Small BGK waves and nonlinear Landau damping (higher dimensions)

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Abstract

Consider Vlasov-Poisson system with a fixed ion background and periodic boundary conditions on the space variables, in dimension $d = 2,3$. First, we show that for general homogeneous equilibrium and any periodic $x-$box, within any small neighborhood in the Sobolev space $W_{x,v}^{s,p}$ $\left(p>1, s<1+\frac{1}{p}\right)$ of the steady distribution function, there exist nontrivial travelling wave solutions (BGK waves) with arbitrary traveling speed. This implies that nonlinear Landau damping is not true in $W^{s,p}$ $\left(s<1+\frac{1}{p}\right)$ space for any homogeneous equilibria and in any period box. The BGK waves constructed are one dimensional, that is, depending only on one space variable. Higher dimensional BGK waves are shown to not exist. Second, we prove that there exist no nontrivial invariant structures in some neighborhood of stable homogeneous equilibria, in the $\left(1+|v|^2\right)^b$-weighted $H_{x,v}^s$ $\left(b>\frac{d-1}{4},\ s>\frac{3}{2}\right)$ space. Since arbitrarilly small BGK waves can also be constructed near any homogeneous equilibria in such weighted $H_{x,v}^s$ $\left(s<\frac{3}{2}\right)$ norm, this shows that $s=\frac{3}{2}$ is the critical regularity for the existence of nontrivial invariant structures near stable homogeneous equilibria. These generalize our previous results in the one dimensional case.

1 Introduction

Consider a collisionless electron plasma with a fixed homogeneous neutralizing ion background. The Vlasov-Poisson system in d dimension is

$$\partial_t f + v\cdot\nabla_x f - \vec{E}\cdot\nabla_v f = 0, \tag{1a}$$

$$\vec{E} = -\nabla_x\phi, \quad -\Delta\phi = -\int_{\mathbf{R}^d} f\ dv + 1, \tag{1b}$$

where $f(t,x,v)\geq 0$ is the distribution function, $\vec{E}(x,t)$ is the electrical field and $\phi(x,t)$ is the electrical potential. We consider the Vlasov-Poisson system in a $x-$periodic box, with periods T_i in x_i. In 1946, Landau [6], looking for analytical solutions of the linearized Vlasov-Poisson system around Maxwellian $\left(e^{-\frac{1}{2}|v|^2}, 0\right)$, pointed out that the electric field is subject to time decay even in the absence of collisions. The effect of this Landau damping, as it is subsequently called, plays a fundamental role in the study of plasma physics. However, Landau's treatment is in the linear regime; that is, only for infinitesimally small initial perturbations. Recently, nonlinear Landau damping was shown ([12]) for analytical perturbations of stable equilibria with linear exponential decay. For general perturbations in Sobolev spaces, the proof of nonlinear damping remains open. We refer to [9] [12] for more discussions and references on this topic. In [9], the following results were obtained for 1D Vlasov-Poisson system. First, we show that for general homogeneous equilibria, within any small neighborhood in the Sobolev space $W^{s,p}$ $\left(p > 1, s < 1 + \frac{1}{p}\right)$ of the steady distribution function, there exist nontrivial travelling wave solutions (BGK waves) with arbitrary minimal period and traveling speed. This implies that nonlinear Landau damping is not true in $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$ space for any homogeneous equilibria and any spatial period. Second, it is shown that for homogeneous equilibria satisfying Penrose's linear stability condition, there exist no nontrivial travelling BGK waves and unstable homogeneous states in some $W^{s,p}$ $\left(p > 1, s > 1 + \frac{1}{p}\right)$ neighborhood. Furthermore, we prove that there exist no nontrivial invariant structures in the H^s $\left(s > \frac{3}{2}\right)$ neighborhood of stable homogeneous states. In particular, these results suggest the contrasting long time dynamics in the H^s $\left(s > \frac{3}{2}\right)$ and H^s $\left(s < \frac{3}{2}\right)$ neighborhoods of a stable homogeneous state.

In this paper, we generalize above results to higher dimensions ($d = 2, 3$). Denote the fractional order Sobolev spaces by $W^{s,p}\left(\mathbf{R}^d\right)$ or $W^{s,p}_{x_1,v}\left((0, T_1) \times \mathbf{R}^d\right)$ with $p > 1, s \geq 0$. These spaces are the complex interpolation of of L^p space and Sobolev spaces $W^{m,p}$ (m positive integer). Our first result is to construct 1D BGK waves in $W^{s,p}_{x_1,v}$ $\left(s < 1 + \frac{1}{p}\right)$ spaces.

Theorem 1.1 *Assume the homogeneous distribution function*

$$f_0(v) \in W^{s,p}\left(\mathbf{R}^d\right) \qquad \left(d \geq 2,\ p > 1, s \in [0, 1 + \frac{1}{p})\right)$$

satisfies

$$f_0(v) \geq 0,\ \int f_0(v)\, dv = 1,\ \int v^2 f_0(v)\, dv < +\infty.$$

*Fix $T_1 > 0$ and $c \in \mathbf{R}$. Then for any $\varepsilon > 0$, there exist travelling BGK wave solutions of the form $f = f_\varepsilon(x_1 - ct, v)$, $\vec{E} = E_\varepsilon(x_1 - ct)\vec{e}_1$ to (**??**), such that $(f_\varepsilon(x_1, v), E_\varepsilon(x_1))$ has minimal period T_1 in x_1, $f_\varepsilon(x_1, v) \geq 0$, $E_\varepsilon(x_1)$ is not identically zero, and*

$$\|f_\varepsilon - f_0\|_{L^1_{x_1,v}} + \int_0^{T_1}\int_{\mathbf{R}^d} |v|^2 |f_\varepsilon - f_0|\, dx_1 dv + \|f_\varepsilon - f_0\|_{W^{s,p}_{x_1,v}} < \varepsilon. \qquad (2)$$

In Proposition 2.1, we show that there exist no 2D and 3D BGK waves. Therefore, the form of 1D BGK waves in Theorem 1.1 is in some sense necessary.

For any $b > \frac{d-1}{4}$, we denote $H_v^{s,b}\left(\mathbf{R}^d\right)$ to be the $\left(1+|v|^2\right)^b$ weighted H^s space, that is,

$$H_v^{s,b}\left(\mathbf{R}^d\right) = \left\{ f \mid \left\| \left(1+|v|^2\right)^b (1-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbf{R}^d)} < \infty \right\}. \tag{3}$$

and

$$\|f\|_{H_v^{s,b}} = \left\| \left(1+|v|^2\right)^b (1-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbf{R}^d)}.$$

Let $\mathbf{T}^d$ be a periodic box with periods T_i in x_i $(i = 1, \cdots, d)$, and

$$\mathbf{Z}^d = \left\{ \left(\frac{2\pi}{T_1} j_1, \cdots, \frac{2\pi}{T_d} j_d \right) \mid j_1, \cdots, j_d \text{ are integers} \right\}.$$

We define the space $H_x^{s_x} H_v^{s_v,b}\left(\mathbf{T}^d \times \mathbf{R}^d\right)$ by

$$h = \sum_{\vec{k} \in \mathbf{Z}^d} e^{i\vec{k}\cdot x} h_{\vec{k}}(v) \in H_x^{s_x} H_v^{s_v,b}$$

if

$$\|h\|_{H_x^{s_x} H_v^{s_v,b}} = \left(\left\|h_{\vec{0}}\right\|^2_{H_v^{s_v,b}} + \sum_{0 \neq \vec{k} \in \mathbf{Z}^d} \left|\vec{k}\right|^{2s_x} \left\|h_{\vec{k}}\right\|^2_{H_v^{s_v,b}} \right)^{\frac{1}{2}} < \infty.$$

Given a homogeneous profile $f_0(v) \in H^{s,b}\left(\mathbf{R}^d\right)$ $\left(d \geq 2,\ s > \frac{3}{2},\ b > \frac{d-1}{4}\right)$, we have the following generalization of Penrose's stability condition in higher dimensions

$$\left|\vec{k}\right|^2 - \max_{v_i \in S_{\vec{k}/|\vec{k}|}} \int_{\mathbf{R}} \frac{f'_{0,\vec{k}/|\vec{k}|}(\alpha)}{\alpha - v_i} d\alpha > 0, \text{ for any } \vec{k} \in Z^d, \tag{4}$$

where $f'_{0,\vec{k}/|\vec{k}|}(\alpha)$ is the projection of $f_0(v)$ to $\vec{k}$ direction, as defined by (22) and $S_{\vec{k}/|\vec{k}|}$ is the set of all critical points of $f_{0,\vec{k}/|\vec{k}|}(\alpha)$. By Corollary 3.1, one only need to check the condition (4) for finitely many $\vec{k} \in Z^d$ with $\left|\vec{k}\right|^2 \leq C(d,s,b)\|f_0\|_{H^{s,b}(\mathbf{R}^d)}$. In particular, for a single humped profile $f_0(v) = \mu\left(\frac{1}{2}|v|^2\right)$ with $\mu'(e) < 0$, the stability condition (4) is satisfied for any period set $(T_1, \cdots, T_d)$.

The following Theorem excludes any nontrivial invariant structures (steady, time periodic, quasi-periodic etc.) near stable homogeneous equilibria satisfying (4), in the $H_x^{s_x} H_v^{s_v,b}$ spaces of high $v-$regularity.

Theorem 1.2 *Consider a homogeneous profile*

$$f_0(v) \in H^{s_0,b}\left(\mathbf{R}^d\right) \quad \left(d \geq 2,\ s_0 > \frac{3}{2},\ b > \frac{d-1}{4}\right), \tag{5}$$

with $f_0(v) \geq 0$ *and* $\int f_0(v)\,dv = 1$*. Let* T^d *be a periodic box with periods* T_i *in* x_i $(i = 1, \cdots, d)$*. Assume that* $f_0(v)$ *satisfies the Penrose stability condition (4) for* $(T_1, \cdots, T_d)$*. Let* $\left(f(x,v,t), \vec{E}(x,v,t)\right)$ *be the solution of (**??**) in* T^d*.*

For any (s_x, s_v) *satisfying*

$$s_x \geq 0,\ s_x > \frac{d-3}{2},\ \text{and}\ \frac{3}{2} < s_v \leq s_0, \tag{6}$$

there exists $\varepsilon_0 > 0$*, such that if*

$$\|f(t) - f_0\|_{H_x^{s_x} H_v^{s_v,b}} < \varepsilon_0,\ \text{for all}\ t \in \mathbf{R}, \tag{7}$$

then $\vec{E}(t) \equiv \vec{0}$ *for all* $t \in \mathbf{R}$*.*

In the above Theorem, the assumption $s_x > \frac{d-3}{2},\ s_x \geq 0$ is to make $H_x^{s_x}$ an algebra which is needed in the proof of Lemma 3.3. The use of weighted Sobolev space $H_v^{s,b}$ in Theorem 1.2 is rather natural in higher dimensions. Indeed, even to state the Penrose's stability condition (4), we need to assume that the homogeneous equilibrium $f_0(v) \in H^{s_0,b}\left(\mathbf{R}^d\right)$ with (s_0, b) satisfying (5). This is because that linear instability (stability) of homogeneous equilibria of Vlasov-Poisson is longitudinal along the wave direction of perturbation. The weighted Sobolev space (5) is needed to ensure that the projected steady distribution function in any wave direction is in $H^s(\mathbf{R})$ $\left(s > \frac{3}{2}\right)$ which is necessary to get the 1D Penrose stability criterion. Moreover, in Theorem 2.1 we also construct (1D) BGK waves arbitrarily near any homogeneous equilibrium in $H_x^{s_x} H_v^{s_v,b}(d \geq 2,\ b > \frac{d-1}{4})$ for any $s_x > 0$ and $s_v < \frac{3}{2}$. Combined with Theorem 1.2, this shows that for weighted Sobolev spaces $H_x^{s_x} H_v^{s_v,b}$, the critical $v-$regularity for the existence of nontrivial invariant structures near a stable homogeneous equilibrium is $s_v = \frac{3}{2}$. This generalizes our 1D results in [9] to higher dimensions. We note that the critical regularity $s_v = \frac{3}{2}$ does not depend on the dimension. This illustrates again the longitudinal (1D) nature of Landau damping, which is obvious in the linear regime.

We briefly mention some differences of the long time behaviors of Vlasov-Poisson in 1D and higher dimensions. For the 1D case, numerical simulations (e.g. [4] [3]) indicated that for certain small initial data near a stable homogeneous state including Maxwellian, there is no decay of electric fields and the asymptotic state is a BGK wave or the superposition of BGK waves. Moreover, BGK waves also appear as the asymptotic states for the saturation of an unstable homogeneous state ([1]) in 1D. These suggest that small BGK waves play an important role in understanding the long time behaviors of 1D Vlasov-Poisson system. However, for 2D and 3D Vlasov-Poisson system, numerical simulations ([11] [13]) suggested that when starting near a homogeneous state (both stable and unstable), the electric fields decay eventually. Our Theorems 1.1 and 2.1 on existence of 1D BGK waves show that such decay of electric field is not true for general initial data near homogeneous states. But the numerical simulations seem to suggest that these 1D BGK waves do not appear in the long time dynamics in 2D and 3D cases. To explain these phenomena, it will be important to understand the transversal instability of 1D BGK waves.

This paper is organized as follows. In Section 2, we prove the existence of 1D BGK waves in $W^{s,p}$ $\left(s<1+\frac{1}{p}\right)$ neighborhoods of homogeneous states. In Section 3, we use the linear decay estimate to show that all invariant structures near stable homogeneous equilibria in $H_x^{s_x}H_v^{s_v,b}$ spaces satisfying (6) are trivial. Throughout this paper, we use C to denote a generic constant in the estimates and the dependence of C is indicated only when it matters in the proof.

2 Existence of BGK waves in $W^{s,p}$ $\left(s<1+\frac{1}{p}\right)$

In this Section, we construct nontrivial steady states (BGK waves) near any homogeneous state in the space $W_{x,v}^{s,p}$ $\left(s<1+\frac{1}{p}\right)$. We consider $d=2$ only, since the proof is almost the same for $d=3$. The BGK waves we construct are one-dimensional, that is, the steady distribution $f=f\left(x_1,v_1,v_2\right)$ and the electric field $\vec{E}=E\left(x_1\right)\vec{e}_1$. We will show that such a restriction is necessary by excluding $2D$ and $3D$ BGK waves.

Proof of Theorem 1.1. We adapt the line of proof in [9] to construct BGK wave solutions for $2D$ Vlasov-Poisson equations. First, we modify $f_0\left(v\right)$ to a smooth function $f_1\left(v\right)$ with some additional properties. Let $\eta\left(v\right)$ $\left(v\in\mathbf{R}^2\right)$ be the standard mollifier function. For $\delta_1>0$ define $f_{\delta_1}\left(v\right)=\eta_{\delta_1}\left(v\right)*f_0\left(v\right)$, where $\eta_{\delta_1}\left(v\right)=\frac{1}{\delta_1^2}\eta\left(\frac{v}{\delta_1}\right)$. Then by the properties of mollifiers, we have

$$f_{\delta_1}\in C^{\infty}\left(\mathbf{R}\right),\ f_{\delta_1}\left(v\right)\geq 0,\ \int_{\mathbf{R}^2}f_{\delta_1}\left(v\right)dv=1,$$

and when δ_1 is small enough

$$\left\|f_{\delta_1}-f_0\right\|_{L^1\left(\mathbf{R}^2\right)}+\int_{\mathbf{R}^2}\left|v\right|^2\left|f_{\delta_1}-f_0\right|\ dv+\left\|f_{\delta_1}-f_0\right\|_{W^{s,p}\left(\mathbf{R}^2\right)}\leq\frac{\varepsilon}{6}.$$

Modifying $f_{\delta_1}\left(v\right)$ near infinity by cut-off, we can assume in addition that $f_{\delta_1}\left(v\right)\in H^{2,b}\left(\mathbf{R}^2\right)$ (defined in (3)). In the second step, let $\sigma\left(x_1\right)=\sigma\left(\left|x_1\right|\right)$ be the 1D cut-off function. Let $\delta_2>0$ be a small number, and define

$$\begin{aligned} f_{\delta_1,\delta_2}\left(v_1,v_2\right)&=f_{\delta_1}\left(v_1,v_2\right)\left(1-\sigma\left(\frac{v_1}{\delta_2}\right)\right)+\left(\frac{f_{\delta_1}\left(v_1,v_2\right)+f_{\delta_1}\left(-v_1,v_2\right)}{2}\right)\sigma\left(\frac{v_1}{\delta_2}\right)\\ &=f_{\delta_1}\left(v_1,v_2\right)-\left(\frac{f_{\delta_1}\left(v_1,v_2\right)-f_{\delta_1}\left(-v_1,v_2\right)}{2}\right)\sigma\left(\frac{v_1}{\delta_2}\right).\end{aligned}$$

Then,

$$f_{\delta_1,\delta_2}\in C^\infty\left(\mathbf{R}^2\right),\ f_{\delta_1,\delta_2}(v)>0,\ \int_{\mathbf{R}^2}f_{\delta_1,\delta_2}(v)\,dv=\int_{\mathbf{R}^2}f_{\delta_1}(v)\,dv=1,$$

and $f_{\delta_1,\delta_2}(v_1,v_2)$ is even in v_1 when $v_1\in[-\delta_2,\delta_2]$. We show that: when δ_2 is small enough

$$\|f_{\delta_1,\delta_2}-f_{\delta_1}\|_{L^1(\mathbf{R}^2)}+\int_{\mathbf{R}^2}|v|^2\left|f_{\delta_1,\delta_2}-f_{\delta_1}\right|\,dv+\|f_{\delta_1,\delta_2}-f_{\delta_1}\|_{W^{s,p}(\mathbf{R}^2)}\leq\frac{\varepsilon}{6}. \quad (8)$$

Fist, a minor modification of the proof of Lemma 2.2 in [9] yields that: when $\delta_2\to0$,

$$\|f_{\delta_1}-f_{\delta_1,\delta_2}\|_{L^1(\mathbf{R}^2)}+\int_{\mathbf{R}^2}|v|^2\left|f_{\delta_1}-f_{\delta_1,\delta_2}\right|\,dv+\ \|f_{\delta_1}-f_{\delta_1,\delta_2}\|_{W^{1,p}(\mathbf{R}^2)}\to0.$$

It remains to show that

$$\|\nabla\left(f_{\delta_1}-f_{\delta_1,\delta_2}\right)\|_{W^{s-1,p}(\mathbf{R}^2)}\to0,\text{ when }\delta_2\to0. \quad (9)$$

We have

$$\partial_{v_2}\left(f_{\delta_1}-f_{\delta_1,\delta_2}\right)=\left(\frac{\partial_{v_2}f_{\delta_1}(v_1,v_2)-\partial_{v_2}f_{\delta_1}(-v_1,v_2)}{2}\right)\sigma\left(\frac{v_1}{\delta_2}\right),$$

and

$$\partial_{v_1}\left(f_{\delta_1}-f_{\delta_1,\delta_2}\right)=\left(\frac{\partial_{v_1}f_{\delta_1}(v_1,v_2)+\partial_{v_1}f_{\delta_1}(-v_1,v_2)}{2}\right)\sigma\left(\frac{v_1}{\delta_2}\right)+\sigma'\left(\frac{v_1}{\delta_2}\right)\frac{v_1}{\delta_2}\frac{f_{\delta_1}(v_1,v_2)-f_{\delta_1}(-v_1,v_2)}{2v_1}.$$

By a scaling argument as in the proof of Lemma 2.2 of [9],

$$\left\|\frac{f_{\delta_1}(v_1,v_2)-f_{\delta_1}(-v_1,v_2)}{2v_1}\right\|_{W^{s-1,p}(\mathbf{R}^2)}\leq C\,\|f_{\delta_1}\|_{W^{s,p}(\mathbf{R}^2)}.$$

So (9) follows from Lemma 2.1 below. Thus for any fixed $\varepsilon>0$, by choosing δ_1,δ_2 small enough, we get

$$\|f_{\delta_1,\delta_2}-f_0\|_{L^1(\mathbf{R}^2)}+\int_{\mathbf{R}^2}|v|^2\left|f_{\delta_1,\delta_2}-f_0\right|\,dv+\|f_{\delta_1,\delta_2}-f_0\|_{W^{s,p}(\mathbf{R}^2)}\leq\frac{\varepsilon}{3}.$$

We set $f_1(v_1,v_2)=f_{\delta_1,\delta_2}(v_1,v_2)$, then

$$f_1(v)>0,\quad f_1(v)\in C^\infty\left(\mathbf{R}^2\right)\cap H^{2,b}\left(\mathbf{R}^2\right),\ \int_{\mathbf{R}^2}f_1(v)\,dv=1,$$

$f_1(v)$ is even for v_1 in $[-\delta_2,\delta_2]$ and within $\frac{\varepsilon}{3}$ distance of $f_0(v)$ in the norm of (2). Below, we denote $a=\delta_2/2$.

Fix the x_1-period $T_1 > 0$, we only consider the travel speed $c = 0$ since the construction for any $c \in \mathbf{R}$ follows by the Galilean transform as in [9]. Our strategy is to construct BGK wave solutions of the form $(f_\varepsilon(x_1, v_1, v_2), E_\varepsilon(x_1)\vec{e}_1)$ by bifurcation at a modified homogeneous profile near $f_1(v_1, v_2)$. Denote $\sigma(x) = \sigma(|x|)$ to be the cut-off function such that $\sigma(x) \in C_0^\infty(\mathbf{R})$,

$$0 \leq \sigma(x) \leq 1;\ \sigma(x) = 1 \text{ when } |x| \leq 1;\ \sigma(x) = 0 \text{ when } |x| \geq 2. \tag{10}$$

Similar to Lemma 2.1 in [9], there exists $g_0(x_1, x_2) \in C^\infty(\mathbf{R}^2)$, $g_0 = 0$ when $|x_1| \geq 4a^2$, such that

$$f_1(v_1, v_2)\,\sigma\left(\frac{v_1}{a}\right) = g_0\left(v_1^2, v_2\right).$$

Define $g_+(x_1, x_2),\ g_-(x_1, x_2) \in C^\infty(\mathbf{R}^2)$ by

$$g_\pm(x_1, x_2) = \begin{cases} f_1\left(\pm\sqrt{x_1}, x_2\right)\left(1 - \sigma\left(\frac{\sqrt{x_1}}{a}\right)\right) + g_0(x_1, x_2) & \text{if } x_1 > a^2 \\ g_0(x_1, x_2) & \text{if } -4a^2 < x_1 \leq a^2 \\ 0 & \text{if } x_1 \leq -4a^2 \end{cases}.$$

Then

$$f_1(v_1, v_2) = \begin{cases} g_+\left(v_1^2, v_2\right) & \text{if } v_1 > 0 \\ g_-\left(v_1^2, v_2\right) & \text{if } v_1 \leq 0 \end{cases}.$$

Since $\partial_{v_1} f_1(0, v_2) = 0$, $f_1 \in C^\infty(\mathbf{R}^2) \cap H^{2,b}(\mathbf{R}^2)$, we have

$$\left|\int_{\mathbf{R}^2} \frac{\partial_{v_1} f_1(v)}{v_1} dv\right| < \infty.$$

Indeed, let $\bar{f}_1(v_1) = \int_{\mathbf{R}} f_1(v_1, v_2)\ dv_2$, then since $\bar{f}_1'(0) = 0$, by Corollary 3.1,

$$\left|\int_{\mathbf{R}^2} \frac{\partial_{v_1} f_1(v)}{v_1} dv\right| = \left|\int_{\mathbf{R}} \frac{\bar{f}_1'(v_1)}{v_1} dv_1\right| \leq C\,\|f_1\|_{H^{2,b}(\mathbf{R}^2)}.$$

We consider three cases.

Case 1: $\int_{\mathbf{R}^2} \frac{\partial_{v_1} f_1(v)}{v_1} dv < \left(\frac{2\pi}{T_1}\right)^2$. Let

$$F_1(v_1) = \exp\left(-\frac{(v_1 - v_0)^2}{2}\right) + \exp\left(-\frac{(v_1 + v_0)^2}{2}\right) = G_1\left(v_1^2\right),$$

and $F_2(v_2) = e^{-\frac{1}{2}v_2^2}$, where v_0 is a large positive constant such that

$$\int_{\mathbf{R}} \frac{F_1'(v_1)}{v_1} dv_1 > 0.$$

Let $\gamma, \delta > 0$ be two small parameters to be fixed, define

$$f_{\gamma,\delta}(v_1, v_2) = \frac{1}{1 + C_0\gamma^2}\left[f_1(v_1, v_2) + \frac{\gamma}{\delta} F_1\left(\frac{v_1}{\gamma\delta}\right) F_2(v_2)\right], \tag{11}$$

where $C_0 = \int F_1(v_1) F_2(v_2)\, dv > 0$. The rest of the proof is similar to the proof of Proposition 2.1 in [9]. We sketch it below. There exists $0 < \delta_1 < \delta_2$ such that for $\gamma_0 > 0$ small enough

$$0 < \int_{\mathbf{R}^2} \frac{\partial_{v_1} f_{\gamma,\delta_2}(v_1, v_2)}{v_1} dv < \left(\frac{2\pi}{T_1}\right)^2 < \int_{\mathbf{R}^2} \frac{\partial_{v_1} f_{\gamma,\delta_1}(v_1, v_2)}{v_1} dv, \text{ when } 0 < \gamma < \gamma_0. \tag{12}$$

Let $\beta(x_1)$ be a T_1 periodic function and denote $e = \frac{1}{2}v_1^2 - \beta(x_1)$. We look for 1D BGK wave solution

$$f^0 = f^{\beta}_{\gamma,\delta}(x_1, v), \quad \vec{E}^0 = E^0(x_1)\, \vec{e}_1$$

near $(f_{\gamma,\delta}, 0)$, where

$$f^{\beta}_{\gamma,\delta}(x_1, v) = \begin{cases} \frac{1}{1+C_0\gamma^2}\left[g_+(2e, v_2) + \frac{\gamma}{\delta} G_1\left(\frac{2e}{(\gamma\delta)^2}\right) F_2(v_2)\right] & \text{if } v_1 > 0 \\ \frac{1}{1+C_0\gamma^2}\left[g_-(2e, v_2) + \frac{\gamma}{\delta} G_1\left(\frac{2e}{(\gamma\delta)^2}\right) F_2(v_2)\right] & \text{if } v_1 \le 0 \end{cases} \tag{13}$$

and $E^0(x_1) = -\beta'(x_1)$. The steady Vlasov-Poisson equation is reduced to the ODE

$$\begin{aligned} \beta'' &= \int_{\mathbf{R}^2} f^{\beta}_{\gamma,\delta}(x, v)\ dv - 1 \\ &= \frac{1}{1+C_0\gamma^2}\{ \int_{v_1>0} g_+(2e, v_2)\, dv + \int_{v_1 \le 0} g_-(2e, v_2)\, dv \\ &\qquad + \int_{\mathbf{R}^2} \frac{\gamma}{\delta} G_1\left(\frac{2e}{(\gamma\delta)^2}\right) F_2(v_2)\, dv\} - 1 \\ &:= h_{\gamma,\delta}(\beta)\,. \end{aligned} \tag{14}$$

Since

$$h_{\gamma,\delta}(0) = \int_{\mathbf{R}^d} f_{\gamma,\delta}(v)\ dv - 1 = 0$$

and

$$\begin{aligned} & h'_{\gamma,\delta}(0) \\ &= \frac{-2}{1+C_0\gamma^2}\{ \int_{v_1>0} \partial_1 g_+\left(v_1^2, v_2\right) dv + \int_{v_1 \le 0} \partial_1 g_-\left(v_1^2, v_2\right) dv \\ &\qquad + \int_{\mathbf{R}^2} \frac{\gamma}{\delta} \frac{1}{(\gamma\delta)^2} G'\left(\frac{v_1^2}{(\gamma\delta)^2}\right) F_2(v_2)\, dv\ \} \\ &= -\int_{\mathbf{R}^2} \frac{\partial_{v_1} f_{\gamma,\delta}(v_1, v_2)}{v_1} dv < 0, \text{ when } 0 < \gamma < \gamma_0,\ \delta_1 < \delta < \delta_2, \end{aligned}$$

so $\beta = 0$ is a center for the ODE (14) and there exist bifurcation of periodic solutions. More precisely, for any fixed $\gamma \in (0, \gamma_0)$, there exists $r_0 > 0$ (independent of $\delta \in (\delta_1, \delta_2))$, such that for each $0 < r < r_0$, there exists a $T(\gamma, \delta; r)$ $-$periodic solution $\beta_{\gamma,\delta;r}$ to the ODE (14) with $\|\beta_{\gamma,\delta;r}\|_{H^2(0,T(\gamma,\delta;r))} = r$. Moreover,

$$\left(\frac{2\pi}{T(\gamma,\delta;r)}\right)^2 \to \int_{\mathbf{R}^2} \frac{\partial_{v_1} f_{\gamma,\delta}(v_1,v_2)}{v_1} dv, \text{ when } r \to 0.$$

To get a solution with the given period T_1, we adjust $\delta \in [\delta_1, \delta_2]$ by using the inequality (12) and the fact that $T(\gamma,\delta;r)$ is continuous to δ. So for each $\gamma, r > 0$ small enough, there exists $\delta_{T_1}(\gamma, r) \in (\delta_1, \delta_2)$, such that $T(\gamma, \delta_{T_1}; r) = T_1$. Define $f_{\gamma,r}(x_1, v) = f^{\beta}_{\gamma,\delta_{T1}}(x, v)$, $\beta_{\gamma,r}(x_1) = \beta_{\gamma,\delta_{T1};r}$ and let $\vec{E}_{\gamma,r}(x_1) = -\beta'_{\gamma,r}(x_1)\vec{e}_1$. Then $\left(f_{\gamma,r}(x_1,v), \vec{E}_{\gamma,r}(x_1)\right)$ is a nontrivial steady solution to (**??**) with x_1-period T_1. For any fixed $\gamma > 0$, let

$$\delta(\gamma) = \lim_{r\to 0} \delta_{T_1}(\gamma, r) \in [\delta_1, \delta_2].$$

By the dominant convergence theorem, it is easy to show that

$$\left\|f_{\gamma,r}(x_1,v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{L^1_{x_1,v}} + \int_0^{T_1}\int_{\mathbf{R}^2} |v|^2 \left|f_{\gamma,r}(x_1,v) - f_{\gamma,\delta(\gamma)}(v)\right| dx_1 dv$$
$$+ \left\|f_{\gamma,r}(x,v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{W^{2,p}_{x_1,v}} \to 0,$$

when $r = \|\beta_{\gamma,r}(x_1)\|_{H^2(0,T_1)} \to 0$. So for any $\gamma > 0$ and $\varepsilon > 0$, there exists $r = r(\gamma, \varepsilon) > 0$ such that

$$\left\|f_{\gamma,r}(x_1,v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{L^1_{x_1,v}} + \int_0^{T_1}\int_{\mathbf{R}^2} |v|^2 \left|f_{\gamma,r}(x_1,v) - f_{\gamma,\delta(\gamma)}(v)\right| dx_1 dv$$
$$+ \left\|f_{\gamma,r}(x,v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{W^{2,p}_{x_1,v}} < \frac{\varepsilon}{3}.$$

Since

$$f_1(v) - f_{\gamma,\delta(\gamma)}(v) = \frac{1}{1 + C_0\gamma^2}\left[-C_0\gamma^2 f_1(v) - \frac{\gamma}{\delta} F_1\left(\frac{v_1}{\gamma\delta}\right) F_2(v_2)\right].$$

and $\delta(\gamma) \in [\delta_1, \delta_2]$, by using Lemma 2.1, for $s < 1 + \frac{1}{p}$,

$$\left\|f_1(v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{W^{s,p}(\mathbf{R}^2)} \to 0, \quad \text{when } \gamma \to 0.$$

It is also easy to show that

$$\left\|f_1(v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{L^1} + \int_{\mathbf{R}^2} |v|^2 \left|f_1(v) - f_{\gamma,\delta(\gamma)}(v)\right| dv \to 0, \quad \text{when } \gamma \to 0.$$

Thus we can choose $\gamma > 0$ small enough such that

$$T_1 \left\|f_1(v) - f_{\gamma,\delta(\gamma)}(v)\right\|_{L^1} + T_1 \int_{\mathbf{R}^2} |v|^2 \left|f_1(v) - f_{\gamma,\delta(\gamma)}(v)\right| dv$$

$$+\left\|f_1(v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{W^{s,p}(\mathbf{R}^2)}<\frac{\varepsilon}{3}.$$

So the nontrivial steady solution $\left(f_{\gamma,r}(x_1,v), \vec{E}_{\gamma,r}(x_1)\right)$ is within ε distance of the homogeneous state $\left(f_0(v), \vec{0}\right)$ in the norm of (2).

Case 2: $\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_1(v)}{v_1}dv<\left(\frac{2\pi}{T_1}\right)^2$. Choose $F_1(v_1)=\exp\left(-\frac{v_1^2}{2}\right)$ and $F_2(v_2)$ is the same as before. Define $f_{\gamma,\delta}(v)$ as in Case 1 (see (11)). Then there exists $0<\delta_1<\delta_2$ such that

$$0<\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_{\gamma,\delta_1}(v_1,v_2)}{v_1}dv<\left(\frac{2\pi}{T_1}\right)^2<\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_{\gamma,\delta_2}(v_1,v_2)}{v_1}dv.$$

The rest of the proof is the same as in Case 1.

Case 3: $\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_1(v)}{v_1}dv=\left(\frac{2\pi}{T_1}\right)^2$. For $\delta>0$, define $f_\delta(v_1,v_2)=\frac{1}{\delta}f_1\left(\frac{v_1}{\delta},v_2\right)$. For any $\varepsilon>0$, there exist $0<\delta_1(\varepsilon)<1<\delta_2(\varepsilon)$ such that

$$0<\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_{\delta_2}(v)}{v_1}dv<\left(\frac{2\pi}{T_1}\right)^2<\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_{\delta_1}(v)}{v}dv,$$

and when $\delta\in(\delta_1(\varepsilon),\delta_2(\varepsilon))$,

$$T_1\left\|f_1(v)-f_\delta(v)\right\|_{L^1(\mathbf{R}^2)}+\ T_1\int_{\mathbf{R}^2}|v|^2\left|f_1(v)-f_\delta(v)\right|dv$$

$$+\left\|f_1(v)-f_\delta(v)\right\|_{W^{s,p}(\mathbf{R}^2)}<\frac{\varepsilon}{3}.$$

We construct steady BGK waves near $\left(f_\delta(v), \vec{0}\right)$, which are of the form

$$f_\delta^\beta(x_1,v)=\begin{cases}\frac{1}{\delta}g_+\left(\frac{2e}{\delta^2},v_2\right) & \text{if } v_1>0\\ \frac{1}{\delta}g_-\left(\frac{2e}{\delta^2},v_2\right) & \text{if } v_1\le 0\end{cases},\ e=\frac{1}{2}v_1^2-\beta(x_1), \tag{15}$$

and $\vec{E}^0=-\beta'(x_1)\vec{e}_1$. The existence of BGK waves is then reduced to solve the ODE

$$\beta''=\int_{\mathbf{R}^2}f_\delta^\beta(x_1,v)\ dv-1:=h_\delta(\beta). \tag{16}$$

As in Case 1, for any $\delta\in(\delta_1(\varepsilon),\delta_2(\varepsilon))$, $\exists\ r_0(\varepsilon)>0$ (independent of δ) such that for each $0<r<r_0$, there exists a $T(\delta;r)-$periodic solution $\beta_{\delta;r}$ to the ODE (16), satisfying $\|\beta_{\delta;r}\|_{H^2(0,T(\delta;r))}=r$ and

$$\left(\frac{2\pi}{T(\delta;r)}\right)^2\to\int_{\mathbf{R}^2}\frac{\partial_{v_1}f_\delta(v)}{v_1}dv,\text{ when } r\to 0.$$

For r small enough, again there exists $\delta_{T_1}(r,\varepsilon)\in(\delta_1(\varepsilon),\delta_2(\varepsilon))$ such that $T(\delta_{T_1};r)=T_1$. Define $f_{r,\varepsilon}(x_1,v)=f_{\delta_{T_1}}^\beta(x_1,v)$ and $\vec{E}_{r,\varepsilon}(x)=-\beta'_{\delta_{T1};r}(x_1)\vec{e}_1$.

Then $\left(f_{r,\varepsilon}(x_1,v), \vec{E}_{r,\varepsilon}(x)\right)$ is a nontrivial steady solution to (**??**) with x_1-period T_1. As in Cases 1 and 2, by choosing r small enough, $f_{r,\varepsilon}(x_1,v)$ is within ε distance of the homogeneous state $(f_0(v), 0)$ in the norm of (2). This finishes the proof of the Theorem 1.1. ■

Lemma 2.1 *(i) Given* $f \in W^{\frac{1}{p},p}(\mathbf{R}) \cap L^\infty(\mathbf{R})$, *and*

$$g \in W^{s,p}\left(\mathbf{R}^2\right) \left(p > 1, 0 \leq s < \frac{1}{p}\right).$$

Then for $\delta > 0$,

$$\left\| f\left(\frac{v_1}{\delta}\right) g(v_1,v_2) \right\|_{W^{s,p}(\mathbf{R}^2)} \to 0, \text{ when } \delta \to 0. \tag{17}$$

(ii) Given $f, g \in W^{s,p}(\mathbf{R})$ $\left(p > 1, 0 \leq s < \frac{1}{p}\right)$. *Then for* $\delta > 0$,

$$\left\| f\left(\frac{v_1}{\delta}\right) g(v_2) \right\|_{W^{s,p}(\mathbf{R}^2)} \to 0, \text{ when } \delta \to 0.$$

Proof. Proof of (i): First we consider $g \in C_0^\infty\left(\mathbf{R}^2\right)$. By Fubini Theorem for $W^{s,p}\left(\mathbf{R}^2\right)$ norm (see [15]), we have

$$\begin{aligned} &\left\| f\left(\frac{v_1}{\delta}\right) g(v_1,v_2) \right\|_{W^{s,p}(\mathbf{R}^2)} \\ &\leq C\left(\left\| \left\| f\left(\frac{v_1}{\delta}\right) g(v_1,v_2) \right\|_{W^{s,p}_{v_1}(\mathbf{R})} \right\|_{L^p_{v_2}} + \left\| f\left(\frac{v_1}{\delta}\right) \left\| g(v_1,v_2) \right\|_{W^{s,p}_{v_2}(\mathbf{R})} \right\|_{L^p_{v_1}} \right). \end{aligned}$$

By the estimates in the proof of Theorem 3.2 of [15], for any $p > 1,\ s < \frac{1}{p}$, when $h_1 \in W^{s,p}(\mathbf{R}),\ h_2 \in W^{\frac{1}{p},p}(\mathbf{R}) \cap L^\infty(\mathbf{R})$, we have

$$\|h_1 h_2\|_{W^{s,p}} \leq C \|h_1\|_{W^{s,p}} \left(\|h_2\|_{W^{\frac{1}{p},p}} + \|h_2\|_{L^\infty} \right).$$

So

$$\begin{aligned} &\left\| \left\| f\left(\frac{v_1}{\delta}\right) g(v_1,v_2) \right\|_{W^{s,p}_{v_1}(\mathbf{R})} \right\|_{L^p_{v_2}} \\ &\leq C \left\| f\left(\frac{v_1}{\delta}\right) \right\|_{W^{s,p}(\mathbf{R})} \left\| \|g\|_{W^{\frac{1}{p},p}_{v_1}(\mathbf{R})} + \|g\|_{L^\infty_{v_1}(\mathbf{R})} \right\|_{L^p_{v_2}} \\ &\leq C \left\| f\left(\frac{v_1}{\delta}\right) \right\|_{W^{s,p}(\mathbf{R})} \left\| \|g\|_{W^{1,p}_{v_1}(\mathbf{R})} \right\|_{L^p_{v_2}} \\ &\leq C \left\| f\left(\frac{v_1}{\delta}\right) \right\|_{W^{s,p}(\mathbf{R})} \|g\|_{W^{1,p}(\mathbf{R}^2)} \to 0, \end{aligned}$$

when $\delta \to 0$.Since $\left\| f\left(\frac{v_1}{\delta}\right)\right\|_{W^{s,p}(\mathbf{R})} \to 0$ under the assumption $s < \frac{1}{p}$ (see [9] for a proof). By the trace Theorem, we also have

$$\left\| f\left(\frac{v_1}{\delta}\right) \|g(v_1, v_2)\|_{W^{s,p}_{v_2}(\mathbf{R})} \right\|_{L^p_{v_1}} \le C \left\| f\left(\frac{v_1}{\delta}\right)\right\|_{L^p} \|g\|_{W^{2,p}(\mathbf{R}^2)} \to 0,$$

when $\delta \to 0$. This proves (17) for $g \in C_0^\infty\left(\mathbf{R}^2\right)$. When $g \in W^{s,p}\left(\mathbf{R}^2\right)$, (17) can be proved by using $C_0^\infty\left(\mathbf{R}^2\right)$ functions as approximations.

Proof of (ii): By Fubini Theorem for $W^{s,p}\left(\mathbf{R}^2\right)$ norm,

$$\begin{aligned}
&\left\| f\left(\frac{v_1}{\delta}\right) g(v_2)\right\|_{W^{s,p}(\mathbf{R}^2)} \\
&\le C\left(\left\| f\left(\frac{v_1}{\delta}\right)\right\|_{W^{s,p}_{v_1}(\mathbf{R})} \|g(v_2)\|_{L^p_{v_2}} + \|g(v_2)\|_{W^{s,p}_{v_2}(\mathbf{R})} \left\| f\left(\frac{v_1}{\delta}\right)\right\|_{L^p_{v_1}}\right) \\
&\to 0, \text{ when } \delta \to 0.
\end{aligned}$$

■

By the similar proof of Theorem 1.1, we can get the following.

Theorem 2.1 *Assume the homogeneous distribution function*

$$f_0(v) \in H^{s_v,b}\left(\mathbf{R}^d\right) \quad \left(d \ge 2,\ b > \frac{d-1}{4},\ s_v \in [0, \frac{3}{2})\right)$$

satisfies

$$f_0(v) \ge 0, \quad \int f_0(v)\, dv = 1, \quad \int v^2 f_0(v)\, dv < +\infty.$$

*Fix $T_1 > 0$ and $c \in \mathbf{R}$. Then for any $\varepsilon > 0$, $s_x \ge 0$, there exist travelling wave solutions of the form $f = f_\varepsilon(x_1 - ct, v)$, $\vec{E} = E_\varepsilon(x_1 - ct)\,\vec{e}_1$ to (**??**), such that $(f_\varepsilon(x_1, v), E_\varepsilon(x_1))$ has minimal period T_1 in x_1, $f_\varepsilon(x_1, v) \ge 0$, $E_\varepsilon(x_1)$ is not identically zero, and*

$$\|f_\varepsilon - f_0\|_{L^1_{x_1,v}} + \int_0^{T_1}\int_{\mathbf{R}^d} |v|^2 \left| f_\varepsilon(x_1, v) - f_0(v)\right| dx_1 dv + \|f_\varepsilon - f_0\|_{H^{s_x}_x H^{s_v,b}_v} < \varepsilon. \tag{18}$$

Proof. The construction of BGK waves follows the same line of the proof of Theorem 1.1. First, we modify $f_0(v)$ to a smooth profile $f_1(v)$. Then by adding proper perturbations in a scaling form to $f_1(v)$, we get the modified profile $f_{\gamma,\delta}(v)$. The BGK waves $(f_\varepsilon(x_1, v), E_\varepsilon(x_1))$ are obtained by bifurcation near $(f_{\gamma,\delta}(v), 0)$. To show the estimate (18), we need to control three deviations in the norm of (18): i) $f_\varepsilon(x_1, v) - f_{\gamma,\delta}(v)$; ii) $f_{\gamma,\delta}(v) - f_1(v)$; and iii) $f_1(v) - f_0(v)$. For the estimate of i), we choose integers $\bar{s}_x \ge s_x$, $\bar{s}_v \ge s_v$, and $\bar{b} \ge b$ and it is easy to show that

$$\|f_\varepsilon(x_1, v) - f_{\gamma,\delta}(v)\|_{H^{s_x}_x H^{s_v,b}_v} \le C\, \|f_\varepsilon(x_1, v) - f_{\gamma,\delta}(v)\|_{H^{\bar{s}_x}_x H^{\bar{s}_v,\bar{b}}_v}$$

and the right hand side can be made arbitrarily small by using the dominant convergence Theorem. For estimates of ii) and iii), we use the following analogue of Lemma 2.1. ■

Lemma 2.2 *(i) Given* $f(v_1) \in H^{\frac{1}{2}}(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$, $g(v_1, v_2) \in H^{s,b}(\mathbf{R}^2)$ $\left(0 \le s < \frac{1}{2}, b > \frac{1}{4}\right)$. *For* $\delta > 0$,

$$\left\| f\left(\frac{v_1}{\delta}\right) g(v_1, v_2) \right\|_{H^{s,b}(\mathbf{R}^2)} \to 0, \text{ when } \delta \to 0.$$

(ii) Given $f, g \in H^{s,b}(\mathbf{R})$ $\left(0 \le s < \frac{1}{2}, b > \frac{1}{4}\right)$. *For* $\delta > 0$,

$$\left\| f\left(\frac{v_1}{\delta}\right) g(v_2) \right\|_{H^{s,b}(\mathbf{R}^2)} \to 0, \text{ when } \delta \to 0.$$

Proof. First, we show that for any function $h \in H^{s,b}(\mathbf{R}^d)$ $\left(d \ge 1, 0 \le s \le 2, b > \frac{1}{4}\right)$, the norm $\|h\|_{H^{s,b}(\mathbf{R}^d)}$ defined by (3) is equivalent to both

$$\left\| \left(1 + |v|^2\right)^b f \right\|_{H^s(\mathbf{R}^d)} \tag{19}$$

and

$$\left\| \left(1 + |v_1|^{2b} + \cdots + |v_d|^{2b}\right) f \right\|_{H^s(\mathbf{R}^d)}. \tag{20}$$

We only need to prove the equivalence of the norms (3) and (19) for $s = 0$ and $s = 2$, since then for $0 < s < 2$ it follows from interpolation. For $s = 0$, it is trivial. For $s = 2$, by choosing $a > 0$ small enough, we have

$$\begin{aligned}
&\left\| \left(1 + a|v|^2\right)^b (1 - \Delta) f - (1 - \Delta)\left(1 + a|v|^2\right)^b f \right\|_{L^2(\mathbf{R}^d)} \\
&= \left\| f\ \Delta\left(\left(1 + a|v|^2\right)^b - 1\right) + 2\nabla f \cdot \nabla\left(\left(1 + a|v|^2\right)^b - 1\right) \right\|_{L^2} \\
&\le \frac{1}{2} \left\| \left(1 + a|v|^2\right)^b (1 - \Delta) f \right\|_{L^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2} \left\| \left(1 + a|v|^2\right)^b (1 - \Delta) f \right\|_{L^2} &\le \left\| (1 - \Delta)\left(1 + a|v|^2\right)^b f \right\|_{L^2} \\
&\le \frac{3}{2} \left\| \left(1 + a|v|^2\right)^b (1 - \Delta) f \right\|_{L^2}.
\end{aligned}$$

The equivalence of (3) and (20) can be proved in the same way.

Proof of (i): By Lemma 2.1 (i),

$$\left\| f\left(\frac{v_1}{\delta}\right) g(v_1, v_2) \right\|_{H^{s,b}(\mathbf{R}^2)} \le C \left\| \left(1 + |v|^2\right)^b f\left(\frac{v_1}{\delta}\right) g(v_1, v_2) \right\|_{H^s(\mathbf{R}^2)} \to 0,$$

when $\delta \to 0$. Since $f(v_1) \in H^{\frac{1}{2},}(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$ and

$$\left\| \left(1 + |v|^2\right)^b g(v_1, v_2) \right\|_{H^s} \le C \|g\|_{H^{s,b}(\mathbf{R}^2)} < \infty.$$

Proof of (ii): By using the equivalent norm (20) and Lemma 2.1 (ii),

$$\begin{aligned}
&\left\| f\left(\frac{v_1}{\delta}\right) g(v_2)\right\|_{H^{s,b}(\mathbf{R}^2)} \\
&\leq C\left\|\left(1+|v_1|^{2b}+|v_2|^{2b}\right) f\left(\frac{v_1}{\delta}\right) g(v_2)\right\|_{H^{s}(\mathbf{R}^2)} \\
&\leq C\{\left\| f\left(\frac{v_1}{\delta}\right) g(v_2)\right\|_{H^{s}(\mathbf{R}^2)} + \delta^{2b}\left\|\left|\frac{v_1}{\delta}\right|^{2b} f\left(\frac{v_1}{\delta}\right) g(v_2)\right\|_{H^{s}(\mathbf{R}^2)} \\
&\qquad + \left\| f\left(\frac{v_1}{\delta}\right) |v_2|^{2b} g(v_2)\right\|_{H^{s}(\mathbf{R}^2)}\} \\
&\rightarrow 0, \text{ when } \delta \rightarrow 0.
\end{aligned}$$

■

In the following, we show that there exist no truly 2D or 3D BGK solutions. Therefore, the $1D$ BGK form of solutions constructed in Theorem 1.1 is in some sense necessary.

Proposition 2.1 *(i) $(d=2)$ Assume $\mu \in C^1(\mathbf{R}) \cap L^1(\mathbf{R}^+)$, $\mu \geq 0$. If*

$$f_0(x,v) = \mu\left(\frac{1}{2}|v|^2 - \beta(x)\right), \qquad \vec{E}_0(x) = -\nabla\beta$$

is a solution of the Vlasov-Poisson system, then $\vec{E}_0 \equiv 0$.

(ii) $(d=3)$ Assume $\mu \in C^1(\mathbf{R}) \cap L^1(\mathbf{R}^+)$, $\mu \geq 0$. If

$$f_0(x_1,x_2,v_1,v_2,v_3) = \mu\left(\frac{1}{2}\left(v_1^2+v_2^2\right) - \beta(x_1,x_2), v_3\right),$$
$$\vec{E}_0(x_1,x_2) = \left(-\partial_{x_1}\beta, -\partial_{x_2}\beta, 0\right)$$

is a solution of the Vlasov-Poisson system, then $\vec{E}_0 \equiv 0$.

(iii) $(d=3)$ Assume $\mu \in C^1(\mathbf{R})$, $\mu \geq 0$, $\mu(r)\sqrt{r} \in L^1(\mathbf{R}^+)$. If

$$f_0(x,v) = \mu\left(\frac{1}{2}|v|^2 - \beta(x)\right), \qquad \vec{E}_0(x) = -\nabla\beta$$

is a solution of the Vlasov-Poisson system, then $\vec{E}_0 \equiv 0$.

Proof. We only prove (i) since the proof of (ii) and (iii) is similar. The electric potential β satisfies

$$-\Delta\beta = -\int_{\mathbf{R}^2} \mu\left(\frac{1}{2}|v|^2 - \beta\right) dv + 1 = g(\beta). \tag{21}$$

By the assumptions on μ, we have $g(\beta) \in C^1(\mathbf{R})$ and

$$\begin{aligned}
g'(\beta) &= \int_{\mathbf{R}^2} \mu'\left(\frac{1}{2}|v|^2 - \beta(x)\right) dv \\
&= 2\pi\int_0^\infty \mu'(s-\beta)\, ds = -2\pi\mu(-\beta) \leq 0,
\end{aligned}$$

Taking x_1 derivative of (21) and integrating with β_{x_1}, we have

$$\int_{T^2} \left|\nabla \beta_{x_1}\right|^2 \ dx = \int_{\mathbf{T}^2} g'(\beta) \left|\beta_{x_1}\right|^2 dx \leq 0.$$

So $\int_{T^2} \left|\nabla \beta_{x_1}\right|^2 \ dx = 0$ and β_{x_1} is a constant C. By the periodic assumption of β, $C = 0$ and thus $\beta_{x_1} \equiv 0$. Similarly, $\beta_{x_2} \equiv 0$. ∎

Remark 2.1 *In 2D and 3D, the function $g(\beta)$ defined in (21) always satisfies $g'(\beta) \leq 0$ and thus the elliptic problem (21) only has trivial solutions. For 1D, the function $g'(\beta)$ can change signs and thus the existence of 1D BGK waves is possible. We note that Proposition 2.1 does not exclude steady (travelling wave) solutions not of BGK types. It would be interesting to construct or exclude nontrivial steady solutions not of BGK types.*

3 Non-existence of Invariant structures

In this section, we prove that there exist no nontrivial invariant structures near stable equilibria in spaces $H_x^{s_x} H_v^{s_v,b}$ $\left(s_v > \frac{3}{2}\right)$. First, we derive the optimal linear decay estimate in a space-time norm under the Penrose stability condition (4). By assuming the uniform in time bound close to the stable homogeneous state above the regularity threshold $(s_v > \frac{3}{2})$, it is shown that the solutions of nonlinear equation is dominated by the linearized equation and a decay estimate in space-time is obtained for the nonlinear solution. By the time translation property of Vlasov-Poisson system, such a space-time decay implies that the invariant solution must be trivial, that is, homogeneous.

Lemma 3.1 *Given $f(v) \in H^{s,b}\left(\mathbf{R}^d\right)$ $\left(d \geq 2,\ s \geq 0,\ b > \frac{d-1}{4}\right)$. For any unit vector $\vec{e} \in \mathbf{R}^d$, let $v = \alpha\vec{e} + w$ where $v \in \mathbf{R}^d$ and $w \perp \vec{e}$. Define*

$$f_{\vec{e}}(\alpha) = \int_{\mathbf{R}^{d-1}} f(\alpha\vec{e} + w) \ dw. \tag{22}$$

Then

$$\left\|f_{\vec{e}}(\alpha)\right\|_{H^s(\mathbf{R})} \leq C \left\|f\right\|_{H^{s,b}(\mathbf{R}^d)}$$

for some constant C independent of $\vec{e}$.

Proof. To simplify notations, we only consider $\vec{e} = (1, 0, \cdots, 0)$. Then $\alpha = v_1$ and

$$f_{\vec{e}}(v_1) = \int_{\mathbf{R}^{d-1}} f(v) \ dv_2 \cdots dv_d.$$

Let $\xi = (\xi_1, \cdots \xi_d)$ be the Fourier variable. Then

$$\begin{aligned}
\|f_{\vec{e}}(v_1)\|_{H^s(\mathbf{R})} &= \left\| \left(1+|\xi_1|^2\right)^{\frac{s}{2}} \hat{f}_{\vec{e}}(\xi_1) \right\|_{L^2(\mathbf{R})} \\
&= \left\| \left(1+|\xi_1|^2\right)^{\frac{s}{2}} \hat{f}(\xi_1, 0, \cdots, 0) \right\|_{L^2(\mathbf{R})} \\
&\leq C \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{f}(\xi) \right\|_{H^{2b}(\mathbf{R}^d)} \\
&= C \left\| \left(1+|v|^2\right)^{b} (1-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbf{R}^d)} = C\|f\|_{H^{s,b}(\mathbf{R}^d)}.
\end{aligned}$$

Here, the first inequality above is due to the trace theorem and that $2b > \frac{d-1}{2}$.
■

Corollary 3.1 *Given $f(v) \in H^{s,b}\left(\mathbf{R}^d\right)$ $\left(d \geq 2,\ s > \frac{3}{2},\ b > \frac{d-1}{4}\right)$. For any unit vector $\vec{e} \in \mathbf{R}^d$, we have*

(i) If α_0 is a critical point of $f_{\vec{e}}(\alpha)$, then

$$\int_{\mathbf{R}} \left| \frac{f'_{\vec{e}}(\alpha)}{\alpha - \alpha_0} \right| d\alpha \leq C(d,s,b)\|f\|_{H^{s,b}(\mathbf{R}^d)}.$$

(ii) For any $\alpha' \in \mathbf{R}$,

$$\left| P\int_{\mathbf{R}} \frac{f'_{\vec{e}}(\alpha)}{\alpha - \alpha'} d\alpha \right| \leq C(d,s,b)\|f\|_{H^{s,b}(\mathbf{R}^d)},$$

where $P\int_{\mathbf{R}}$ is the principal value integral.

Proof. (i) follows from Lemma 3.1 and the following Hardy inequality (see Lemma 3.1 of [9]): If $u(v) \in W^{s,p}(\mathbf{R})$ $\left(p > 1, s > \frac{1}{p}\right)$, and $u(0) = 0$, then

$$\int_{\mathbf{R}} \left| \frac{u(v)}{v} \right| dv \leq C\|u\|_{W^{s,p}(\mathbf{R})}, \tag{23}$$

for some constant C.

Proof of (ii): Since

$$P\int_{\mathbf{R}} \frac{f'_{\vec{e}}(\alpha)}{\alpha - \alpha'} d\alpha = \frac{1}{2}\int_{\mathbf{R}} \frac{\frac{d}{d\alpha}\left(f_{\vec{e}}(\alpha+\alpha') + f_{\vec{e}}(-\alpha+\alpha')\right)}{\alpha} d\alpha,$$

by Hardy inequality (23)

$$\begin{aligned}
\left| P\int_{\mathbf{R}} \frac{f'_{\vec{e}}(\alpha)}{\alpha - \alpha'} d\alpha \right| &\leq C\left(\|f_{\vec{e}}(\alpha+\alpha')\|_{H^s(\mathbf{R})} + \|f_{\vec{e}}(-\alpha+\alpha')\|_{H^s(\mathbf{R})}\right) \\
&\leq C\|f_{\vec{e}}(\alpha)\|_{H^s(\mathbf{R})} \leq C\|f\|_{H^{s,b}(\mathbf{R}^d)} \text{ (by Lemma 3.1).}
\end{aligned}$$

■

The next lemma generalizes the one dimensional result in [9] about linear decay estimates in a space-time norm. The linearized Vlasov-Poisson system at an homogeneous state $\left(f,\vec{E}\right)=\left(f_0\left(v\right),\vec{0}\right)$ is

$$\partial_t f+v\cdot\nabla_x f-\vec{E}\cdot\nabla_v f_0=0, \tag{24a}$$

$$\vec{E}=-\nabla_x\phi,\quad -\Delta\phi=-\int_{\mathbf{R}^d} f\ dv, \tag{24b}$$

Lemma 3.2 *Assume* $f_0\left(v\right)\in H^{s_0,b}\left(\mathbf{R}^d\right)$ $\left(d\geq 2,\ s_0>\frac{3}{2},\ b>\frac{d-1}{4}\right)$ *satisfies the Penrose stability condition (4) for* $x-$*period tuple* $\left(T_1,\cdots,T_d\right)$*. Let* $\left(f\left(x,v,t\right),\vec{E}\left(x,t\right)\right)$ *be a solution of the linearized Vlasov-Poisson system (24a)-(24b) with* $x-$*period tuple* $\left(T_1,\cdots,T_d\right)$*. If* $f\left(x,v,0\right)\in H_x^{s_x}H_v^{s_v,b}$ *with* $\left|s_v\right|\leq s_0-1$*, then*

$$\left\|t^{s_v}\vec{E}\left(x,t\right)\right\|_{L_t^2H_x^{\frac{3}{2}+s_x+s_v}}\leq C_0\left\|f\left(x,v,0\right)\right\|_{H_x^{s_x}H_v^{s_v,b}}, \tag{25}$$

Proof. First, we reduce the linearized problem to the one dimensional case. Since the homogeneous component of $f\left(x,v,t\right)$ remains steady for the linearized equation and therefore has no effect on $\vec{E}\left(x,t\right)$, we assume that f has no homogeneous component. Let

$$f\left(x,v,t\right)=\sum_{\vec{0}\neq\vec{k}\in\mathbf{Z}^d}e^{i\vec{k}\cdot x}f_{\vec{k}}\left(v,t\right)$$

and the electric potential

$$\phi\left(x,t\right)=\sum_{\vec{0}\neq\vec{k}\in\mathbf{Z}^d}e^{i\vec{k}\cdot x}\phi_{\vec{k}}\left(t\right).$$

Then

$$\vec{E}\left(x,t\right)=-\nabla_x\phi=-\sum_{\vec{0}\neq\vec{k}\in\mathbf{Z}^d}i\vec{k}\phi_{\vec{k}}\left(t\right)e^{i\vec{k}\cdot x}=\sum_{\vec{0}\neq\vec{k}\in\mathbf{Z}^d}\vec{E}_{\vec{k}}\left(t\right)e^{i\vec{k}\cdot x},$$

where $\vec{E}_{\vec{k}}\left(t\right)=-i\vec{k}\phi_{\vec{k}}\left(t\right)$. Denote $\vec{e}=\vec{k}/\left|\vec{k}\right|$, then

$$\vec{E}_{\vec{k}}\left(t\right)=-i\vec{k}\phi_{\vec{k}}\left(t\right)=\tilde{E}_{\vec{k}}\left(t\right)\vec{e}$$

where $\tilde{E}_{\vec{k}}\left(t\right)=-i\left|\vec{k}\right|\phi_{\vec{k}}\left(t\right)$. Let $v=\alpha\vec{e}+w$ where $\alpha\in\mathbf{R}$, $w\perp\vec{e}$, and

$$\tilde{f}_{\vec{k}}\left(\alpha,t\right)=f_{\vec{k},\vec{e}}\left(\alpha,t\right)=\int_{\mathbf{R}^{d-1}}f_{\vec{k}}\left(\alpha\vec{e}+w,t\right)\ dw$$

The linearized Vlasov equation implies that

$$\begin{aligned} 0 &= \partial_t f_{\vec{k}} + v \cdot i\vec{k}\ f_{\vec{k}} - \vec{E}_{\vec{k}} \cdot \nabla_v f_0 \\ &= \partial_t f_{\vec{k}} + i\alpha \left|\vec{k}\right| f_{\vec{k}} - \tilde{E}_k \partial_\alpha f_0 \end{aligned}$$

An integration of the w variable on above equation yields

$$\partial_t \tilde{f}_{\vec{k}}(\alpha, t) + i\alpha \left|\vec{k}\right| \tilde{f}_{\vec{k}}(\alpha, t) - \tilde{E}_k f'_{0,\vec{e}}(\alpha) = 0. \tag{26}$$

The Poisson equation implies

$$\left|\vec{k}\right|^2 \phi_{\vec{k}}(t) = -\int_{\mathbf{R}^d} f_{\vec{k}}(v, t)\, dv,$$

and thus

$$i\left|\vec{k}\right| \tilde{E}_{\vec{k}}(t) = -\int_{\mathbf{R}} \tilde{f}_{\vec{k}}(\alpha, t)\, d\alpha. \tag{27}$$

Equations (26) and (27) imply that $\left(\tilde{f}_{\vec{k}}(\alpha, t), \tilde{E}_k(t)\right) e^{i|\vec{k}|x}$ solves the linearized 1D Vlasov-Poisson equations at the homogeneous profile $f_{0,\vec{e}}(\alpha)$. Thus by the 1D representation formula in [9] and the Penrose stability condition (4), we have

$$\tilde{E}_{\vec{k}}(t) = \frac{\left|\vec{k}\right|}{2\pi} \int_{\mathbf{R}} \frac{G_{\vec{k}}(y + i0)}{\left|\vec{k}\right|^2 - F_{\vec{e}}(y + i0)} e^{-i|\vec{k}|yt} dy.$$

Here,

$$G_{\vec{k}}(y + i0) = P\int_{\mathbf{R}} \frac{\tilde{f}_{\vec{k}}(\alpha, 0)}{\alpha - y} d\alpha + i\pi \tilde{f}_{\vec{k}}(y, 0)$$

and

$$F_{\vec{e}}(y + i0) = P\int_{\mathbf{R}} \frac{f'_{0,\vec{e}}(\alpha)}{\alpha - y} d\alpha + i\pi f'_{0,\vec{e}}(y).$$

By the Penrose stability condition (4) and

$$\left|P\int_{\mathbf{R}} \frac{f'_{0,\vec{e}}(\alpha)}{\alpha - y} d\alpha\right| \leq C(d, s, b) \|f_0\|_{H^{s,b}(\mathbf{R}^d)} \ \text{(Corollary 3.1)},$$

there exists $c_0 > 0$ (independent of $\vec{k}$), such that

$$\left|\left|\vec{k}\right|^2 - F_{\vec{e}}(y + i0)\right|^2 \geq c_0 \left|\vec{k}\right|^2.$$

Then by the same proof of Proposition 4.1 in [9],

$$\begin{aligned} \left\|t^{s_v} \tilde{E}_{\vec{k}}(t)\right\|_{L^2}^2 &\leq C\left|\vec{k}\right|^{-3-2s_v} \left\|\tilde{f}_{\vec{k}}(\alpha, 0)\right\|_{H_v^{s_v}}^2 \\ &\leq C\left|\vec{k}\right|^{-3-2s_v} \left\|f_{\vec{k}}(v, 0)\right\|_{H^{s,b}(\mathbf{R}^d)}^2. \end{aligned}$$

So

$$\begin{aligned}
&\left\| t^{s_v} \vec{E}(x,t) \right\|^2_{L^2_t H_x^{\frac{3}{2}+s_x+s_v}} \\
&= \sum_{0 \neq \vec{k} \in \mathbf{Z}^d} \left| \vec{k} \right|^{3+2s_v+2s_x} \left\| t^{s_v} \tilde{E}_{\vec{k}}(t) \right\|^2_{L^2} \\
&\leq C \sum_{0 \neq \vec{k} \in \mathbf{Z}^d} \left| \vec{k} \right|^{2s_x} \left\| f_{\vec{k}}(v,0) \right\|^2_{H^{s,b}(\mathbf{R}^d)} \\
&\leq C \left\| f(x,v,0) \right\|_{H_x^{s_x} H_v^{s_v,b}}.
\end{aligned}$$

■

Lemma 3.3 *Assume* $f_0(v) \in H^{s_0,b}(\mathbf{R}^d)$ $\left(d \geq 2,\ s_0 > \frac{3}{2},\ b > \frac{d-1}{4}\right)$ *and the Penrose stability condition (4) is satisfied for* $x-$*period tuple* $(T_1, \cdots, T_d)$*. Let* $\left(f(x,v,t), \vec{E}(x,t)\right)$ *be a solution of the Vlasov-Poisson system (1a)-(1b) with* $x-$*period tuple* $(T_1, \cdots, T_d)$*.*

For any (s_x, s_v) *satisfying (6), there exists* $\varepsilon_0 > 0$*, such that if*

$$\left\| f(t) - f_0 \right\|_{H_x^{s_x} H_v^{s_v,b}} < \varepsilon_0, \text{ for all } t \geq 0,$$

then

$$\left\| (1+t)^{s_v-1} \vec{E}(x,t) \right\|_{L^2_{\{t \geq 0\}} H_x^{\frac{3}{2}+s_x}} \leq C\varepsilon_0. \tag{28}$$

Proof. Denote L_0 to be the linearized operator corresponding to the linearized Vlasov-Poisson equation at $(f_0(v), 0)$, and $\mathcal{E}$ is the mapping from $f(x,v)$ to $\vec{E}(x)$ by the Poisson equation (24b).It follows from Lemma 3.2 that: For any $0 \leq s_v \leq s_0 - 1$, if $h(x,v) \in H_x^{s_x} H_v^{s_v,b}$,then

$$\left\| (1+t)^{s_v} \mathcal{E}\left(e^{tL_0} h\right) \right\|_{L^2_t H_x^{\frac{3}{2}+s_x}} \leq C \left\| h(x,v) \right\|_{H_x^{s_x} H_v^{s_v,b}}. \tag{29}$$

Denote $f_1(t) = f(t) - f_0$, then

$$\partial_t f_1 = L_0 f_1 + \vec{E} \cdot \partial_v f_1.$$

Thus

$$f_1(t) = e^{tL_0} f_1(0) + \int_0^t e^{(t-u)L_0} \left(\vec{E} \cdot \partial_v f_1\right)(u)\, du = f_{\text{lin}}(t) + f_{\text{non}}(t),$$

and correspondingly

$$\vec{E}(t) = \mathcal{E}(f_{\text{lin}}(t)) + \mathcal{E}(f_{\text{non}}(t)) = \vec{E}_{\text{lin}}(t) + \vec{E}_{\text{non}}(t).$$

By the linear estimate (29),

$$\begin{aligned}
\left\| (1+t)^{s_v-1} \vec{E}_{\text{lin}}(x,t) \right\|_{L^2_{\{t \geq 0\}} H_x^{\frac{3}{2}+s_x}} &= \left\| (1+t)^{s_v-1} \mathcal{E}\left(e^{tL_0} f_1(0)\right) \right\|_{L^2_t H_x^{\frac{3}{2}+s_x}} \\
&\leq C \left\| f_1(0) \right\|_{H_x^{s_x} H_v^{s_v,b}},
\end{aligned}$$

and

$$
\begin{aligned}
&\left\|(1+t)^{s_v-1}\,\vec{E}_{\mathrm{non}}\,(x,t)\right\|^2_{L^2_{\{t\geq 0\}}H_x^{\frac{3}{2}+s_x}}\\
&=\int_0^\infty (1+t)^{2(s_v-1)}\left\|\vec{E}_{\mathrm{non}}\,(x,t)\right\|^2_{H_x^{\frac{3}{2}+s_x}}dt\\
&\leq \int_0^\infty (1+t)^{2(s_v-1)}\left(\int_0^t \left\|\mathcal{E}\left[e^{(t-u)L_0}\left(\vec{E}\partial_v f_1\right)(u)\right]\right\|_{H_x^{\frac{3}{2}+s_x}}du\right)^2 dt\\
&\leq \int_0^\infty (1+t)^{2(s_v-1)}\int_0^t (1+(t-u))^{-2(s_v-1)}\,(1+u)^{-2(s_v-1)}\,du\\
&\qquad\cdot\int_0^t (1+u)^{2(s_v-1)}\,(1+(t-u))^{2(s_v-1)}\left\|\mathcal{E}\left[e^{(t-u)L_0}\left(\vec{E}\cdot\partial_v f_1\right)(u)\right]\right\|^2_{H_x^{\frac{3}{2}+s_x}}\;dudt\\
&\leq C\int_0^\infty\int_0^t (1+u)^{2(s_v-1)}\,(1+(t-u))^{2(s_v-1)}\left\|\mathcal{E}\left[e^{(t-u)L_0}\left(\vec{E}\cdot\partial_v f_1\right)(u)\right]\right\|^2_{H_x^{\frac{3}{2}+s_x}}\;dudt\\
&= C\int_0^\infty (1+u)^{2(s_v-1)}\int_u^\infty (1+(t-u))^{2(s_v-1)}\left\|\mathcal{E}\left[e^{(t-u)L_0}\left(\vec{E}\cdot\partial_v f_1\right)(u)\right]\right\|^2_{H_x^{\frac{3}{2}+s_x}}\;dtdu\\
&\leq C\int_0^\infty (1+u)^{2(s_v-1)}\left\|\left(\vec{E}\cdot\partial_v f_1\right)(u)\right\|^2_{H_x^{s_x}H_v^{s_v-1,b}}du\\
&\leq C\int_0^\infty (1+u)^{2(s_v-1)}\left\|\vec{E}\,(u)\right\|^2_{H_x^{\frac{3}{2}+s_x}}\|f_1\,(u)\|^2_{H_x^{s_x}H_v^{s_v,b}}du\\
&\leq C\varepsilon_0^2\left\|\left(1+t^{s_v-1}\right)\vec{E}\,(x,t)\right\|^2_{L^2_{\{t\geq 0\}}H_x^{\frac{3}{2}+s_x}}.
\end{aligned}
$$

In the above estimate, we use the fact that

$$
\int_0^t (1+(t-u))^{-2(s_v-1)}\,(1+u)^{-2(s_v-1)}\,du\leq C\,(1+t)^{-2(s_v-1)}
$$

because of the assumption that $s_v-1>\frac{1}{2}$. The assumption (6) ensures that the following inequality is true

$$
\left\|\left(\vec{E}\cdot\partial_v f_1\right)(u)\right\|_{H_x^{s_x}H_v^{s_v-1,b}}\leq C\left\|\vec{E}\,(u)\right\|_{H_x^{\frac{3}{2}+s_x}}\|f_1\,(u)\|_{H_x^{s_x}H_v^{s_v,b}}.
$$

Thus

$$
\begin{aligned}
&\left\|\left(1+t^{s_v-1}\right)\vec{E}\,(x,t)\right\|_{L^2_{\{t\geq 0\}}H_x^{\frac{3}{2}+s_x}}\\
&\leq\left\|(1+t)^{s_v-1}\,\vec{E}_{\mathrm{lin}}\,(x,t)\right\|_{L^2_{\{t\geq 1\}}H_x^{\frac{3}{2}+s_x}}+\left\|(1+t)^{s_v-1}\,\vec{E}_{\mathrm{non}}\,(x,t)\right\|_{L^2_{\{t\geq 0\}}H_x^{\frac{3}{2}+s_x}}\\
&\leq C\,\|f_1\,(0)\|_{H_x^{s_x}H_v^{s_v,b}}+C\varepsilon_0\left\|\left(1+t^{s_v-1}\right)\vec{E}\,(x,t)\right\|_{L^2_{\{t\geq 0\}}H_x^{\frac{3}{2}+s_x}}.
\end{aligned}
$$

By taking $\varepsilon_0=\frac{1}{2C}$, we get the estimate (28). ■

Theorem 1.2 follows from Lemma 3.3 and the time translation symmetry of the Vlasov-Poisson equation. Since the arguments are exactly the same as in the 1D case ([9]), we skip the details.

As a corollary of Theorem 1.2, we get the following nonlinear instability result.

Corollary 3.2 *Assume $f_0(v) \in H^{s_0,b}(\mathbf{R}^d)$ $\left(d \geq 2,\ s_0 > \frac{3}{2},\ b > \frac{d-1}{4}\right)$ and the Penrose stability condition (4) is satisfied for the $x-$period tuple $(T_1, \cdots, T_d)$. For any (s_x, s_v) satisfying (6), there exists $\varepsilon_0 > 0$ such that for any solution $\left(f(x,v,t), \vec{E}(x,t)\right)$ of the Vlasov-Poisson system (**??**) with $x-$period tuple $(T_1, \cdots, T_d)$ and $\vec{E}(x,0)$ not identically zero, the following is true:*

$$\|f(T^*) - f_0\|_{H_x^{s_x} H_v^{s_v,b}} \geq \varepsilon_0, \text{ for some } T^* \in \mathbf{R}.$$

We can also study the positive (negative) invariant structures near $(f_0(v), 0)$, which are solutions $\left(f(t), \vec{E}(t)\right)$ of nonlinear Vlasov-Poisson equation satisfying the conditions (7) for all $t \geq 0$ $(t \leq 0)$. The next theorem shows that the electric field of these semi-invariant structures must decay when $t \to +\infty$ $(t \to -\infty)$.

Theorem 3.1 *Given a homogeneous profile*

$$f_0(v) \in H^{s,b}(\mathbf{R}^d) \qquad \left(d \geq 2,\ s_0 > \frac{3}{2},\ b > \frac{d-1}{4}\right).$$

*Assume that $f_0(v)$ satisfies the Penrose stability condition (4) for $(T_1, \cdots, T_d)$. Let $\left(f(x,v,t), \vec{E}(x,v,t)\right)$ be a solution of (**??**) in T^d. For any (s_x, s_v) satisfying (6), there exists $\varepsilon_0 > 0$, such that if*

$$\|f(t) - f_0\|_{H_x^{s_x} H_v^{s_v,b}} < \varepsilon_0, \text{ for all } t \geq 0 \ \ (\text{or } t \leq 0),$$

with

$$\|f(0)\|_{L^\infty_{x,v}} < \infty, \ \int_{T^d}\int_{\mathbf{R}^d} |v|^2 f(0,x,v)\, dvdx < \infty,$$

then $\left\|\vec{E}(t,x)\right\|_{L_x^2} \to 0$ when $t \to +\infty$ $(\text{or } t \to -\infty)$.

Proof. By energy conservation,

$$\begin{aligned}
&\int_{T^d}\int_{\mathbf{R}^d} |v|^2 f(x,v,t)\, dvdx + \left\|\vec{E}(x,t)\right\|_{L_x^2}^2 \\
&= \int_{T^d}\int_{\mathbf{R}^d} |v|^2 f(x,v,0)\, dvdx + \|E(x,0)\|_{L^2}^2 < C.
\end{aligned}$$

Let $j = \int vf\ dv$. When $d = 2$,we have

$$|j(t)| = \left|\int vf\ \ (t)\,dv\right| \le \int_{|v|\le A} |v|\,dv\,\|f(t)\|_{L^\infty_{x,v}} + \frac{1}{A}\int_{|v|\ge A} |v|^2 f dv$$
$$\le C\left(\|f(0)\|_{L^\infty_{x,v}} A^3 + \frac{1}{A}\int |v|^2 f dv\right) \le C\,\|f(0)\|_{L^\infty_{x,v}}^{\frac{1}{4}} \left(\int |v|^2 f dv\right)^{\frac{3}{4}},$$

by choosing

$$A = \left(\int |v|^2 f dv / \|f(0)\|_{L^\infty_{x,v}}\right)^{\frac{1}{4}}.$$

Thus

$$\|j(x,t)\|_{L_x^{\frac{4}{3}}} \le C \int\int |v|^2 f\ dvdx \le C.$$

Since

$$\frac{d}{dt}\left\|\vec{E}(x,t)\right\|^2_{L^2_x} = \int_{T^d} j(x,t)\cdot\vec{E}(x,t)\,dx$$
$$\le \|j(x,t)\|_{L_x^{\frac{4}{3}}} \|E(x,t)\|_{L^4_x} \le C\,\|E(x,t)\|_{H_x^{\frac{3}{2}}},$$

and by Lemma 3.3

$$\int_0^\infty \|E(x,t)\|_{H_x^{\frac{3}{2}}} dt \le \left(\int_0^\infty (1+t)^{-2(s-1)}\,dt\right)^{\frac{1}{2}} \left(\int_0^\infty (1+t)^{2(s-1)} \|E(x,t)\|^2_{H_x^{\frac{3}{2}}}\,dt\right)^{\frac{3}{2}}$$
$$\le C\varepsilon_0,$$

thus $\lim_{t\to\infty}\left\|\vec{E}(x,t)\right\|_{L^2_x}$ exists and equals zero. When $d = 3$, the proof is very similar. The estimates become

$$\|j(x,t)\|_{L_x^{\frac{5}{4}}} \le C,$$

and

$$\frac{d}{dt}\left\|\vec{E}(x,t)\right\|^2_{L^2_x} \le \|j(x,t)\|_{L_x^{\frac{5}{4}}} \|E(x,t)\|_{L^5_x} \le C\,\|E(x,t)\|_{H_x^{\frac{3}{2}+s_x}}.$$

The rest is the same. ■

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