• Let $\Omega \subset R^n$ be a domain (open and connected subset) and $f \in C^1(\Omega, R^n)$. Consider an autonomous ODE system

$$x' = f(x)$$

and let $\phi(t, x)$ be the solution map, i.e. $x(t) = \phi(t, x_0)$ is the solution with $x(0) = x_0$.

(1) Prove if $T > 0$ and $\phi(T, x_0) = x_0$, then $\phi(t + T, x_0) = \phi(t, x_0)$ for all $t \in R$.

(2) Assume $\Omega = R^n$. Let

$$F(x) = \frac{1}{1 + |f(x)|^2} f(x).$$

Let $\psi(t, x)$ be the solution map map of $x' = F(x)$. Prove $\psi(t, x)$ is defined for all $t \in R$ and $x \in R^n$.

(3) Assume $\partial\Omega \neq \emptyset$, let

$$F(x) = \frac{dist(x, \partial\Omega)}{(1 + |f(x)|^2)\big(1 + dist(x, \partial\Omega)\big)} f(x).$$

Let $\psi(t, x)$ be the solution map of $x' = F(x)$. Prove $\psi(t, x)$ is defined for all $t \in R$ and $x \in \Omega$. (Here $dist(x, \partial\Omega)$ is the distance function, which is Lipschitz in x and thus F is Lipschitz and ψ is locally well-defined.)