Sketch

#1.61 (2).

Consider Lyapunov function. $V(x,y) = x^2+y^2$.

Show that $V(x,y)$ is a Lyapunov function, and then apply theorem 1.55 and 1.56. for $\alpha=0$, <0, >0.

#1.184.

convert the system into polar coordinates:

$$\begin{cases} \dot{r} = r(r^2-1)\sin\left(\frac{1}{r^2-1}\right) \\ r\dot{\theta} = -r. \end{cases}$$

① $r=0$, i.e. $(0,0)$.

② $0<r<1$, consider $r_k = \sqrt{1-\frac{1}{\pi k}}$, $\forall k\in\mathbb{N}$.

these set of r_k's gives periodic orbits.

Then show these orbits are limit cycles, i.e. they are isolated, by seeing $\dot{r}>0$, $\dot{r}<0$ in the annulus

$$S_k = \{(x,y)\in\mathbb{R}^2 \mid r_k < x^2+y^2 < r_{k+1}\}.$$

#2.25 (4).

Step 1. Find the fundamental matrix solution:

eigenvalues: 3, -1.

corresponding eigenvectors: $\begin{bmatrix}2\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}$

$$\Phi(t) = \begin{pmatrix} 2e^{3t} & e^{-t} \\ e^{3t} & e^{-t} \end{pmatrix}$$

Principal fundamental matrix solution at $t=0$ is $\Psi(t) = \Phi(t)\,\Phi(0)^{-1}$