Attachment assigned on 09/05/2017

1. By contraction mapping theorem:

$\exists!$ fixed point x_0 of $f^{(n)}$, i.e. $f^{(n)}(x_0) = x_0$ $(*)$

$$(*) \Longrightarrow f(f^{(n)}(x_0)) = f^{(n)}(f(x_0)) = f(x_0).$$

By the uniqueness, $x_0 = f(x_0)$.

Notice: any fixed point of f is also the fixed point of $f^{(n)}$

$\therefore$ f has a unique fixed point.

2. (a) False.

Consider counterexample containing $\sin(\frac{1}{x})$

(b). True. • Step 1: Show f is injective by letting $L_0 = \|T^{-1}\|^{-1}$

Suppose $x_1, x_2 \in X$ s.t. $f(x_1) = f(x_2)$.

$$\text{i.e. } Tx_1 + g(x_1) = Tx_2 + g(x_2).$$

$$\|T^{-1}\|^{-1}|x_1 - x_2| \le |T(x_1 - x_2)| = |g(x_1) - g(x_2)| \le L|x_1 - x_2|$$

$$\Longrightarrow x_1 = x_2.$$

• Step 2: f is surjective.

$\forall y \in Y$. Consider $G: X \longmapsto X$, $G(x) = T^{-1}y - T^{-1}g(x)$

Show that G is contraction. Then use the contraction mapping theorem.

$\Longrightarrow$ f is bijection, and so invertible.

• Step 3: inverse is Lipschitz with Lipschitz constant $\frac{1}{\|T^{-1}\|^{-1} - L}$

let $y_1 = f(x_1)$. $y_2 = f(x_2)$.

show $|y_1 - y_2| \ge (\|T^{-1}\|^{-1} - L)|f^{-1}(y_1) - f^{-1}(y_2)|$

Exercise 1.25.

(a) let $z = x + iy$, the system is equivalent to $\dot{z} = -z^2$.

- $z(t) = \dfrac{1}{c+t}$, $c = \dfrac{1}{z(0)}$. let $c = a+bi$. $a, b \in \mathbb{R}$.

$$z(t) = x(t) + iy(t) = \frac{1}{a+bi+t}$$

$$\Rightarrow \quad x(t) = \frac{a+t}{(a+t)^2+b^2}, \qquad y(t) = -\frac{b}{(a+t)^2+b^2}.$$

- $x_0 = x(0) = \dfrac{a}{a^2+b^2}$, $y_0 = y(0) = -\dfrac{b}{a^2+b^2}$.

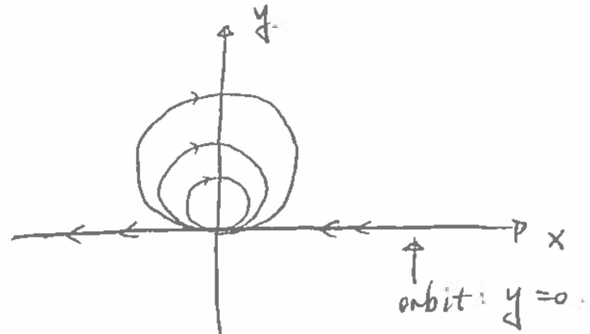
  write a, b in terms of x_0, y_0.

- $$\phi(t, x, y) = \Bigg(\underbrace{\frac{(x^2+y^2)\,\big((x^2+y^2)t + x\big)}{\big((x^2+y^2)t + x\big)^2 + y^2}}_{X(t,x,y)}, \; \underbrace{\frac{(x^2+y^2)\,y}{\big((x^2+y^2)t + x\big)^2 + y^2}}_{Y(t,x,y)} \Bigg)$$

(b).

For $y \neq 0$: $\big(X(t,x,y) - 0 \big)^2 + \left(Y(t,x,y) - \dfrac{x^2+y^2}{2y} \right)^2 = \left(\dfrac{x^2+y^2}{2y} \right)^2$

Orbits:



Thus, the flow exactly gives rational parameterizations for these orbits