Assignment-2 (Solutions)

1.5.9

a) e^{-n^2t} Converges to Zero. Theorem 7 applies, and u Can be differentiated

$$u''(x,t) = \sum_{n=1}^{\infty} -n^2 b_n e^{n^2 t} \sin nx$$

b) $x = 0$: $\sin(nx) = 0 \Rightarrow u(0,t) = 0$

$x = \pi$; $\sin(nx) = 0 \Rightarrow u(\pi,t) = 0$

c) $u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx$

$u(x,0) = \frac{1}{2}\left(f(x+) + f(x-)\right)$

1.6.2 a) $f(x) = x$, $-1 < x < 1$

Parseval's Identity

$$\frac{1}{a}\int_{-a}^{a} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2$$ as $a + b$ represents a odd functions

$$\frac{1}{a}\int_{-1}^{1} f^2(x)\,dx = \int_{-1}^{1} x^2\,dx = 2/3 \rightarrow LHS$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)dx = -\frac{2}{n\pi}(-1)^n$$

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)^2(-1)^n = \frac{4}{\pi} \times \frac{\pi}{6} = 2/3 \rightarrow RHS$$

$LHS = RHS$, Hence Proved.

b) $f(x) = \sin x$; $-\pi < x < \pi$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left(2\pi - \frac{\sin 2x}{2}\Big|_{-\pi}^{\pi}\right)$$

$= \underline{1} \rightarrow$ LHS

$f(x) = \sin x$; $= b_n \sin nx$ $\therefore$ $b_n = 1$ $n = 1$

$\therefore f(x) = 1 \rightarrow$ RHS

Hence Proved

1.9 3 a) Find the Fourier Integral Representation of $f(x) = \frac{1}{1+x^2}$

$$\text{EXP}(-|x|) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\lambda x)}{1+x^2}\,dx$$

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$$f(x) = \int_0^\infty \text{EXP}(-|\lambda|)\cos \lambda x \, dx \quad , \quad 0 < x < \infty$$

b) $f(x) = \frac{\sin(x)}{x}$

$$f(x) = \int_0^\infty \frac{2\sin \lambda}{\pi \lambda} \cos \lambda x \, dx.$$

$$f(\lambda) = 2/\pi \int_0^\infty \frac{\sin x}{x} \cos \lambda x \, dx$$

$$F(x) = \int_0^\infty \cos(\lambda x)\, dx.$$

1.9 6a) Show that if K is positive, then.

$$\int_0^\infty e^{-Kx} \sin(x)\, dx = \frac{1}{1+K^2}.$$

$$\int_0^\infty e^{-Kx} \sin x\, dx = \left[\frac{e^{-Kx}}{-K} \sin x\right]_0^\infty + \int \frac{e^{-Kx}}{K} \cos x\, dx.$$

$$= \frac{-1}{K}\left[e^{-K\infty} \sin \infty - 0\right] + \left. \frac{e^{-Kx}}{K^2} \cos x \right|_0^\infty$$

$$- 1/K^2 \int_0^\infty e^{-Kx} \sin x\, dx$$

$$\left(1 + 1/K^2\right) \int_0^\infty e^{-Kx} \sin(x)\, dx = 1/K^2$$

$$\Rightarrow \int_0^\infty e^{-Kx} \sin(x)\, dx = \frac{1}{1+K^2}$$

1.10.3 a) $$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$C(\lambda) = \frac{1}{2\pi} \int_0^\infty e^{-x} \cdot e^{-i\lambda x}\, dx.$$

$$C(\lambda) = \frac{1}{2\pi}\left(\frac{1}{1+i\lambda}\right) \text{(or) } f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\lambda x}}{1+i\lambda}\, d\lambda$$

3b)

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{else where} \end{cases}$$

$$C(\lambda) = \frac{1}{2\pi} \int_0^{\pi} \sin x \; e^{-i\lambda x} \, dx.$$

$$C(\lambda) = \frac{1}{2\pi} \left[-\frac{e^{-i\lambda x} \sin x}{i\lambda} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos x \; e^{-i\lambda x}}{i\lambda} \, dx.$$

$$C(\lambda) = \frac{1}{2\pi} \left[\frac{\cos x \cdot e^{-i\lambda x}}{\lambda^2} \right]_0^{\pi} + \frac{1}{\lambda^2} C(\lambda)$$

$$C(\lambda)\left(1 - \frac{1}{\lambda^2}\right) = \left[\frac{1 + e^{-i\lambda\pi}}{\lambda^2} \right] \frac{1}{2\pi}$$

$$C(\lambda = \frac{1 + e^{-i\lambda\pi}}{(1 - \lambda^2}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 + e^{-i\lambda\pi})}{(1 - \lambda^2)} \cdot e^{i\lambda x} \, dx.$$