#2.26.

(a). let $y = \dot{x}$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the eigenvalues $\lambda_\pm = (-1 \pm i\sqrt{15})/2$ and the corresponding eigenvecto

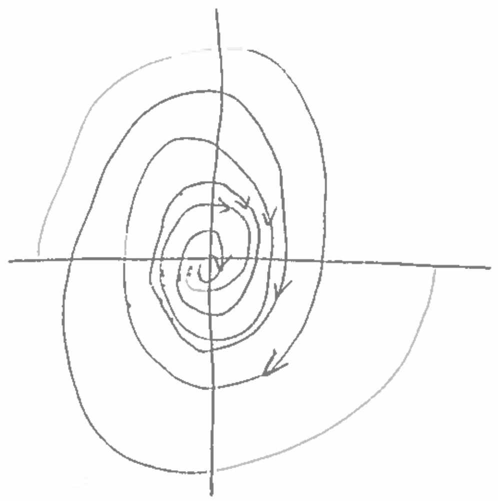
$$\vec{V}_\pm = \begin{pmatrix} \frac{\lambda_\mp}{4} \\ 1 \end{pmatrix}.$$

$$P = \begin{pmatrix} \frac{\lambda_-}{4} & \frac{\lambda_+}{4} \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}.$$

$$e^{tA} = Pe^{tD}P^{-1} = P\begin{pmatrix} e^{t\lambda_+} & 0 \\ 0 & e^{t\lambda_-} \end{pmatrix}P^{-1}$$

Then the flow is $\varphi_t(x) = e^{tA}x$

(b)



#. 2.27.

(a) Try $x(t) = t^r$.

The equation becomes $r^2 + (b-1)r + c = 0$

Quadratic formula implies that there are three different types of solutions.

(b) ① $(b-1)^2-4c>0$,

$$x(t)=C_+t^{r_+}+C_-t^{r_-}\quad,\quad C_\pm\in\mathbb{R}$$

as $t\mapsto 0^+$, $\begin{cases} x(t)\longmapsto \pm\infty & , \quad r_-<0 \text{ or } r_+<0 \\ x(t)\longmapsto 0 & , \quad r_-, \text{ and } r_+\geq 0 \end{cases}$

② $(b-1)^2-4c=0$

$$x(t)=C_+t^r+C_-t^r\ln t\quad,\quad C_\pm\in\mathbb{R}$$

as $t\mapsto 0^+$, $\begin{cases} x(t)\longmapsto \pm\infty & , \quad r\leq 0 \\ x(t)\longmapsto 0 & , \quad r>0 \end{cases}$

③. $(b-1)^2-4c<0$, $\quad r=a+bi$

$$x(t)=C_+t^a\cos(b\ln t)+C_-t^a\sin(b\ln t)\quad,\quad C_\pm\in\mathbb{R}$$

as $t\mapsto 0^+$. $\begin{cases} \text{limit DNE} & , \quad r\leq 0 \\ x(t)\longmapsto 0 & , \quad r>0 \end{cases}$

(c). $y=\dot{x}$,

$$\begin{pmatrix} x \\ y \end{pmatrix}'=\begin{pmatrix} 0 & 1 \\ -ct^{-2} & -bt^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\quad,\quad t>0$$

(d). $b=1$, $c=-1$,

$$\begin{pmatrix} x \\ y \end{pmatrix}'=\begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

By (a), we may immediately write down two solutions:

$$x_1(t)=\begin{pmatrix} t \\ 1 \end{pmatrix},\qquad x_2(t)=\begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}.$$

$\phi(t)=\begin{pmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{pmatrix}$. Then the principle fundamental matrix at $t_0>0$ is: $\Phi(t)=\phi(t)\phi^{-1}(t_0)$.

2.29

(a). $|ABx| = |A(Bx)| \leq \|A\| \, |Bx| \leq \|A\| \, \|B\| \, |x|$

$$\frac{|ABx|}{|x|} \leq \|A\| \, \|B\|$$

By the definition, $\|AB\| \leq \|A\| \, \|B\|$

(b) By induction, $\|A^k\| \leq \|A\| \, \|A^{k-1}\| \leq \cdots \leq \|A\|^k$.