither uw nor xv is contained in two triangles, then $e = uw$ and $f = vx$ show that (iii) holds. So by symmetry, we may assume that uw is contained in a triangle $uwyu$. Now, the triangle $uvwu$ corresponds to a vertex of $T(G)$ with degree at least 2, a contradiction. □

Next we prove an important lemma describing how two cuts in $\mathcal{C}(G)$ can interact.

(3.2) Lemma. *Let $k \geq 5$ be an integer, and let G be a k-connected graph such that $\Delta(T(G)) \leq 1$. Let $S \in \mathcal{C}(G)$ and let H be a nontrivial component of $G - S$. Then for any $S' \in \mathcal{C}(G)$, $V(H) \not\subseteq S'$.*

Proof. Suppose for a contradiction that $V(H) \subseteq S'$ for some $S' \in \mathcal{C}(G)$. Let H' be a nontrivial component of $G - S'$, and let $W' = G - (S' \cup V(H'))$. Also, let $W = G - (S \cup V(H))$. By Lemma (2.5), both W and W' have a nontrivial component, and hence, Lemma (2.7) and Lemma (2.8) may be applied to W and W'.

Denote $V(H \cap H'), V(H) \cap S'$ and $V(H \cap W')$ by H_1, H_2 and H_3, respectively. Denote $S \cap V(H'), S \cap S'$ and $S \cap V(W')$ by Q_1, Q_2 and Q_3, respectively. Denote $V(W \cap H'), V(W) \cap S'$ and $V(W \cap W')$ by W_1, W_2 and W_3, respectively. See Figure 1.

	H	S	W
H'	H_1	Q_1	W_1
S'	H_2	Q_2	W_2
W'	H_3	Q_3	W_3

Figure 1: Cuts and components

Since $V(H) \subseteq S'$, $H_1 = H_3 = \emptyset$ and $H_2 = V(H)$. Therefore, $|H_2| \geq k-2$ when $k \geq 7$ (by Lemma (2.7)) and $|H_2| \geq k-1$ when $k \in \{5,6\}$ (by Lemma (2.8)). Note that $|Q_2 \cup W_2| = |S'| - |H_2|$. Hence, since $|S'| \leq k+1$, we have $|Q_2 \cup W_2| \leq 3$ when $k \geq 7$, and $|Q_2 \cup W_2| \leq 2$ when $k \in \{5,6\}$.

We claim that $W_1 \neq \emptyset$ or $W_3 \neq \emptyset$. For otherwise, $|W_2| = |V(W)| \geq k-2 \geq 5$ when $k \geq 7$ (by Lemma (2.7)) and $|W_2| = |V(W)| \geq k-1 \geq 4$ when $k \in \{5,6\}$ (by Lemma (2.8)), a contradiction.

By symmetry, we may assume that $W_1 \neq \emptyset$. Then $Q_1 \cup Q_2 \cup W_2$ is a cut in G. Since G is k-connected, $|Q_1 \cup Q_2 \cup W_2| \geq k$, and so, $|Q_1| \geq k - |Q_2 \cup W_2|$. Hence, $|Q_1| \geq k-3$ when $k \geq 7$, and $|Q_1| \geq k-2$ when $k \in \{5,6\}$. Therefore, because $|Q_2 \cup Q_3| = |S| - |Q_1|$, $|Q_2 \cup Q_3| \leq 4$ when $k \geq 7$, and $|Q_2 \cup Q_3| \leq 3$ when $k \in \{5,6\}$.

Suppose $W_3 = \emptyset$. Then $Q_3 = V(W')$. Hence, $|Q_3| \geq k-2 \geq 5$ when $k \geq 7$ (by Lemma (2.7)), and $|Q_3| \geq k-1 \geq 4$ when $k \in \{5,6\}$ (by Lemma (2.8)). This shows $|Q_2 \cup Q_3| \geq 5$ when $k \geq 7$ and $|Q_2 \cup Q_3| \geq 4$ when $k \in \{5,6\}$, a contradiction.

So $W_3 \neq \emptyset$. Then $Q_3 \cup Q_2 \cup W_2$ is a cut in G. Since G is k-connected, $|Q_3 \cup Q_2 \cup W_2| \geq k$. On the other hand, we know that $|Q_2 \cup W_2| + |Q_2 \cup Q_3| \leq 7$ when $k \geq 7$, and $|Q_2 \cup W_2| + |Q_2 \cup Q_3| \leq 5$ when $k \in \{5,6\}$. So $Q_2 = \emptyset$, and either $k = 5$, $|Q_2 \cup W_2| = 2$, and $|Q_2 \cup Q_3| = 3$, or $k = 7$, $|Q_2 \cup W_2| = 3$, and $|Q_2 \cup Q_3| = 4$.

Suppose $k = 5$, $|Q_2 \cup W_2| = 2$, and $|Q_2 \cup Q_3| = 3$. This, together with $|Q_1| \geq k-2 = 3$ and $Q_2 = \emptyset$, implies that $|Q_1| = 3 = |Q_3|$. Thus $|S| = k+1$, $S \in \mathcal{C}_T(G)$ for some isolated triangle T, and either $V(T) \subseteq Q_1$ or $V(T) \subseteq Q_3$. This shows that either $Q_1 \cup Q_2 \cup W_2$ or $Q_2 \cup Q_3 \cup W_2$ belongs to $\mathcal{C}_T(G)$, a contradiction because both are 5-cuts while S is a 6-cut.

So $k = 7$, $|Q_2 \cup W_2| = 3$, and $|Q_2 \cup Q_3| = 4$. This, together with $|Q_1| \geq k-3 = 4$ and $Q_2 = \emptyset$, implies that $|S| = 8$ and $|Q_1| = 4 = |Q_3|$. Thus, $S \in \mathcal{C}_T^{k+1}(G)$ for some isolated triangle T or $S \in \mathcal{C}_{e,f}^{k+1}(G)$ for some cohesive edges e and f, and $V(T)$ or $V(e) \cup V(f)$ is contained in either Q_3 or Q_1. Since both $Q_2 \cup Q_3 \cup W_2$ and $Q_1 \cup Q_2 \cup W_2$ are k-cuts, we derive a contradiction to the fact that $S \in \mathcal{C}(G)$. □

In the remainder of this section, we study how cuts in $\mathcal{C}(G)$ (and associated components) interact when one is chosen to minimize a nontrivial component. First, we prove a technical lemma which will be used several times later to simplify proofs.

(3.3) Lemma. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and assume that $\Delta(T(G)) \leq 1$. Let $S \in \mathcal{C}(G)$ and a nontrivial component H of $G - S$ be chosen so that $|V(H)|$ is minimized. Let $uv \in E(H)$ and let $S' \in \mathcal{C}(G)$ such that $S' \in \mathcal{C}_{uv}(G)$ when uv is lonely, or $S' \in \mathcal{C}_{T'}(G)$ when uv is in an isolated triangle T', or $S' \in \mathcal{C}_{e',f'}(G)$ for some cohesive edges e', f' with $\{u, v\} \subseteq V(e') \cup V(f')$. Let H' be a nonempty union of components of $G - S'$ such that $H' \neq G - S'$. Assume $S' \cap V(H) \neq \emptyset$. Then $|(S \cap V(H')) \cup (S \cap S') \cup (V(H) \cap S')| \geq |S'| + 1$, or $V(H) \cap V(H') = \emptyset$, or $|S'| = k$ and $|V(H) \cap V(H')| = 1$.*

Proof. For convenience, let $H_1 := V(H) \cap V(H')$, $H_2 := V(H) \cap S'$, $Q_1 := S \cap V(H')$, $Q_2 := S \cap S'$, and $R := Q_1 \cup Q_2 \cup H_2$. Let $A' := \{u, v\}$ if $S' \in \mathcal{C}_{uv}(G)$; $A' := V(T')$ if $S' \in \mathcal{C}_{T'}(G)$; and $A' := V(e') \cup V(f')$ if $S' \in \mathcal{C}_{e',f'}(G)$. Note that $A' \cap S' \subseteq Q_2 \cup H_2 \subseteq R$.

If $|R| \geq |S'| + 1$ or $H_1 = \emptyset$ then the assertion of the lemma holds. So we may assume $|R| \leq |S'|$ and $H_1 \neq \emptyset$. Then R is a cut in G, and $k \leq |R| \leq |S'| \leq k + 1$.

First, assume $|S'| = k + 1$. Then $S' \in \mathcal{C}^{k+1}_{T'}(G)$ or $S' \in \mathcal{C}^{k+1}_{e',f'}(G)$. Therefore, $A' \subseteq S'$, and $A' = T'$ or $A' = V(e') \cup V(f')$. Hence, because $|R| \leq |S'|$ and $A' = A' \cap S' \subseteq R$, it follows from Lemma (2.6) that $R \in \mathcal{C}(G)$. If $|H_1| \geq 2$ then by Lemma (2.5) H_1 has a nontrivial component H_1', and hence, R and H_1' contradict the choice of S and H (because $S' \cap V(H) \neq \emptyset$). So $|H_1| = 1$, and let x be the only vertex of H_1. Then $N_G(x) \subseteq R$. Since $A' \subseteq R$ and because $A' = T'$ or $A' = V(e') \cup V(f')$, we see that $|N_G(x) \cap A'| \leq 1$ when $A' = T'$ and $|N_G(x) \cap A'| \leq 2$ when $A' = V(e') \cup V(f')$. This shows that $|N_G(x)| \leq k - 1$ (by (ii) and (iii) Proposition (2.4)), contradicting the k-connectivity of G.

Now assume $|S'| = k$. If $|H_1| = 1$ then the assertion of the lemma holds. So we may assume $|H_1| \geq 2$. Then $R \notin \mathcal{C}(G)$; for, otherwise, it follows from Lemmas (2.5) that H_1 has a nontrivial component, say H_1', and so, R and H_1' contradict the choice of S and H (because $S' \cap V(H) \neq \emptyset$). Hence $A' \not\subseteq R$, and so, $A' \not\subseteq S'$, which implies that $S' \in \mathcal{C}^k_{e',f',2}(G)$ or $S' \in \mathcal{C}^k_{e',f',3}(G)$. In particular, $A' = V(e') \cup V(f')$ and $|S' \cap A'| \geq 2$. Therefore, $|R \cap A'| \geq 2$. If $|R \cap A'| = 2$ then $|S \cap A'| = 2$, which implies $S' \in \mathcal{C}^k_{e',f',2}(G)$ and $R \in \mathcal{C}^k_{e',f',2}(G)$, and so, $R \in \mathcal{C}(G)$, a contradiction. So $|R \cap A'| = 3$. Because $R \notin \mathcal{C}(G)$ and by Lemma (2.6), $G - R$ has exactly two components one of which is trivial. However, this is impossible (since $|H_1| \geq 2$, $S' \cap V(H) \neq \emptyset$, $H' \neq G - S'$, and $H \neq G - S$). □

The next two results show that the cuts in $\mathcal{C}(G)$ that we shall work with have size $k + 1$. For convenience, if e and f are cohesive edges in a k-connected graph G such that $G/e/f$ is k-connected, then we say that G has a k-contractible pair of cohesive edges.

(3.4) Lemma. *Let $k \geq 5$ be an integer, let G be a k-connected graph, and assume that $\Delta(T(G)) \leq 1$. Let $S \in \mathcal{C}(G)$ and a nontrivial component H of $G - S$ be chosen so that $|V(H)|$ is minimized. Then $|S| = k + 1$, or G has a k-contractible clique or a k-contractible pair of cohesive edges.*

Proof. Suppose $|S| = k$. Let $W = G - (S \cup V(H))$. By Lemma (2.5), W contains a nontrivial component of $G - S$. Since H is nontrivial, let $uv \in E(H)$. By Lemma (3.1), let $A := \{u, v\}$ if uv is a lonely edge in G; $A := V(T)$ if uv is contained in an isolated triangle T in G; and $A := V(e) \cup V(f)$ if $\{u, v\} \subseteq V(e) \cup V(f)$ for some cohesive edges e and f in G. We may assume that $\mathcal{C}_e(G) \cup \mathcal{C}_T(G) \cup \mathcal{C}_{e,f}(G) \neq \emptyset$; for, otherwise, the assertion of the lemma holds by Lemmas (2.1), (2.2), and (2.3). Let $S' \in \mathcal{C}_{uv}(G) \cup \mathcal{C}_T(G) \cup \mathcal{C}_{e,f}(G)$.

We claim that we may assume $S' \cap V(H) \neq \emptyset$. This is true unless $S' \in \mathcal{C}^k_{e,f,2}(G)$ such that $V(e) = \{u,v\}$ or $V(f) = \{u,v\}$, and $\{u,v\} \cap S' = \emptyset$. In this exceptional case, we assume by symmetry $V(e) = \{u,v\}$, and let $V(f) = \{x,y\}$ such that $ux, vy, uy \in E(G)$. Then we see that ux and vy are cohesive edges in G. If $\mathcal{C}_{ux,vy}(G) = \emptyset$ then the assertion of the lemma holds. So we may assume $\mathcal{C}_{ux,vy}(G) \neq \emptyset$. Then we can choose $S' \in \mathcal{C}_{ux,vy}(G)$, and we see that $S' \cap V(H) \neq \emptyset$.

Let H' be a nontrivial component of $G - S'$, and let $W' = G - (S' \cup V(H'))$. By Lemma (2.5), W' contains a nontrivial component of $G - S'$. Denote $V(H \cap H'), V(H) \cap S'$ and $V(H \cap W')$ by H_1, H_2 and H_3, respectively. Denote $S \cap V(H'), S \cap S'$ and $S \cap V(W')$ by Q_1, Q_2 and Q_3, respectively. Denote $V(W \cap H'), V(W) \cap S'$ and $V(W \cap W')$ by W_1, W_2 and W_3, respectively. See Figure 1. Note that $A \cap S' \subseteq H_2 \cup Q_2$.

By Lemma (3.2), $H_1 \neq \emptyset \neq W_3$ or $H_3 \neq \emptyset \neq W_1$. By symmetry, we may assume that $H_1 \neq \emptyset \neq W_3$. Then $Q_1 \cup Q_2 \cup H_2$ and $Q_3 \cup Q_2 \cup W_2$ are cuts in G. Since G is k-connected, we have $|Q_3 \cup Q_2 \cup W_2| \geq k$.

Note that $|Q_1 \cup Q_2 \cup H_2| \leq |S'|$; for, otherwise, $k + |S'| = |S| + |S'| = |Q_1 \cup Q_2 \cup H_2| + |Q_2 \cup Q_3 \cup W_2| \geq |S'| + 1 + k$, a contradiction. Hence, by Lemma (3.3), $|S'| = k$ and $|H_1| = 1$. Therefore, $2k = |S| + |S'| = |Q_1 \cup Q_2 \cup H_2| + |Q_3 \cup Q_2 \cup W_2| \geq 2k$. This implies $|Q_1 \cup Q_2 \cup H_2| = k$ and $|Q_3 \cup Q_2 \cup W_2| = k$. Let $H_1 = \{x\}$. Note that $N_G(x) = Q_1 \cup Q_2 \cup H_2$.

Suppose $H_3 \neq \emptyset$. Then $|Q_3 \cup Q_2 \cup H_2| \geq k$. First, let us consider the case when $|Q_3 \cup Q_2 \cup H_2| = k$. By Lemma (3.3), $|H_3| = 1$. Let y be the only vertex in H_3. By k-connectivity, $N_G(y) = Q_3 \cup Q_2 \cup H_2$. Therefore, $A \cap S' \subseteq N_G(y) \cap N_G(x)$. This contradicts the choice of $S' \in \mathcal{C}(G)$ (that uv is lonely, or T is isolated, or e and f are cohesive). So $|Q_3 \cup Q_2 \cup H_2| \geq |S'| + 1$. If $W_1 \neq \emptyset$, then $Q_1 \cup Q_2 \cup W_2$ is a cut in G, and so, $|Q_1 \cup Q_2 \cup W_2| \geq k$. Thus $|S'| + |S| = |Q_1 \cup Q_2 \cup W_2| + |Q_3 \cup Q_2 \cup H_2| \geq |S'| + 1 + k = |S'| + |S| + 1$, a contradiction. Hence $W_1 = \emptyset$. Since $|Q_3 \cup Q_2 \cup H_2| \geq |S'| + 1 \geq k + 1$ and $|Q_1 \cup Q_2 \cup Q_3| = k$, we have $|H_2| > |Q_1|$. This implies $|V(H)| > |V(H')|$, and so, S' and H' contradict the choice of S and H.

So $H_3 = \emptyset$. Therefore, $|H_2| = |V(H)| - 1$, and hence, $|H_2| \geq k - 3$ when $k \geq 7$ (by Lemma (2.7)), and $|H_2| \geq k - 2$ when $k \in \{5,6\}$ (by Lemma (2.8)). Because $|Q_2 \cup W_2| = k - |H_2|$ and $|Q_1 \cup Q_2| = k - |H_2|$, $|Q_2 \cup W_2| = |Q_1 \cup Q_2| \leq 3$ when $k \geq 7$, and $|Q_2 \cup W_2| = |Q_1 \cup Q_2| \leq 2$ when $k \in \{5,6\}$. Since $|W_1| = |V(H')| - |H_1| - |Q_1|$, $|W_1| \geq (k-2) - 1 - 3 \geq 1$ when $k \geq 7$ and $|W_1| \geq (k-1) - 1 - 2 \geq 1$ when $k \in \{5,6\}$. Hence $W_1 \neq \emptyset$. This implies that $Q_1 \cup Q_2 \cup W_2$ is a cut in G, and so, $|Q_1 \cup Q_2 \cup W_2| \geq k$. On the other hand, $|Q_1 \cup Q_2 \cup W_2| \leq |Q_2 \cup W_2| + |Q_1 \cup Q_2| \leq 6$ when $k \geq 7$, and $|Q_1 \cup Q_2 \cup W_2| \leq |Q_2 \cup W_2| + |Q_1 \cup Q_2| \leq 4$ when $k \in \{5,6\}$; and we arrive at a contradiction. □

(3.5) Lemma. *Let $k \geq 5$ be an integer and let G be a k-connected graph such that $\Delta(T(G)) \leq 1$. Let $S \in \mathcal{C}(G)$ and a nontrivial component H of $G - S$ be chosen so that $|V(H)|$ is minimized. Then either G has a k-contractible clique or a k-contractible pair of cohesive edges, or for each $uv \in E(H)$ there exists $S' \in \mathcal{C}(G)$ such that $|S'| = k+1$, and either $S' \in \mathcal{C}_{T'}(G)$ for some isolated triangle T' with $\{u,v\} \subseteq V(T')$, or $S' \in \mathcal{C}_{e',f'}(G)$ for some cohesive edges e', f' with $\{u,v\} \subseteq V(e') \cup V(f')$.*

Proof. By Lemma (3.1), let $A' := \{u,v\}$ when uv is lonely; $A' := V(T')$ when uv is contained in some isolated triangle T' in G; and $A' := V(e') \cup V(f')$ when $\{u,v\} \subseteq V(e') \cup V(f')$ for some cohesive edges e', f'. Suppose G contains no k-contractible cliques or pairs of cohesive edges. Then $\mathcal{C}_{uv}(G) \cup \mathcal{C}_{T'}(G) \cup \mathcal{C}_{e',f'}(G) \neq \emptyset$. Therefore, by the same argument as in the proof of Lemma (3.4), we may choose $S' \in \mathcal{C}_{uv}(G) \cup \mathcal{C}_{T'}(G) \cup \mathcal{C}_{e',f'}(G)$ such that $S' \cap V(H) \neq \emptyset$. If $|S'| = k+1$ then the assertion of this lemma holds. So we may assume $|S'| = k$.

Let $W = G - (S \cup V(H))$. Let H' be a nontrivial component of $G - S'$, and let $W' = G - (S' \cup V(H'))$. By Lemma (2.5), both W and W' have a nontrivial component. Denote $V(H \cap H'), V(H) \cap S'$ and $V(H \cap W')$ by H_1, H_2 and H_3, respectively. Denote $S \cap V(H'), S \cap S'$ and $S \cap V(W')$ by Q_1, Q_2 and Q_3, respectively. Denote $V(W \cap H'), V(W) \cap S'$ and $V(W \cap W')$ by W_1, W_2 and W_3, respectively. See Figure 1. Note that $A' \cap S' \subseteq Q_2 \cup H_2$.

By Lemma (3.4), we have $|S| = k+1$. Thus, $S \in \mathcal{C}_T^{k+1}(G)$ for some isolated triangle T in G, or $S \in \mathcal{C}_{e,f}^{k+1}(G)$ for some cohesive edges e and f in G. Let $A := V(T)$ if $S \in \mathcal{C}_T^{k+1}(G)$ and let $A := V(e) \cup V(f)$ when $S \in \mathcal{C}_{e,f}^{k+1}(G)$. Then $A \subseteq Q_1 \cup Q_2$, or $A \subseteq Q_2 \cup Q_3$, or $A \cap Q_i \neq \emptyset$ for all $1 \leq i \leq 3$. Therefore, we consider two cases.

Case 1. $A \subseteq Q_1 \cup Q_2$ or $A \subseteq Q_2 \cup Q_3$.

By symmetry, we may assume that $A \subseteq Q_1 \cup Q_2$.

Suppose $H_1 \neq \emptyset$. Then $Q_1 \cup Q_2 \cup H_2$ is a cut in G. Since $A \subseteq Q_1 \cup Q_2$ and because $|S| = k+1$ and $S \in \mathcal{C}(G)$, we see that $|Q_1 \cup Q_2 \cup H_2| \geq k+1$. Suppose $|Q_1 \cup Q_2 \cup H_2| = k+1$. Then since $|S| = k+1$ and $S \in \mathcal{C}(G)$, it follows from Lemma (2.6) that $Q_1 \cup Q_2 \cup H_2 \in \mathcal{C}(G)$. Therefore, because $|Q_1 \cup Q_2 \cup H_2| = k+1$, it follows from Lemma (2.5) that H_1 has a nontrivial component, say H_1'. Then $Q_1 \cup Q_2 \cup H_2$ and H_1' contradict the choice of S and H (since $S' \cap V(H) \neq \emptyset$). So $|Q_1 \cup Q_2 \cup H_2| \geq k+2$. This implies $|H_2| > |Q_3|$. Thus, if $W_3 = \emptyset$ then $|V(H)| > |W'|$, and S' and a nontrivial component of W' contradict the choice of S and H. So $W_3 \neq \emptyset$. Then $Q_2 \cup Q_3 \cup W_2$ is a cut in G, and hence, $|Q_2 \cup Q_3 \cup W_2| \geq k$. This shows $2k+1 = |S| + |S'| = |Q_1 \cup Q_2 \cup H_2| + |Q_3 \cup Q_2 \cup W_2| \geq 2k+2$, a contradiction.

Thus $H_1 = \emptyset$. By Lemma (3.2), $H_3 \neq \emptyset \neq W_1$. So $Q_1 \cup Q_2 \cup W_2$ and $Q_3 \cup Q_2 \cup H_2$ are cuts in G. Since $A \subseteq Q_1 \cup Q_2$ and because $|S| = k+1$ and $S \in \mathcal{C}(G)$, we see that $|Q_1 \cup Q_2 \cup W_2| \geq k+1$. Also, by connectivity of G, $|Q_3 \cup Q_2 \cup H_2| \geq k$. Since $2k+1 = |S| + |S'| = |Q_1 \cup Q_2 \cup W_2| + |Q_2 \cup Q_3 \cup H_2| \geq 2k+1$, we have $|Q_1 \cup Q_2 \cup W_2| = k+1$ and $|Q_3 \cup Q_2 \cup H_2| = k$. Hence by Lemma (3.3), $|H_3| = 1$.

Therefore, $|H_2| = |V(H)| - |H_1| - |H_3| = |V(H)| - 1$. Hence $|H_2| \geq k-3$ when $k \geq 7$ (by Lemma (2.7)), and $|H_2| \geq k-2$ when $k \in \{5, 6\}$ (by Lemma (2.8)). Since $|Q_2 \cup W_2| = |S'| - |H_2| = k - |H_2|$ and $|Q_2 \cup Q_3| = k - |H_2|$, we see $|Q_2 \cup W_2| = |Q_2 \cup Q_3| \leq 3$ when $k \geq 7$, and $|Q_2 \cup W_2| = |Q_2 \cup Q_3| \leq 2$ when $k \in \{5, 6\}$.

If $W_3 = \emptyset$, then since $|Q_3| \geq |V(W')| - |H_3| = |V(W')| - 1$, we have $|Q_3| \geq 4$ when $k \geq 7$ (by Lemma (2.7)) and $|Q_3| \geq 3$ when $k \in \{5, 6\}$ (by Lemma (2.8)), a contradiction. So $W_3 \neq \emptyset$. Then $Q_2 \cup Q_3 \cup W_2$ is cut in G, and so, $|Q_2 \cup Q_3 \cup W_2| \geq k$. Hence $|Q_2 \cup W_2| + |Q_2 \cup Q_3| \geq k \geq 7$ when $k \geq 7$ and $|Q_2 \cup W_2| + |Q_2 \cup Q_3| \geq k \geq 5$ when $k \in \{5, 6\}$, a contradiction.

Case 2. $A \cap Q_i \neq \emptyset$ for all $1 \leq i \leq 3$.

In this case, $S \in \mathcal{C}_{e,f}^{k+1}(G)$. Without loss of generality, let $A \cap Q_1 = \{a\}$, $A \cap Q_2 = \{b, d\}$, and $A \cap Q_3 = \{c\}$. Then $ab, bc, cd, da, bd \in E(G)$ and $ac \notin E(G)$. By Lemma (3.2), $H_1 \neq \emptyset \neq W_3$ or $H_3 \neq \emptyset \neq W_1$. By symmetry, we may assume that $H_1 \neq \emptyset \neq W_3$.

Then $Q_1 \cup Q_2 \cup H_2$ is a cut in G containing $\{a, b, d\}$, and $Q_3 \cup Q_2 \cup W_2$ is a cut in G containing $\{c, b, d\}$. Suppose $|Q_1 \cup Q_2 \cup H_2| = k$. Then by Lemma (3.3), $|H_1| = 1$. Let x denote the unique vertex in H_1. By connectivity of G, $\{xa, xb, xd\} \subseteq E(G)$, which implies that e or f is contained in two triangles, a contradiction. So $|Q_1 \cup Q_2 \cup H_2| \geq k+1$.

If $|Q_3 \cup Q_2 \cup W_2| \geq k+1$, then $2k+1 = |S| + |S'| = |Q_1 \cup Q_2 \cup H_2| + |Q_3 \cup Q_2 \cup W_2| \geq 2k+2$, a contradiction. So $|Q_3 \cup Q_2 \cup W_2| = k$. Because $|S| = k+1$, $Q_3 \cup Q_2 \cup W_2 \notin \mathcal{C}_{e,f}(G)$. Hence, by the definition of $\mathcal{C}^k_{e,f,3}(G)$, $G-(Q_3\cup Q_2\cup W_2)$ has exactly two components, one of which consists of a only. However, this is impossible since $W_3\neq\emptyset\neq H_1$. □

4 Proof of the main result

In this section, we complete the proof of Theorem (1.1), using an argument similar to that of [2].

Let $k\geq 5$ be an integer and let G be a k-connected graph such that $\Delta(T(G))\leq 1$.

If $\mathcal{C}_e(G)=\emptyset$ for some lonely edge e, or $\mathcal{C}_T(G)=\emptyset$ for some isolated triangle T, or $\mathcal{C}_{e,f}(G)=\emptyset$ for some cohesive edges e and f in G, then we see that Theorem (1.1) follows from Propositions (2.1), (2.2), and (2.3).

So we may assume that $\mathcal{C}_e(G)\neq\emptyset$ for all lonely edges e, $\mathcal{C}_T(G)\neq\emptyset$ for all isolated triangles T, and $\mathcal{C}_{e,f}(G)\neq\emptyset$ for all cohesive edges e and f in G. Choose $S\in\mathcal{C}(G)$ and a nontrivial component H of $G-S$ such that

(1) $|V(H)|$ is minimum.

Let $W:=G-(S\cup V(H))$. By Lemma (2.5), W contains a nontrivial component of $G-S$. By Lemma (3.4), we have

(2) $|S|=k+1$.

We pick a spanning tree of H and label its edges as $g_1,\dots,g_m$ such that

(3) $G[\{g_1,\dots,g_i\}]$ is connected for all $1\leq i\leq m$.

By Lemma (3.5), for each $1\leq i\leq m$, there is some $S_i\in\mathcal{C}(G)$ such that

(4) $|S_i|=k+1$, and $S_i\in\mathcal{C}^{k+1}_{T_i}(G)$ for some isolated triangle T_i with $V(g_i)\subseteq V(T_i)$, or $S_i\in\mathcal{C}^{k+1}_{e_i,f_i}(G)$ for some cohesive edges e_i and f_i with $V(g_i)\subseteq V(e_i)\cup V(f_i)$. (Hence, $S_i\cap V(H)\neq\emptyset$.)

Let $A_i=V(T_i)$ if $S_i\in\mathcal{C}^{k+1}_{T_i}(G)$; and let $A_i=V(e_i)\cup V(f_i)$ if $S_i\in\mathcal{C}^{k+1}_{e_i,f_i}(G)$. Let H^i be a nontrivial component of $G-S_i$, and let $W^i=G-(S_i\cup V(H^i))$. By Lemma (2.5), W^i contains a nontrivial component of $G-S_i$. Write $H^i_1=V(H\cap H^i)$, $H^i_2=V(H)\cap S_i$, $H^i_3=V(H\cap W^i)$, $Q^i_1=S\cap V(H^i)$, $Q^i_2=S\cap S_i$, $Q^i_3=S\cap V(W^i)$, $W^i_1=V(W\cap H^i)$, $W^i_2=V(W)\cap S_i$, and $W^i_3=V(W\cap W^i)$. Then by (4) and by the definition of $\mathcal{C}^{k+1}_{e_i,f_i}(G)\cup\mathcal{C}^{k+1}_{T_i}(G)$, we have

(5) $A_i\subseteq H^i_2\cup Q^i_2$.

Since $S\in\mathcal{C}(G)$ and $|S|=k+1$, $S\in\mathcal{C}^{k+1}_T(G)$ for some isolated triangle T in G, or $S\in\mathcal{C}^{k+1}_{e,f}(G)$ for some cohesive edges e and f of G. Let $A=V(T)$ if $S\in\mathcal{C}^{k+1}_T(G)$, and $A=V(e)\cup V(f)$ if $S\in\mathcal{C}^{k+1}_{e,f}(G)$. We claim that

(6) $A\subseteq Q^i_1\cup Q^i_2$, or $A\subseteq Q^i_2\cup Q^i_3$.

Suppose (6) fails. Then $S\in\mathcal{C}^{k+1}_{e,f}(G)$ and $A\cap Q^i_j\neq\emptyset$ for all $1\leq j\leq 3$. Hence $A=V(e)\cup V(f)$, and we may write $A\cap Q^i_1=\{a\}$, $A\cap Q^i_2=\{b,d\}$, and $A\cap Q^i_3=\{c\}$, where $ab,bc,cd,da,bd\in E(G)$ and $ac\notin E(G)$.

By Lemma (3.2), $H^i_1\neq\emptyset\neq W^i_3$ or $H^i_3\neq\emptyset\neq W^i_1$. By symmetry, we may assume that $H^i_1\neq\emptyset\neq W^i_3$. Then $Q^i_1\cup Q^i_2\cup H^i_2$ is a cut in G containing $\{a,b,d\}$, and $Q^i_2\cup Q^i_3\cup W^i_2$ is a cut in G containing $\{c,b,d\}$. Since $|S_i|=k+1$, it follows from Lemma (3.3) that $|Q^i_1\cup Q^i_2\cup H^i_2|\geq k+2$.

We claim that $|Q_2^i \cup Q_3^i \cup W_2^i| \geq k+1$. Suppose $|Q_2^i \cup Q_3^i \cup W_2^i| = k$. Then by (2) and since $S \in \mathcal{C}(G)$, $Q_2^i \cup Q_3^i \cup W_2^i \notin \mathcal{C}_{e,f}(G)$. So by the definition of $\mathcal{C}^k_{e,f,3}(G)$, $G-(Q_2^i \cup Q_3^i \cup W_2^i)$ has exactly two components, one of which consists of a only. But this is impossible since $H_1^i \neq \emptyset \neq W_3^i$.

Hence $2k+2 = |S|+|S_i| = |Q_1^i \cup Q_2^i \cup H_2^i| + |Q_2^i \cup Q_3^i \cup W_2^i| \geq 2k+3$, a contradiction. Thus (6) holds.

By (6) and by symmetry, we may assume (by relabeling H^i and W^i if necessary) that $A \subseteq Q_1^i \cup Q_2^i$ for all $1 \leq i \leq m$. Next we show

(7) $H_3^i = \emptyset$, $|Q_1^i \cup Q_2^i \cup H_2^i| = k+2$, and $|Q_3^i \cup Q_2^i \cup W_2^i| = k$.

Suppose $H_3^i \neq \emptyset$. Then since $|S_i| = k+1$ it follows from Lemma (3.3) that $|Q_3^i \cup Q_2^i \cup H_2^i| \geq k+2$. First, assume $W_1^i \neq \emptyset$. Then $Q_1^i \cup Q_2^i \cup W_2^i$ is a cut in G containing A, and hence $|Q_1^i \cup Q_2^i \cup W_2^i| \geq k+1$ (because $S \in \mathcal{C}(G)$ and $|S| = k+1$). This shows $2k+2 = |S|+|S_i| = |Q_1^i \cup Q_2^i \cup W_2^i| + |Q_2^i \cup Q_3^i \cup H_2^i| \geq 2k+3$, a contradiction. Now assume $W_1^i = \emptyset$. Since $|Q_3^i \cup Q_2^i \cup H_2^i| \geq k+2$ and $|Q_1^i \cup Q_2^i \cup Q_3^i| = k+1$, $|H_2^i| \geq |Q_1^i|+1$. This implies that $|V(H^i)| = |Q_1^i \cup H_1^i| < |H_2^i \cup H_1^i| \leq |V(H)|$, and S_i and H^i contradict the choice of S and H.

So $H_3^i = \emptyset$. Therefore, it follows from Lemma (3.2) that $H_1^i \neq \emptyset$ and $W_3^i \neq \emptyset$. So $Q_1^i \cup Q_2^i \cup H_2^i$ is a cut in G. Since $|S'| = k+1$, it follows from Lemma (3.3) that $|Q_1^i \cup Q_2^i \cup H_2^i| \geq k+2$. Since G is k-connected, $|Q_3^i \cup Q_2^i \cup W_2^i| \geq k$. Therefore, $2k+2 \geq |Q_1^i \cup Q_2^i \cup H_2^i| + |Q_3^i \cup Q_2^i \cup W_2^i| = |S|+|S_i| = 2k+2$. Hence, $|Q_1^i \cup Q_2^i \cup H_2^i| = k+2$ and $|Q_3^i \cup Q_2^i \cup W_2^i| = k$. This completes the proof of (7).

(8) $|V(H)| \geq k$.

Since $|H_2^i| \geq 2$ and $H_3^i = \emptyset$ (by (7)), it suffices to prove $|H_1^i| \geq k-2$ for some $1 \leq i \leq m$. Suppose $|H_1^i| = 1$ for some $1 \leq i \leq m$, and let x denote the only vertex in H_1^i. Note that $|A_i \cap A| \leq 2$. First, assume $|A_i \cap A| \leq 1$. Then x has at most two neighbors in $A_i \cup A$ when $A_i = V(T_i)$ and $A = V(T)$; x has at most three neighbors in $A_i \cup A$ when $A_i = V(e_i) \cup V(f_i)$, and $A = V(T)$ or $A_i = V(T_i)$ and $A = V(e) \cup V(f)$; and x has at most four neighbors in $A_i \cup A$ when $A_i = V(e_i) \cup V(f_i)$ and $A = V(e) \cup V(f)$. Therefore, since $|Q_1^i \cup Q_2^i \cup H_2^i| = k+2$, we see that the number of neighbors of x in G is at most $(k+2)-5+2$, or $(k+2)-6+3$, or $(k+2)-7+4$. This shows $d_G(x) \leq k-1$, contradicting the assumption that G is k-connected. So $|A_i \cap A| = 2$, and $A_i = V(e_i) \cup V(f_i)$ and $A = V(e) \cup V(f)$. Hence x has at most three neighbors in $A_i \cup A$, and we see that the number of neighbors of x in G is at most $(k+2)-6+3$, a contradiction. So we have (8).

(9) $|N(U) \cap V(H)| \geq |U|+1$ for all nonempty subsets U of $S-A$.

Suppose that there is a nonempty set U of $S-A$ such that $|N(U) \cap V(H)| \leq |U|$. Then $|N(U) \cap V(H)| \leq |U| \leq |S-A| \leq k-2 < k \leq |V(H)|$. So $H - N(U) \neq \emptyset$. Hence $S^* := (S-U) \cup (N(U) \cap V(H))$ is a cut containing A and separating $H-N(U)$ from $W \cup U$. Since $S \in \mathcal{C}_T^{k+1}(G)$ or $S \in \mathcal{C}_{e,f}^{k+1}(G)$, we see that $S^* \in \mathcal{C}_T^{k+1}(G)$ or $S^* \in \mathcal{C}_{e,f}^{k+1}(G)$. Hence $H-N(U)$ has a nontrivial component H^*. Because $|H-N(U)| < |V(H)|$, S^* and H^* contradict the choice of S and H.

(10) $|\bigcup_{i=1}^j (N(Q_3^i) \cap V(H))| \leq |\bigcup_{i=1}^j Q_3^i| + 1$ for all $1 \leq j \leq m$.

We prove (10) by induction on j. Since $|Q_1^i \cup Q_2^i \cup Q_3^i| = |S| = k+1$ and $|H_2^i| \geq 2$, it follows from (7) that $|H_2^i| = |Q_3^i|+1$ and $|Q_3^i| \geq 1$. Since $N(Q_3^i) \cap V(H) \subseteq H_2^i$, it follows from (9) that $N(Q_3^i) \cap V(H) = H_2^i$. Hence (10) holds when $j=1$.

Assume $j \geq 2$. Let $P_j = (N(Q_3^j) \cap V(H)) \cap (\bigcup_{i=1}^{j-1} N(Q_3^i) \cap V(H))$. Since $G[\{g_1, \ldots, g_j\}]$ is connected for all $1 \leq j \leq m$, we have $P_j \neq \emptyset$ for all $2 \leq j \leq m$. Let $R_j = Q_3^j \cap (\bigcup_{i=1}^{j-1} Q_3^i)$. If $R_j = \emptyset$, then clearly $|P_j| \geq |R_j| + 1$. If $R_j \neq \emptyset$, then since $N(R_j) \cap V(H) \subseteq P_j$, it follows from (9) that $|P_j| \geq |R_j| + 1$. Hence we have $|\bigcup_{i=1}^{j}(N(Q_3^i) \cap V(H))| \leq (|\bigcup_{i=1}^{j-1} Q_3^i| + 1) + (|Q_3^j| + 1) - (|R_j| + 1) = |\bigcup_{i=1}^{j} Q_3^i| + 1$. This proves (10)

Since $G[\{g_1, \ldots, g_m\}]$ is a spanning tree of H and because for each $1 \leq j \leq m$, $V(g_j) \subseteq A_j \cap V(H) \subseteq H_2^j$, we have $\bigcup_{i=1}^{m}(N(Q_3^i) \cap V(H)) = V(H)$. By (10), $|V(H)| \leq |\bigcup_{i=1}^{m} Q_3^i| + 1 \leq |S - A| + 1 \leq k - 1$, contradicting (8). This completes the proof of Theorem (1.1).

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