1. Use the Fourier transform to solve the equation
   \[ u_t = t^2 u_{xx} \quad x \in \mathbb{R}, \ t > 0, \]
   with initial condition \( u(x, 0) = f(x) \).

2. Evans, p87, #15 (in the second edition). If you are using the first edition, it is #13 there.

3. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Assume \( u \in C^{2,1}(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty)) \) satisfies
   \[
   \begin{cases}
   u_t = \Delta u & x \in \Omega, \ t > 0, \\
   u(x, 0) = g(x) & x \in \Omega, \\
   u(x, t) = 0 & x \in \partial \Omega, t > 0,
   \end{cases}
   \]
   where \( g \in C_c(\Omega) \). Prove that \(|u(x, t)| \leq \frac{\|g\|_{L^1}}{(4\pi t)^{n/2}}\) for all \( x \in \Omega, t > 0 \).
   (Hint: Compare \( u \) with \( v \), where \( v \) solves the heat equation in the whole space with initial condition \(|g|\) in \( \Omega \), and initial condition 0 outside \( \Omega \).)

4. Consider the Cauchy problem
   \[ u_t - \Delta u - u^2(x, t) = f(x, t) \quad \text{in} \ \mathbb{R}^n \times (0, T) \]
   with initial condition \( u(x, 0) = 0 \). Prove that there is at most one bounded solution \( u \in C^{2,1}(\mathbb{R}^n \times (0, T)) \).

5. The following example (Tychonoff, 1935) says that there exists a solution \( u \in C^\infty(\mathbb{R} \times (0, \infty)) \cap C(\mathbb{R} \times [0, \infty)) \) to the heat equation, which is identically zero at \( t = 0 \) but not for \( t > 0 \). (That’s why we need to impose growth conditions at infinity to show uniqueness of solutions to the Cauchy problem.)
   (a) Let \( g(t) \) be a function to be specified later. Consider the power series
   \[ u(x, t) = \sum_{j=0}^{\infty} \frac{g^{(j)}(t)}{(2k)!} x^{2k}. \]
   If we are allowed to differentiate the power series term by term, formally check that \( u \) solves the heat equation \( u_t = u_{xx} \) in \( \mathbb{R} \times (0, \infty) \).
   (b) We now choose \( g \) as
   \[ g(t) = \begin{cases} 
   e^{-1/t^2} & t > 0, \\
   0 & t = 0.
   \end{cases} \]
   Check that the \( u \) defined in part (a) is indeed in \( C^\infty(\mathbb{R}^n \times (0, T)) \), and \( \lim_{t \to 0^+} u(x, t) = 0 \) for all \( x \in \mathbb{R} \).
   (You may directly use the estimate that \(|g^{(k)}(t)| \leq \frac{k!}{(Ct)^k} e^{-\frac{1}{2t^2}}\) for all \( t > 0 \), where \( C > 0 \) is some fixed constant.)