HW 2 4542

Chapter 8

#5 (a) $\begin{cases} x^2+y=0 \\ x-y+a=0 \end{cases} \Rightarrow \begin{cases} y=-x^2 \\ x^2+x+a=0 \end{cases}$

$\Delta = 1-4a$

- If $a>\frac{1}{4}$, then no solution for above system, no equilibrium points for the original system
- If $a=\frac{1}{4} \Rightarrow (-\frac{1}{2},-\frac{1}{4})$ one equilibrium point

Let $F(x,y)=\begin{pmatrix} x^2+y \\ x-y+a \end{pmatrix} \Rightarrow DF(x,y)=\begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$

$DF(-\frac{1}{2},-\frac{1}{4})=\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

- If $a<\frac{1}{4} \Rightarrow x_{\pm}=\frac{-1\pm\sqrt{1-4a}}{2}$, equilibria:

$\left(\frac{-1+\sqrt{1-4a}}{2}, -\frac{2-4a-2\sqrt{1-4a}}{4}\right)$, $\left(\frac{-1-\sqrt{1-4a}}{2}, -\frac{2-4a+2\sqrt{1-4a}}{4}\right)$

$DF(x_+,-x_+^2)$

$=\begin{pmatrix} -1+\sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}$

$DF(x_-,-x_-^2)$

$=\begin{pmatrix} -1-\sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}$

(b) • $a=\frac{1}{4}$ $\lambda_1=0,\ \lambda_2=-2$ for $DF(-\frac{1}{2},-\frac{1}{4})$

one stable direction, one center direction

$y+\frac{1}{4}=x+\frac{1}{2}$ line of equilibria

• For $a < \frac{1}{4}$.

$(x_+, -x_+^2)$ $\det(DF(x_+, -x_+^2)) = -\sqrt{1-4a} < 0$,

$\Rightarrow$ linearized system is of saddle type

$(x_-, -x_-^2)$ $\det(DF(x_-, -x_-^2)) = \sqrt{1-4a} > 0$

$tr(DF(x_-, -x_-^2)) = -2 - \sqrt{1-4a} < 0$

$\Rightarrow \lambda_1 < \lambda_2 < 0$, linearized system has a sink at $(x_-, -x_-^2)$

(c) $a = \frac{1}{4}$, there is a bifurcation point

$a < \frac{1}{4}$. two equilibria with one saddle, one sink

$a = \frac{1}{4}$. one equilibrium.

$a > \frac{1}{4}$, no equilibrium.

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#9 (a) $\begin{cases} r - r^2 = 0 & \Rightarrow \quad r = 0 \text{ or } r = 1 \\ \sin\theta + a = 0 & \Rightarrow \quad \sin\theta = -a \end{cases}$

$(0,0)$ is an equilibrium ; $r=1$ is invariant.

$a=-1$, $a=1$, bifurcation occurs

(b) • $a<-1$, $\sin\theta + a < 0$ for $\forall\,\theta$

every solution will move clockwise

if $0<r<1$, then $r'>0$, $r\uparrow$

$r>1$ then $r'<0$, $r\downarrow$

All solutions will approach $r=1$ (Limit cycle)

• $a=-1$, $\sin\theta + a < 0$ if $\theta \neq \frac{\pi}{2} + 2k\pi$, $k\in\mathbb{Z}$

$\sin\theta + a = 0$ if $\theta = \frac{\pi}{2} + 2k\pi$, $k\in\mathbb{Z}$.

an equilibrium point $(r,\theta) = (1, \frac{\pi}{2})$ occurs

• $-1<a<1$ $\sin\theta + a = 0$ has two solutions for $\theta\in[0,2\pi)$

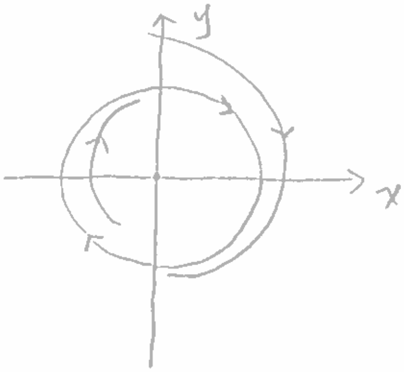
we have two equilibria on $r=1$, one is of saddle type. the other one is sink.

• $a=1$ $\sin\theta + a = 0$ has one solution for $\theta\in[0,2\pi)$

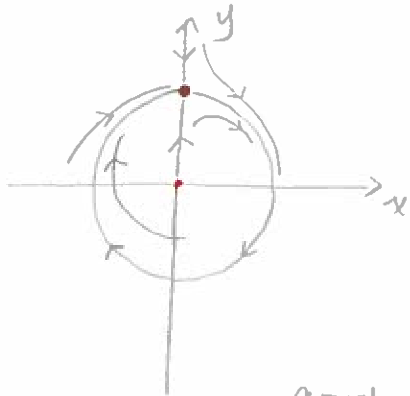
one equilibrium $(r,\theta) = (1, \frac{3\pi}{2})$

• $a>1$ $\sin\theta + a > 0$ every solution will move counterclockwise

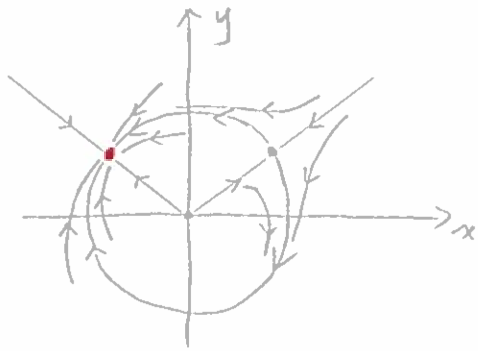
(c)



$a<-1$

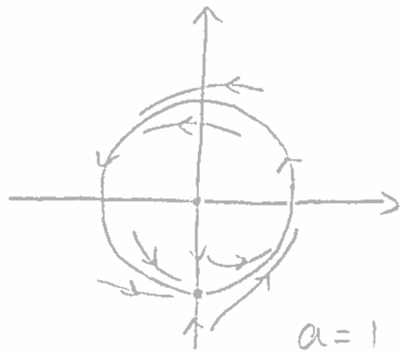


$a=-1$

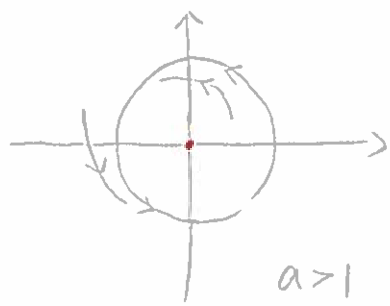


$-1<a<1$

(equilibria symmetric about y axis maybe above or below or on x axis)



$a=1$



$a>1$

(d) $a<-1$ or $a>1$, only 1 equilibrium

$a=\pm 1$, 2 equilibria

$-1<a<1$, 3 equilibria

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