4542 HW 6

Chap 11 #4

(a) $y' = 0 \Rightarrow y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$

If $y=0$, $x'=0 \Rightarrow x(1-x) = 0$ $\quad x=1$ or $x=0$

$(0,0)$ or $(1,0)$

If $y=1$, $x'=0 \Rightarrow x\left(1-x-\frac{1}{x+b}\right) = 0$

$\Rightarrow x=0$ or $x = \frac{1-b \pm \sqrt{(1-b)(-3-b)}}{2}$

If $b>1$, we have

$(0,1)$ and $\left(\frac{1-b \pm \sqrt{(1-b)(-3-b)}}{2}, 1\right)$, $(0,0)$, $(1,0)$

as equilibria in the region $\{x \geq 0, y \geq 0\}$.

If $b \leq 1$, we have

$(0,1)$, $(0,0)$, $(1,0)$ as equilibria in $\{x \geq 0, y \geq 0\}$

Let $F(x,y) = \begin{pmatrix} x(1-x) - \frac{xy}{x+b} \\ y(1-y) \end{pmatrix}$, then

$$DF(x,y) = \begin{pmatrix} 1-2x-\frac{y}{x+b}+\frac{xy}{(x+b)^2} & -\frac{x}{x+b} \\ 0 & 1-2y \end{pmatrix}$$

$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad (0,0)$ unstable

$DF(1,0) = \begin{pmatrix} -1 & -\frac{1}{1+b} \\ 0 & 1 \end{pmatrix}$ $\quad (1,0)$ saddle

$$DF(0,1) = \begin{pmatrix} 1-\frac{1}{b} & 0 \\ 0 & -1 \end{pmatrix}$$

For $b<1$, $1-\frac{1}{b}<0$

$(0,1)$ stable

For $b>1$, $(0,1)$ saddle

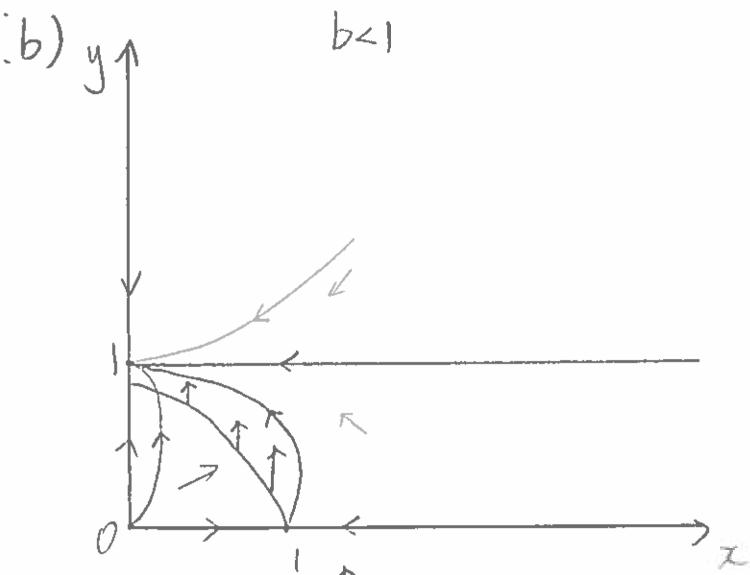
$$DF\left(\frac{1-b+\sqrt{(1-b)(-3-b)}}{2}, 1\right)$$

$$= \begin{pmatrix} \frac{4}{(1+b+\sqrt{(1-b)(-3-b)})^2} \cdot \frac{1}{4}\left[3-5b-\sqrt{(b-1)(b+3)}\right] & -\frac{x}{x+b} \\ 0 & -1 \end{pmatrix}$$

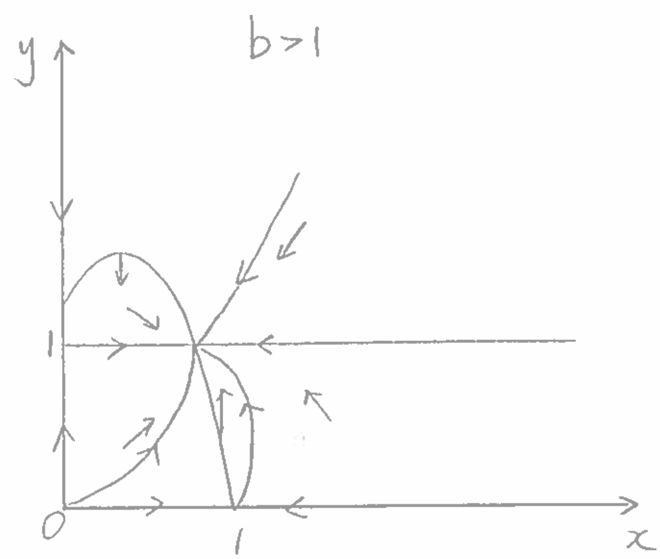
$\lambda_1<0$, $\lambda_2<0$ when $b>1$.

$\left(\frac{1-b+\sqrt{(b+3)(b-1)}}{2}, 1\right)$ stable

(b)



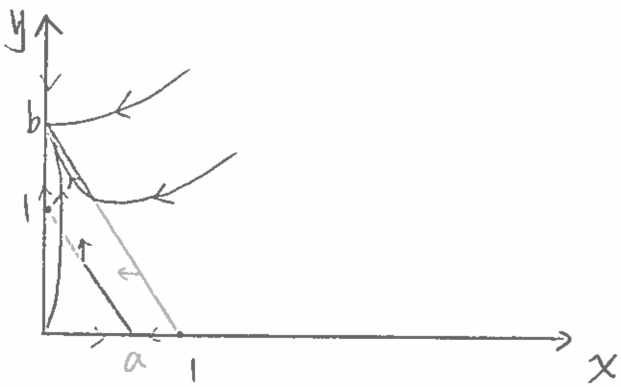
b=1 similar to this



(c) A bifurcation occurs when $b=1$. from $b<1$ to $b>1$ we have a new equilibrium and $(0,1)$ undergoes a stability change.

#6 $\begin{cases} x' = x(a-x-ay) \\ y' = y(b-bx-y) \end{cases}$ $F(x,y) = \begin{pmatrix} x(a-x-ay) \\ y(b-bx-y) \end{pmatrix}$

① $a<1,\ b>1$



eq:

$(a,0)$ $(0,0)$ $(0,b)$

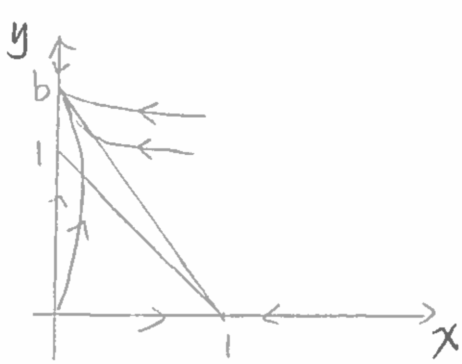
$$DF(x,y) = \begin{pmatrix} a-2x-ay & -ax \\ -by & b-2y-bx \end{pmatrix}$$

$DF(a,0) = \begin{pmatrix} -a & -a^2 \\ 0 & b-ab \end{pmatrix}$ saddle

$DF(0,0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ unstable

$DF(0,b) = \begin{pmatrix} a-ab & 0 \\ -b^2 & -b \end{pmatrix}$ stable

② $a=1,\ b>1$

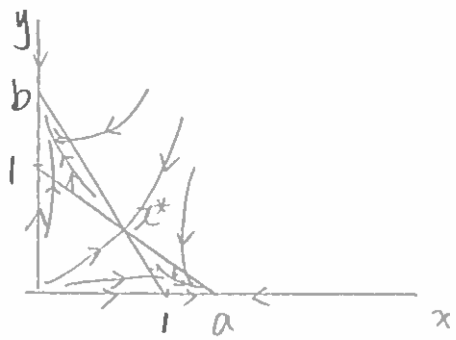


$(0,0)$ unstable

$(0,b)$ stable

$(1,0)=(a,0)$ unstable by phase portrait

③ $a>1 \quad b>1$



(0,0) unstable

(a,0) stable

(0,b) stable

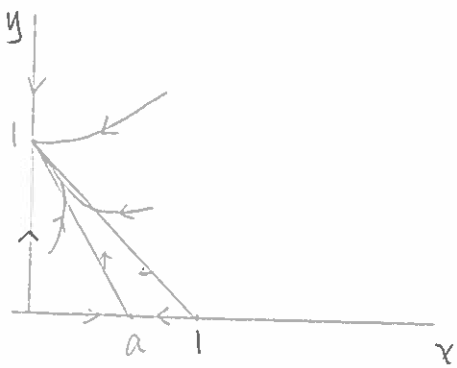
New eq. $\left(\frac{a(b-1)}{ab-1}, \frac{b(a-1)}{ab-1}\right) = x^*$

$$DF\left(\frac{a(b-1)}{ab-1}, \frac{b(a-1)}{ab-1}\right) = \begin{pmatrix} \frac{a-ab}{ab-1} & -\frac{a^2(b-1)}{ab-1} \\ -\frac{b^2(a-1)}{ab-1} & \frac{b-ab}{ab-1} \end{pmatrix}$$

$$\det = \frac{1}{(ab-1)^2}(a-1)(b-1)\left[ab - a^2b^2\right] < 0$$

x^* saddle pt.

④ $a<1 \, , \, b=1$

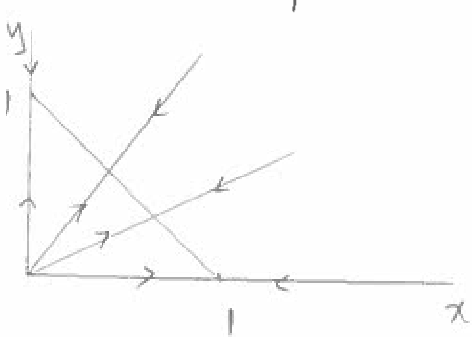


(0,0) unstable

(a,0) saddle

(0,1) = (0, b) stable by phase portrait

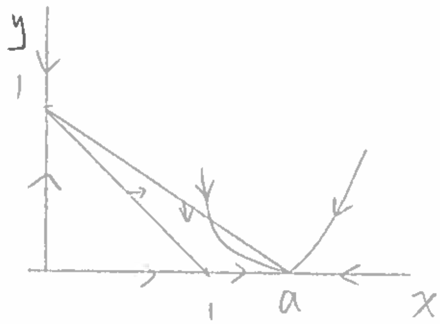
⑤ $a=1 \quad b=1$



(0,0) unstable

∞ many equilibria

⑥ $a>1 \quad b=1$

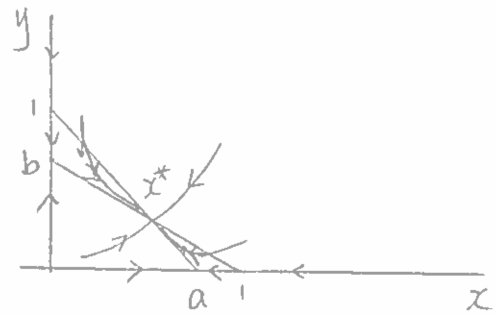


$(0,0)$ unstable

$(a,0)$ stable

$(0,1)=(0,b)$ unstable

⑦ $a<1 \quad b<1$



Eq: $x^* = \left(\dfrac{a(b-1)}{ab-1}, \dfrac{b(a-1)}{ab-1} \right)$

$\det(DF(x^*)) > 0$

$\mathrm{tr}(DF(x^*)) < 0$

x^* stable eq.

$(0,0)$ unstable $\quad (a,0)$ saddle $\quad (0,b)$ saddle

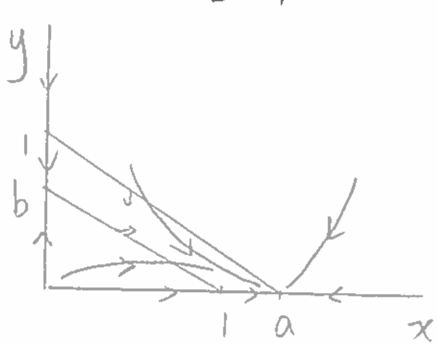
⑧ $a=1 \quad b<1$

$(0,0)$ unstable

$(1,0)$ stable

$(0,b)$ saddle

⑨ $a>1 \quad b<1$



$(0,0)$ unstable

$(a,0)$ stable

$(0,b)$ saddle

Bifurcation occurs when a, b go from smaller than 1 to larger than 1

Number of equilibria changes

Stability of equilibria also changes.