1. Consider the following wave equation in the sector $-t < x < t$:

$$\begin{cases}
    u_{tt}(x,t) - u_{xx}(x,t) = 0 & x \in (-t,t), \ t > 0 \\
    u(x,x) = f(x), \ u(x,-x) = g(x), & x \geq 0
\end{cases}$$

where $f, g : [0, \infty) \to \mathbb{R}$ are smooth functions with $f(0) = g(0)$. Does there exist a solution? If so, find it. (Correction: the $g$ boundary condition should be $u(-x,x) = g(x)$).

2. Evans, Sec 2.5, #24 (b). (We already proved (a) in class, so you only need to do (b)). It is #17 (b) in the first edition.

3. In this exercise, you will show that nonlinear wave equation has finite speed of propagation too. Assume $u(x,t)$ is a smooth solution to the nonlinear wave equation

$$u_{tt} - \Delta u + f(\nabla u, u_t, u) = 0 \text{ in } \mathbb{R}^n \times (0, \infty),$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(\vec{0}, 0, 0) = 0$.

For any $x_0 \in \mathbb{R}^n, t_0 \in (0, \infty)$, show that $u(x_0, t_0) = 0$ if $u(\cdot, 0) = u_t(\cdot, 0) = 0$ on $B(x_0, t_0)$.

(Hint: Consider the energy $e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t^2 + \|
abla u\|^2 + u^2) dx$ for $t \in [0, t_0]$, and show that $e'(t) \leq Ce(t)$.)

4. Let $u$ be a $C^2$ solution of

$$\begin{cases}
    u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
    u(x,0) = 0 & \text{in } \mathbb{R}^2, \\
    u_t(x,0) = h(x) & \text{in } \mathbb{R}^2,
\end{cases}$$

where the support of $h$ lies in $B(0,a)$.

(a) Show that there exists a constant $C$ such that

$$|u(x,t)| \leq \frac{C}{t} \quad \text{for } t \geq 2(|x| + a).$$

(b) Show that for each $x \in \mathbb{R}^2$,

$$\lim_{t \to \infty} tu(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(y) dy.$$