HW5 4542

Chap 10

#6 Consider

$$E(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}$$

$$E_x = x^3 - x \qquad E_y = y$$

$$\begin{cases} x' = -E_y - E E_x \\ y' = E_x - E E_y \end{cases}$$

(a) $\begin{cases} x' = 0 \\ y' = 0 \end{cases}$

If $E(x,y)=0$, then $y=0$, $x^3-x=0$

we get 3 equilibria. $(0,0)$, $(0,1)$, $(0,-1)$

If $E(x,y) \neq 0$, then $E_y = -E^2 E_y \Rightarrow y = -E^2 y$

$\Rightarrow y=0 \Rightarrow x^3-x=0$

Hence we only have 3 equilibria above.

(b) Consider Linearized system.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -(3x^2-1)(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}) - (x^3-x)^2 & -1 - y(x^3-x) \\ (3x^2-1) - (x^3-x)y & -y^2 - (\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

@ $(0,0)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda = \pm 1$

$(0,0)$ is a saddle

@ $(1,0)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 2 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} \Rightarrow \lambda = \frac{6 \pm \sqrt{36 - 4\times 8\times 17}}{16}$$

Complex conjugate eigenvalues with positive real parts
$\Rightarrow (1,0)$ spiral source

@ $(-1,0)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 2 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x+1 \\ y \end{pmatrix}$$

Same matrix as above $\Rightarrow$ $(-1,0)$ spiral source too.

(c). For any sol'n $(x(t), y(t))$

$$\frac{d}{dt} E(x(t), y(t)) = E_x\, x'(t) + E_y\, y'(t)$$

$$= -E_x (E_y + E E_x) + E_y (E_x - E E_y)$$

$$= -E (E_x^2 + E_y^2)$$

$E_x^2 + E_y^2 = 0$ iff $(x(t), y(t)) = (0,0)$ or $(1,0)$ or $(-1,0)$

Otherwise $E_x^2 + E_y^2 > 0$

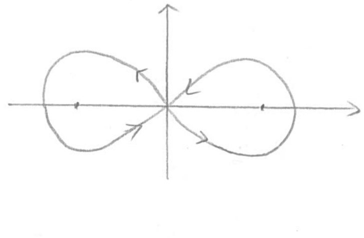
Then exclude the equilibria, if $E > 0$, then $\frac{d}{dt}E < 0$

if $E < 0$, then $\frac{d}{dt}E > 0$

if $E = 0$, then $\frac{d}{dt}E = 0$

$E = 0 \Rightarrow \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} = 0$

We have two homoclinic orbits plus $(0,0)$



(direction coming from the stable & unstable direction at $(0,0)$ for linearized system)

For (x,y) outside these homoclinic orbits, we have $E>0$

$\frac{d}{dt}E<0$, $\omega(x,y) = \{\text{homoclinic orbits}\} \cup \{(0,0)\}$

For (x,y) inside one of these homoclinic orbits, $E<0$,

$\frac{d}{dt}E>0$ $\omega(x,y) = \{\text{corresponding homoclinic orbit}\} \cup \{(0,0)\}$

Phase portrait as in P_{225}. #

#9 (a)

Four more cases if the rotating directions on T_1 and P_2 are changed at the same time.

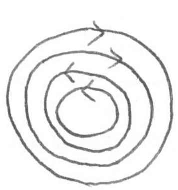
(b) For any $X \in \bar{A}$, since $\bar{A}$ compact $\omega(X) \subset \bar{A}$, $\omega(X) \neq \phi$.

Note that we cannot have any limit cycles because of area-preserving and Coro 1 in P_{226}.

Suppose no equilibrium point in A, then $\omega(X)$ is a closed orbit by Poincaré - Bendixson. Then we must have $X \in \omega(X)$.

Every X in A lies on ~~an~~ a periodic orbit.

Suppose we have a periodic orbit as left then $\exists$ an equilibrium in A by Coro 4 in P_{227}. Contradiction!



Suppose otherwise, then there will be a jump of rotating direction, contradiction!

Therefore, A contains an equilibrium. #