HW 4 4542

Chap 9 #9

(a) If $X' = AX$ is a gradient system, then $\exists\, V(X)$ s.t.

$$-\nabla V(X) = AX \quad , \quad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\Rightarrow \frac{\partial V}{\partial x} = -ax - by \quad , \quad \frac{\partial V}{\partial y} = -cx - dy$$

Then, we must have $\dfrac{\partial^2 V}{\partial y\, \partial x} = \dfrac{\partial^2 V}{\partial x\, \partial y}$

$\Rightarrow -b = -c \Rightarrow b = c$

In this case $\dfrac{\partial V}{\partial x} = -ax - by$, $\dfrac{\partial V}{\partial y} = -bx - dy$

$\Rightarrow V(x,y) = -\frac{1}{2}ax^2 - bxy - \frac{d}{2}y^2 + C_0$. C_0 constant

One gradient function is $V(x,y) = -\frac{1}{2}ax^2 - bxy - \frac{d}{2}y^2$.

(b) If $X' = AX$ is a Hamiltonian system, then $\exists\, H$ s.t.

$$\frac{\partial H}{\partial y} = ax + by \quad , \quad -\frac{\partial H}{\partial x} = cx + dy$$

$$\frac{\partial^2 H}{\partial x\, \partial y} = a = \frac{\partial^2 H}{\partial y\, \partial x} = -d \qquad \text{Hence} \quad a = -d$$

In this case $\dfrac{\partial H}{\partial y} = ax + by$, $\dfrac{\partial H}{\partial x} = -cx + ay$

$$H(x,y) = \frac{1}{2}by^2 + axy - \frac{1}{2}cx^2 + c_1 \quad , \quad c_1 \text{ constant}$$

One hamiltonian is $H(x,y) = \frac{1}{2}by^2 + axy - \frac{1}{2}cx^2$.

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Chap 10 #1(c)

$\begin{cases} r' = \sin r \\ \theta' = -1 \end{cases} \Rightarrow \quad \sin r = 0 \Rightarrow r = k\pi, \; k \in \mathbb{N}$

$r = 0$, $(0,0)$ is an equilibrium point, unstable

$r = k\pi$, $k \in \mathbb{Z}^+$, is periodic orbit.

When $r \in (2k\pi, 2k\pi + \pi)$ $\quad r' > 0$. $\quad (k \in \mathbb{N})$

$r \in (2k\pi + \pi, 2k\pi + 2\pi)$ $\quad r' < 0$ $\quad (k \in \mathbb{N})$

- $\omega((0,0)) = \alpha((0,0)) = (0,0)$
- On periodic orbits: $r = k\pi$
  $\omega((k\pi, \theta)) = \alpha((k\pi, \theta)) = \{(r, \theta) \mid r = k\pi\}$
- When (r, θ) satisfy $r \in (2k\pi, 2k\pi + \pi)$ $\quad k \in \mathbb{N}$
  $\omega((r, \theta)) = \{(r, \theta) \mid r = 2k\pi + \pi\}$
  $\alpha((r, \theta)) = \{(r, \theta) \mid r = 2k\pi\}$
- When (r, θ) satisfy $r \in (2k\pi + \pi, 2k\pi + 2\pi)$ $\quad k \in \mathbb{N}$
  $\omega((r, \theta)) = \{(r, \theta) \mid r = 2k\pi + \pi\}$
  $\alpha((r, \theta)) = \{(r, \theta) \mid r = 2k\pi + 2\pi\}$.

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#4. Linearized system at $(\frac{\pi}{2}, \frac{\pi}{2})$ $\quad \begin{pmatrix} x - \frac{\pi}{2} \\ y - \frac{\pi}{2} \end{pmatrix}' = \begin{pmatrix} 0.1 & 1 \\ -1 & 0.1 \end{pmatrix} \begin{pmatrix} x - \frac{\pi}{2} \\ y - \frac{\pi}{2} \end{pmatrix}$

eigenvalues $\lambda = 0.1 \pm i$, $(\frac{\pi}{2}, \frac{\pi}{2})$ is a spiral source.

Now consider $L_0(x,y) = \sin x \sin y$ for $x \in (0, \pi)$, $y \in (0, \pi)$

$\frac{d}{dt} L_0(x,y) = \sin x \sin y \, (-0.1 \cos^2 x - 0.1 \cos^2 y)$, let $L(x,y) = \ln(\sin x \sin y)$

$\frac{d}{dt} L(x,y) = -0.1 (\cos^2 x + \cos^2 y) < 0$ unless $x = \frac{\pi}{2}, y = \frac{\pi}{2}$

Hence, we have no periodic solution in the square $(0,\pi) \times (0,\pi)$

$L_0(x,y)$ decreases along ~~at~~ any solution emanating from $(\frac{\pi}{2}, \frac{\pi}{2})$

Hence, solutions will spiral toward the boundary.

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