Take-home final policy: You can use the textbook, notes from this class, or other books / online resources. If you are quoting any results not from the textbook, please clearly identify the source. If you are not sure whether a result can be used, please email the instructor for clarification. Discussing any questions with anyone other than the instructor is strictly forbidden, and is a violation of the Georgia Tech academic honor code.

Total points: 40 (+5 for the bonus question in the end)

Pick any 4 problems from the questions 1–5. (Do NOT do more than four.) Each problem is worth 10 points. Please show your steps and explain your reasoning.

1. Let $B_0 := \{ x \in \mathbb{R}^3 : 0 < |x| < 1 \}$ be the unit ball in $\mathbb{R}^3$ minus the origin. Assume $u : B_0 \to \mathbb{R}$ is in $C^2(B_0)$, and harmonic in $B_0$. If we in addition have that $\lim_{|x| \to 0} |x|u(x) = 0$, is it guaranteed that $u(x)$ can be extended to a harmonic function on $\{ x \in \mathbb{R}^3 : |x| < 1 \}$? If so, prove it; if not, give a counterexample.

2. Let $\vec{v}(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^\infty$ vector field, which satisfies $|\nabla \cdot \vec{v}(x)| \leq M$ for all $x \in \mathbb{R}^n$. Let $u(x, t) : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ be a smooth and bounded solution to
   \[ u_t = \Delta (u^2) + \nabla \cdot (\vec{v}(x)u), \]
   with initial condition $0 \leq u(x, 0) \leq 1$ in $\mathbb{R}^n$. Prove that $u(x, t) \leq e^{Mt}$ for all $x \in \mathbb{R}^n, t > 0$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Consider the following initial-boundary-value problem
   \[
   \begin{cases}
   u_{tt} = \Delta u + \nabla \cdot f(\nabla u_t) & \text{in } \Omega \times [0, \infty) \\
   u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty) \\
   u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{in } \Omega,
   \end{cases}
   \]
   where $g, h \in C^\infty_c(\Omega)$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given $C^1$ function such that
   \[ (f(z) - f(y)) \cdot (z - y) \geq 0 \]
   for any $z, y \in \mathbb{R}^n$. Prove that this problem has at most one classical solution $u \in C^2(\bar{\Omega} \times [0, \infty))$.

4. The following is an innocent-looking linear PDE that has surprisingly few solutions. Let $B(0, 1)$ be the closed unit disk in $\mathbb{R}^2$, and consider the equation
   \[ a(x, y)u_x + b(x, y)u_y = -u \quad \text{in } B(0, 1), \]
   where the coefficients $a, b \in C^\infty(B(0, 1))$ satisfying that $xa(x, y) + yb(x, y) > 0$ on $\partial B(0, 1)$. (Note that no boundary condition is imposed on $u$.) Show that the only solution in $C^1(B(0, 1))$ is $u \equiv 0$.  

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5. Consider the Burgers’ equation

\[ u_t + uu_x = 0. \]

Let the spatial domain be the interval \([0, 1]\) with **periodic** boundary condition, and the initial data is given by

\[ u(x, 0) = \begin{cases} 
1 & 0 < x < \frac{1}{2} \\
0 & \frac{1}{2} < x < 1.
\end{cases} \]

Find the entropy solution \(u(x, t)\) for any \(x \in [0, 1], t > 0.\)

Below is an optional bonus question that you can try. If you wish, you can replace a regular question by this question (however please beware of this option since I think it’s more difficult than the previous ones). Or, if you do it in addition to the 4 questions above, you will get an extra 5 points for a correct answer.

0. Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with a smooth boundary. Consider the equation

\[
\begin{align*}
\Delta u + c(1 + u^2) &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(c > 0\) is a constant. Prove that if \(c\) is a sufficiently large constant, there does not exist any solution \(u \in C^2(\Omega)\) to the above equation.