HW 3

Chapter 3

#2 (vi)

(b) $0 = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 2 \Rightarrow \lambda_1 = \sqrt{2},\ \lambda_2 = -\sqrt{2}$

$V_1 = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$ $\quad V_2 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 1-\sqrt{2} \\ \sqrt{2}-1 & 1 \end{pmatrix}$ $\quad T^{-1}AT = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$

(c) $X' = AX$

general sol'n $X(t) = c_1 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} + c_2 e^{-\sqrt{2}t}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$

$Y' = (T^{-1}AT)Y$

general sol'n $Y(t) = c_1 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-\sqrt{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ #

#4 (a) $b^2 < 4k \Rightarrow$ complex eigenvalue $\quad \lambda = \frac{-b \pm \sqrt{b^2-4k}}{2}$

$b^2 = 4k \Rightarrow$ repeated eigenvalue

$b^2 > 4k \Rightarrow$ distinct real eigenvalue

(b) Complex

$\lambda = -\frac{b}{2} \pm \frac{i}{2}\sqrt{4k-b^2}$

$X(t) = c_1 e^{-\frac{b}{2}t}\begin{pmatrix} \cos\frac{1}{2}\sqrt{4k-b^2}\,t \\ -\frac{b}{2}\cos\frac{1}{2}\sqrt{4k-b^2}\,t - \frac{1}{2}\sqrt{4k-b^2}\sin\frac{1}{2}\sqrt{4k-b^2}\,t \end{pmatrix}$

$+ c_2 e^{-\frac{b}{2}t}\begin{pmatrix} \sin\frac{1}{2}\sqrt{4k-b^2}\,t \\ -\frac{b}{2}\sin\frac{1}{2}\sqrt{4k-b^2}\,t + \frac{1}{2}\sqrt{4k-b^2}\cos\frac{1}{2}\sqrt{4k-b^2}\,t \end{pmatrix}$

repeated

$\lambda = -\frac{b}{2}$

$$X(t) = c_1 e^{-\frac{b}{2}t}\begin{pmatrix}1\\-\frac{b}{2}\end{pmatrix} + c_2 e^{-\frac{b}{2}t}\begin{pmatrix}t\\1-\frac{b}{2}t\end{pmatrix}$$

real distinct

$\lambda_1 = \frac{-b+\sqrt{b^2-4k}}{2}$ $\quad\lambda_2 = \frac{-b-\sqrt{b^2-4k}}{2}$

$$X(t) = c_1\begin{pmatrix}e^{\lambda_1 t}\\\lambda_1 e^{\lambda_1 t}\end{pmatrix} + c_2\begin{pmatrix}e^{\lambda_2 t}\\\lambda_2 e^{\lambda_2 t}\end{pmatrix}$$

(c) complex : oscillate $|x| \to 0$ as $t \to \infty$

repeated : $|x| \to 0$ as $t \to \infty$

$|x|$ may increase once then decrease to 0.

real distinct: $|x| \to 0$ strictly as $t \to \infty$ #

#12. $X_1(t) = e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix}$ $\quad X_1'(t) = \begin{pmatrix}\lambda e^{\lambda t}\\0\end{pmatrix} = \begin{pmatrix}\lambda & 1\\0 & \lambda\end{pmatrix} X_1(t)$

$\Rightarrow X_1(t)$ is a sol'n of $X' = \begin{pmatrix}\lambda & 1\\0 & \lambda\end{pmatrix}X$

Similarly $X_2(t)$ is a sol'n of $X' = \begin{pmatrix}\lambda & 1\\0 & \lambda\end{pmatrix}X$

Note Wronskian $W[X_1, X_2](t) = \begin{vmatrix}e^{\lambda t} & te^{\lambda t}\\0 & e^{\lambda t}\end{vmatrix} = e^{2\lambda t} \neq 0$

$X_1(t), X_2(t)$ linearly independent

$\Rightarrow \alpha X_1(t) + \beta X_2(t)$ is the general sol'n of $X' = \begin{pmatrix}\lambda & 1\\0 & \lambda\end{pmatrix}X$

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Chapter 4

#5(a) $X' = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} X$ let $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$

$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ $T_1^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$

$T_1^{-1} A T_1 = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

$Y' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} Y$ let $B = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$

$T_2 = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$ $T_2^{-1} = \begin{pmatrix} -\frac{1}{3} & 1 \\ \frac{1}{3} & 0 \end{pmatrix}$

$T_2^{-1} B T_2 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

let $g_1(x_1) = \begin{cases} x_1^2 & x_1 \geq 0 \\ 1 - x_1^2 & x_1 < 0 \end{cases}$, $g_2(x_2) = \begin{cases} \sqrt{y}, & y \geq 0, \\ 1 - \sqrt{|y|}, & y < 0. \end{cases}$

Define

$G(x_1, x_2) = (g_1(x_1), g_2(x_2))$

$\Rightarrow G \phi^{T_1^{-1} A T_1}(t, (x_1, x_2)) = \phi^{T_2^{-1} B T_2}(t, G(x_1, x_2))$

$\Rightarrow G T_1^{-1} \phi^A (t, T_1(x_1, x_2)) = T_2^{-1} \phi^B (t, T_2 G(x_1, x_2))$

$\Rightarrow (T_2 G T_1^{-1}) \phi^A (t, T_1(x_1, x_2)) = \phi^B (t, (T_2 G T_1^{-1}) T_1 (x_1, x_2))$

let $H = T_2 G T_1^{-1}$ then H is a conjugacy between two original systems. #