STABILITY AND INSTABILITY OF TRAVELLING SOLITONIC BUBBLES

ZHIWU LIN
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY

ABSTRACT. We study the nonlinear Schrödinger equation with general nonlinearity of competing type. This equation have travelling waves with nonvanishing condition at infinity in one dimension. We give a sharp condition for the stability and instability of these solutions. This justifies the previous prediction posed in physical literature.

1. Introduction

We consider the so-called cubic-quintic(or ψ^3-ψ^5) nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi - \alpha_1\psi + \alpha_3\psi|\psi|^2 - \alpha_5\psi|\psi|^4 = 0, x \in \mathbf{R}^D \tag{1.1}$$

Here $\Delta \equiv \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_D^2$, $\alpha_1$is a real constant, α_3, α_5 are positive constants and D denotes the spatial dimension.

Equation(1.1) describes the quasiclassical model of a gas of bosons interacting via the two-body attractive and three-body repulsive δ-function potential. It also arises in various fields of physics(see [1] [2] [3] [4]), according to which the corresponding space dimension D varies from 1 to 3.

As in [2], we assume that the range of parameters is such that

$$\frac{3}{16} < \alpha_1\alpha_5/\alpha_3^2 < \frac{1}{4} \ .$$

In this case, using some scale transformations(see [2]), (1.1) can be rewritten as

$$i\phi_t + \Delta\phi + (|\phi|^2 - \rho_0)(2A + \rho_0 - 3|\phi|^2)\phi = 0, \tag{1.2}$$

where $0 < A < \rho_0$. The potential

$$V(|\phi|^2) = (|\phi|^2 - \rho_0)^2(|\phi|^2 - A)$$

corresponding to the nonlinear interaction in (1.2), has two minima describing different phases. One is $\phi \equiv 0$ which is stable, and another is $\phi \equiv \sqrt{\rho_0}$ which is metastable.

In [2], [4] and [11], Barashenkov et al. considered the stationary solution to (1.2) with nonvanishing condition at infinity. The solution has the form of stationary rarefaction bubble, which is physically interpreted as a nucleus of the stable phase

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in the metastable one. It was shown that these "bubbles" exist for any D, and they are always unstable.

In space dimension one, the solution to (1.2) of the form

$$\phi(x,t) = \phi_v(x - vt)$$

corresponding to "bubble" travelling with speed v have also been found(see [3]). The "boundary condition" is then

$$\lim_{x\to\pm\infty} \phi(x,t) = \sqrt{\rho_0} e^{\mp i\mu} \ ,$$

where μ is real number depending on the speed v, with $\mu = 0$ when $v = 0$. An interesting problem is concerned with the stability of these moving "bubbles". The heuristic argument and the numerical experiment suggest that there exists a certain critical speed v_{cr} such that the "bubbles" are stable for $v > v_{cr}$ and unstable for $v < v_{cr}$(see [1], [3] and [5]). More precisely, it is conjectured that: if

$$\frac{dP_v}{dv} < 0$$

then $\phi_v(x - vt)$ is stable , otherwise it is unstable , where

$$P_v = Im \int_{-\infty}^{+\infty} \phi_{v_x}^* \phi_v \left(1 - \frac{\rho_0}{|\phi_v|^2}\right) dx \ .$$

This conjecture was presented from physical observation of energy minimizing , and it was also supported by numerical experiments (see [1], [3] and [5]).

The aim of this paper is to give a rigorous proof of this conjecture.In fact, we consider

$$i\phi_t + \Delta\phi + F(|\phi|^2)\phi = 0 \qquad (1.3)$$

under the following general assumptions on F:

$F(r) \in C^2_{loc}(\mathbf{R}^+), \quad U(r) = -\int_{\rho_0}^r F(s)ds, \quad \rho_0 > 0$

and

(F.1) $F(\rho_0) = 0, \eta_0 \equiv \sup\{\eta \mid 0 < \eta < \rho_0, U(\eta) = 0\}$ exists, $0 < \eta_0 < \rho_0$ and $F(\eta_0) < 0$,

(F.2) $F'(\rho_0) < 0$.

We assume that F satisfies (F.1) and (F.2) throughout this paper. Now w e state the main theorem of this paper.

Theorem 1.1 (Main theorem)**.** *Under the above assumptions (F.1) and (F.2) on F, for any*

$$v \in [0, v_c), v_c = \sqrt{-2\rho_0 F'(\rho_0)} \ ,$$

there exists travelling bubble

$$\phi_v(x - vt) = a_v e^{i\theta_v}(x - vt)$$

to (1.3)*. The bubble is stable when*

$$\frac{dP_v}{dv} < 0 \ ,$$

and unstable when

$$\frac{dP_v}{dv} > 0 \ .$$

Here the stability means that: foe all $\varepsilon > 0$,there exists $\delta > 0$ such that if initial data

$$\phi_0 = a_0 e^{i\theta_0}$$

satisfies

$$\inf_{s\in\mathbf{R}} (||\tau_s a_0^2 - a_v^2||_{1,2} + ||\tau_s \theta_{0_x} - \theta_{v_x}||_2) < \delta \ ,$$

then

$$\inf_{s\in\mathbf{R}} (||\tau_s a(t)^2 - a_v^2||_{1,2} + ||\tau_s \theta_x(t) - \theta_{v_x}||_2) < \varepsilon \ ,$$

for $0 < t < \infty$.
Here

$$\phi(t) = a(t)e^{i\theta(t)}$$

is the solution to (1.3) with $\phi(0) = \phi_0$ and τ_s denotes the translation by s, i.e.,

$$\tau_s f(x) = f(x+s) \ .$$

The instability means that $\phi_v(x - vt)$ is not stable.

Remark 1.1. The more precise definition of stability and instability will be given in definition 4.1 of section 4.

The solitary wave of (1.3) has been extensively studied with vanishing condition at infinity. And an abstract theory concerning stability of solitary wave in an abstract Hamiltonian system was established by Grillakis, Shatah and Strauss [13]. But this theory does not seem applicable to our case directly because of the non-vanishing condition at infinity.

Instead of treating (1.3) itself, we study its hydrodynamic form, that is, if

$$\phi = (\rho_0 - r)^{\frac{1}{2}} e^{i\theta}$$

is solution of (1.3), then r and $u \equiv \theta_x$ will formally satisfy

$$\begin{cases} r_t &= \dfrac{\partial}{\partial x}\left(2(\rho_0 - r)u\right), \\ u_t &= -\dfrac{\partial}{\partial x}\left(u^2 + \dfrac{r_x^2}{4(\rho_0 - r)^2} - \dfrac{r_{xx}}{2(\rho_0 - r)} - F(\rho_0 - r)\right). \end{cases} \tag{1.4}$$

We show that (1.3) and (1.4) are equivalent near the bubble orbit. Then (1.4) fits in with the framework in [13]. So our stability proof follows from the theorem in [13] through the detailed spectral analysis.

But we cannot use the instability theorem in [13], because the skew-adjoint operator

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

is not onto. In fact, our proof of instability follows the strategy in [9]. But by modifying the decreasing direction used in constructing the Liapunov functional, we get boundness of Liapunov functional more easily. So we can prove the instability without the estimate of the time growth of the solution, which is indispensable to the proofs in [13] and [19]. This paper is organized as follows. In Section 2 we state the result on the existence of travelling bubbles. In section 3 we study the Cauchy problem to (1.3), and establish the equivalence between (1.3) and (1.4) around the bubble orbit. In section 4 we analyze the spectral properties of the linearized Hamiltonian operator. Then we prove the stability result by using theorem in [13] under the assumption

$$\frac{dP_v}{dv} < 0 \ .$$

The instability result is proved in section 5 under the assumption

$$\frac{dP_v}{dv} > 0 \ .$$

2. **Existence of the travelling bubbles**

In this section, we give the existence results of travelling bubbles to (1.3) under assumptions on F stated in section 1. We have the following theorem on the existence of travelling bubbles.

Theorem 2.1. *For any*

$$v \in [0, v_c), v_c = \sqrt{-2\rho_0 F'(\rho_0)} \ ,$$

there exists a travelling bubble

$$\phi_v(x - vt) = a(x - vt)e^{i\theta(x-vt)}$$

to (1.3) *such that*

$$a(x) \geq 0, \quad a(x) = a(-x), \quad \theta_x = \frac{1}{2}v(1 - \frac{\rho_0}{a^2}),$$
$$a(x) \to \sqrt{\rho_0}, \qquad a_x, \quad \theta_x \to 0$$

as $x \to \infty$.
Furthermore,

$$\partial^\alpha(a - \sqrt{\rho_0}), \partial^\alpha(\theta_x)$$

exponentially decay for $|\alpha| \leq 2$.

Proof. If we substitute $\phi_v(x - vt)$ into (1.3), then we get the following equivalent system

$$(-2\theta_x + v)a_x - \theta_{xx}a = 0 \quad , \tag{2.1}$$

$$-a_{xx} + (\theta_x)^2 a - F(a^2)a - v\theta_x a = 0. \tag{2.2}$$

Equation (2.1) implies

$$-2\theta_x + v = \frac{v\rho_0}{a^2}$$

by integration. So

$$\theta_x = \frac{1}{2}v\left(1 - \frac{\rho_0}{a^2}\right) \ ,$$

then (2.2) is rewritten as follows.

$$-a_{xx} = \left(\frac{v^2}{4}(1 - \frac{\rho_0^2}{a^4}) - F(a^2)\right)a := F_v(a^2)a, \tag{2.3}$$

where
$$F_v(r) = \frac{v^2}{4}(1 - \frac{\rho_0^2}{r^2}) - F(r),\; U_v(r) := -\int_{\rho_0}^{r} F_v(s)ds = U(r) - \frac{v^2(r-\rho_0)^2}{4r}.$$
Let $\overline{a} = \sqrt{\rho_0} - a$, then $\overline{a}$ satisfies
$$-\overline{a}_{xx} = -F_v((\sqrt{\rho_0} - \overline{a})^2)(\sqrt{\rho_0} - \overline{a}) := g(\overline{a}) \tag{2.4}$$
if $\overline{a} \in [0, \sqrt{\rho_0}]$. Set
$$G(\xi) := \int_0^{\xi} g(s)ds = -\frac{1}{2}U_v((\sqrt{\rho_0} - \xi)^2)\ .$$
According to Theorem 5 of [7], the necessary and sufficient condition for existence of (2.4) is

$\xi_0 := \inf\{\xi \mid 0 < \xi < \sqrt{\rho_0}, G(\xi) = 0\}$ exists, and $\xi_0 \in (0, \sqrt{\rho_0})$, $g(\xi_0) > 0$.

This is equivalent to :

$\eta_0 := \sup\{\eta \mid 0 < \eta < \rho_0, U_v(\eta) = 0\}$ exists, and $\eta_0 \in (0, \rho_0)$, $F_v(\eta_0) < 0$.

Set

$v_c = \sup\{v \mid \eta_0 \text{ exists and } \eta_0 \in (0, \rho_0), F_v(\eta_0) < 0\}$.

From assumption (F.1), we have $v_c > 0$.

We suppose
$$U(r) = (r - \rho_0)^2 \overline{U}(r)\ ,$$
then
$$\overline{U}(\rho_0) = -\frac{1}{2}F'(\rho_0) > 0 \quad .$$
If
$$v < \sqrt{-2\rho_0 F'(\rho_0)} = \sqrt{4\overline{U}(\rho_0)\rho_0}\ ,$$
we will have $U_v(r) > 0$ when r is near ρ_0. So η_0 exists and $U_v'(\eta_0) > 0$, or $F_v(\eta_0) < 0$. And the condition for existence is satisfied. But if
$$v > \sqrt{-2\rho_0 F'(\rho_0)} \quad ,$$
we get the contrary and there exists no solution for (2.4). So we have
$$v_c = \sqrt{-2\rho_0 F'(\rho_0)} \quad .$$
The rest part of theorem just follows from in [7].

□

Remark 2.1. The nonlinearity we consider here includes the case when several minimas of $U(r)$ coexist, which is typical of models describing competing interactions. In the special case (1.2), the explicit form of travelling bubbles was already explicitly found with $v_c = \sqrt{4\rho_0(\rho_0 - A)}$(see [4]).

Remark 2.2. For repulsive ψ^3 nonlinearity, the Gross-Pitaevskii equation
$$i\psi_t + \frac{1}{2}\Delta\psi + (1 - |\psi|^2)\psi = 0. \tag{2.5}$$
In the two dimensional case, the existence of travelling waves with boundary condition
$$\lim_{x\to+\infty} \psi(x,t) = 1$$

has been obtained by Betheul and Saut([8]), when the speed is small enough. There are also some results concerning stability of the travelling waves of this equation in physical literature(see [15], [16] and [17]). For competing potential in two and three dimension, the author proved that in small speed case the travelling wave exists and no vortex appears(see [18]).

Note that the repulsive and the competitive cases often exhibit entirely different features. For example,the GP equation (2.5) has no stationary finite energy solutions (see [15] and [8]), while (1.2) possess a positive stationary solution with finite energy in any dimension.

3. Cauchy problem and the hydrodynamic interpretation of NLS

In this section, we first state a result of Zhidkov concerning the Cauchy problem of (1.3) with initial data nonvanishing at infinity. Then we give a hydrodynamic interpretation of NLS. We see that the two are equivalent near the bubble orbit. The latter one will be the target system in our study afterwards.

Definition 3.1. Denote $X^k(k = 1, 2, 3, \cdots)$ the sets of functions $\phi(x)(x \in \mathbf{R})$, absolutely continuous with derivatives of order $1, 2, \cdots, k-1$ on any finite interval with finite norm

$$||\phi||_k := |\phi|_c + \sum_{i=1}^{k} ||\frac{d^i\phi}{dx^i}||_2 \quad (|\phi|_c := \sup_{x\in\mathbf{R}} |\phi(x)|) \ .$$

Consider the following Cauchy problem

$$iu_t + u_{xx} + f(|u|^2)u = 0 \tag{3.1}$$

with $u(x, 0) = u_0(x)$.
The following theorem established the local existence of Cauchy problem (3.1), when $u_0(x) \in X^k$.

Theorem 3.1 (Zhidkov [20], [21])**.** *Assume $f(r) \in C^{k+1}_{loc}(\mathbf{R}_+)$ is real ($k = 1, 2, \cdots$), then for any $u_0 \in X^k$, there exists a unique solution*

$$u(x, t) \in C([0, T]; X^k)$$

to the problem (3.1) for some $T > 0$, and either $u(., t)$ exists globally in time or for some finite $T_1 > 0$,

$$\lim_{t\to T_1-0} \sup ||u(., t)||_k = \infty \ .$$

Now we give the hydrodynamic interpretation of the NLS. This is obtained from the Madelung transformation. Set $\phi = ae^{i\theta}$, where $a \geq 0$, θ is real. Substitute it into (1.3), separate real and imaginary parts, and introduce fluid density $\rho = a^2$ and fluid speed $u = \theta_x$. In this way we recover the usual mass continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial t}(2\rho u) = 0 \tag{3.2}$$

and an equivalence of Bernoulli equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u^2 + \frac{\rho_x^2}{4\rho^2} - \frac{\rho_{xx}}{2\rho} - F(\rho)\right) = 0 \tag{3.3}$$

We will use $r = \rho_0 - \rho$ instead of ρ, then (r, u) satisfies (1.4), where

$$(r, u) \in X := H^1(\mathbf{R}) \times L^2(\mathbf{R}) \ .$$

Afterwards, all the analysis will be done on system (1.4). So we need the equivalence between it and (1.3) near the bubble orbit.

Definition 3.2. For any $\phi = ae^{i\theta} \in X^1$ with $|\phi(x)| - \sqrt{\rho_0} \in L^2(\mathbf{R})$, denote

$$\rho(\phi, \phi_v) := \inf_{s \in \mathbf{R}} (||\tau_s a^2 - a_v^2||_{1,2} + ||\tau_s \theta_x - \theta_{v_x}||_2) \ ,$$

where τ_s denote translation by s acting on functions of one real variable, that is $(\tau_s f)(x) = f(x+s)$. Here, $||.||_{1,2}$ denotes H^1-norm, and

$$\phi_v(x - vt) = a_v e^{i\theta_v}(x - vt)$$

is the travelling bubble. Denote

$$U_\varepsilon := \{\phi \in X^1, |\phi| - \sqrt{\rho_0} \in L^2 \mid \rho(\phi, \phi_v) < \varepsilon\} \ .$$

Lemma 3.1. *There exists $\varepsilon_0 > 0$, for all $\varepsilon < \varepsilon_0$, if $\phi_0 = a_0 e^{i\theta_0} \in U_\varepsilon$, denoting*

$$\phi(x,t) = a(x,t)e^{i\theta(x,t)}$$

the solution of (1.3) *with initial data ϕ_0, then there exists $T > 0$, such that*

$$(r(x,t), u(x,t)) := (\rho_0 - a(x,t)^2, \theta_x(x,t))$$

satisfies (1.4) *with initial data*

$$(r_0, u_0) = (\rho_0 - a_0^2, \theta_{0_x}),$$

in $0 \le t < T$.

Proof. Recall that $H^1(\mathbf{R}^1) \hookrightarrow L^\infty(\mathbf{R}^1)$ and $\inf_{x \in \mathbf{R}} a_v(x) > 0$, so if we choose ε_0 small enough, then $\phi_0 \in U_\varepsilon$ implies that $\inf_{x \in \mathbf{R}} a_0(x) > 0$. Then by Theorem 3.1, there exists $T > 0$ such that $\inf_{x \in \mathbf{R}} a(x,t) > 0$, for $t \in [0, T)$. So all the formal computation in deducing (1.4) becomes rigorous, and the equivalence follows. □

Remark 3.1. The above Lemma implies the local existence of the Cauchy problem (1.4) with initial data near bubble orbit.

Next we define several conservation quantities of (1.3) or (1.4).
If

$$\phi(x) = ae^{i\theta} \in X^1, |a| - \sqrt{\rho_0} \in L^2(\mathbf{R}), r = \rho_0 - a^2, u = \theta_x \ .$$

we define four conservation laws as follows:
(1)**Energy**

$$E(\phi) = \int_{-\infty}^{+\infty} (|\phi_x|^2 + U(|\phi|^2))dx$$

or

$$E(r, u) = \int_{-\infty}^{\infty} \left(\frac{r_x^2}{4(\rho_0 - r)} + u^2(\rho_0 - r) + U_1(r) \right) dx \ .$$

Here $U_1(r) = U(\rho_0 - r)$.
(2)**Momentum**

$$P(\phi) = Im \int_{-\infty}^{+\infty} \phi_x^* \phi \left(1 - \frac{\rho_0}{|\phi|^2}\right) dx$$

or

$$P(r, u) = -\int_{-\infty}^{+\infty} ru dx \ .$$

(3)**Number of particles**

$$N(\phi) = \int_{-\infty}^{+\infty} (|\phi|^2 - \rho_0) dx$$

or

$$N(r, u) = -\int_{-\infty}^{+\infty} r dx \ .$$

(4)**Twisting angle**

$$\Theta(\phi) = \int_{-\infty}^{+\infty} \theta_x dx$$

or

$$\Theta(r, u) = \int_{-\infty}^{+\infty} u dx \ .$$

Remark 3.2. The definition of momentum is not usual. But if we want the travelling bubble to satisfy $E' - vP' = 0$, where $'$ denote the Fréchet derivative, then this definition is needed(see [1] for details). Throughout this paper, when talking about (1.4), we always mean the solution is near the bubble orbit. In that case, since $\rho_0 - r$ is away from 0, $E(r, u)$ and $P(r, u)$ are well defined and conserved.

Remark 3.3. Denoting

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} ,$$

we can rewrite (1.4) as

$$\frac{\partial}{\partial t}(r, u) = JE'(r, u) \tag{3.4}$$

This fits in with the framework of [13]. From (3.4), it is easy to see that the four quantities defined above are indeed conserved for (1.3) or (1.4). In particular, P is the conserved quantity corresponding to the translation invariance of energy. The stability analysis use only the first two,while the last two play an important role in our instability proof.

4. **Stability**

In this section, we prove the stability in the case

$$\frac{dP_v}{dv} < 0 \ .$$

This is done by applying the theorem in[13] after the detailed study of the spectral property of the linearized Hamiltonian operator. First we define stability of the travelling bubbles.

Definition 4.1. We say $\phi_v(x-vt)$ is stable, if for any $\varepsilon>0$, there exists $\delta>0$, such that for $\phi_0\in U_\delta$, we have $\phi(t)\in U_\varepsilon$, $0<t<\infty$. Here $\phi(t)$ is the solution to (1.3), and U_ε is defined in definition 3.2.

Because of the equivalence between (1.3) and (1.4), we only have to prove that (r_v,u_v) is stable to (1.4), where

$$\phi_v(x-vt)=a_v e^{i\theta_v}(x-vt),\quad r_v=\rho_0-a_v^2,\quad u_v=\theta_{v_x}.$$

That is, we want to show that:
If

$$\frac{dP_v}{dv}<0\ ,$$

then for any $\varepsilon>0$, there exists $\delta>0$, such that if

$$\inf_s(||\tau_s r_0-r_v||_{1,2}+||\tau_s u_0-u_v||_2)<\delta$$

then

$$\inf_s(||\tau_s r(t)-r_v||_{1,2}+||\tau_s u(t)-u_v||_2)<\varepsilon$$

for $0<t<\infty$. Here $(r(t),u(t))$ is the solution to (1.4) with initial (r_0,u_0).

As mentioned in the last section, system (1.4) fits in with the abstract framework of [13], Precisely, we can write (1.4) as

$$\frac{\partial}{\partial t}(r,u)=JE'(r,u) \tag{4.1}$$

Here

$$J=\begin{pmatrix}0 & \partial_x\\ \partial_x & 0\end{pmatrix}$$

is skew-symmetric,

$$(r,u)\in X:=H^1(\mathbf{R})\times L^2(\mathbf{R}),\langle(r_1,u_1),(r_2,u_2)\rangle:=\int_{-\infty}^{\infty}(r_1r_2+u_1u_2)dx\ .$$

$$E'(r,u)=\left(-u^2-\frac{r_x^2}{4(\rho_0-r)^2}+\frac{r_{xx}}{2(\rho_0-r)}+F(\rho_0-r),2(\rho_0-r)u\right)\ ,$$

$$P'=(-u,-r)\ .$$

In our problem the symmetry is the translation group $\{\tau_s\}(s\in\mathbf{R})$ and

$$P=-\int_{-\infty}^{\infty}rudx=\frac{1}{2}\langle Bu,u\rangle,B:=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\ .$$

Recall that the stability theorem in [13] has following three basic assumptions:

Assumption 1(Existence of Solutions)

For each $u_0\in X$ there exists $t_0>0$ depending only on μ, where $||u_0||\le\mu$, and there exists a solution of (4.1) in the interval $[0,t_0)$ such that
(a) $u(0)=u_0$ and
(b) $E(u(t))=E(u_0),P(u(t))=P(u_0),t\in[0,t_0)$.
We have established the local existence near bubble orbit in section 3, which is enough for stability analysis.

Assumption 2(Existence of Bound States)

There exists real $v_1 < v_2$ and a mapping
(a) $v \mapsto \psi_v$ from the open interval (v_1, v_2) into X which is C^1 and for each

$$v \in (v_1, v_2)$$

(b) $E'(\psi_v) = vP'(\psi_v)$,
(c) $\psi_v \in D(\tau'(0)^3) \cap D(JI\tau'(0)^2)$,
(d) $\tau'(0)\psi_v \neq 0$.
The existence of travelling solution have been obtained in section 2.

Before stating assumption 3, we need several notations:

$$\psi_v := (r_v, u_v)$$

is the travelling wave solution to (1.4), here $v \in [0, v_c), v_c$ is the maximal speed defined in section 2. And

$$d(\psi) := E(\psi_v) - vP(\psi_v)$$

is the Hamiltonian to (1.4).

$$H_v := (E'' - vP'')|_v$$

is the Hessian of functional d.

Because of the translation symmetry, we have(see [13]):

$$H_v(\tau'(0)\psi_v) = 0,$$

or

$$\frac{\partial}{\partial x}\psi_v \in ker(H_v) \ .$$

Assumption 3(Spectral decomposition of H_v)
(1) There exists $\chi \in X$ such that

$$\langle H\chi, \chi\rangle < 0 \ .$$

(2) There exists a closed subspace $P \subset X$ such that

$$\langle Hp, p\rangle \geq \delta ||p||_X^2 \ ,$$

for all $p \in P$.
(3) For all $u \in X$,there exists $a, b \in \mathbf{R}$ and $p \in P$,such that

$$u = a\chi + b\frac{\partial}{\partial x}\psi_v + p \ .$$

Remark 4.1. This is the assumption 3B in[13,section 5]. Here it may not be orthogonal decomposition.

First suppose that Assumption 3 holds, then we have

Theorem 4.1. *The travelling bubbles are stable in the case*

$$\frac{dP_v}{dv} < 0 \ .$$

Proof. It follows directly from Theorem 3 in [13], by noticing that

$$d''(v) = -\frac{dP_v}{dv} \ .$$

□

Remark 4.2. It is easy to show the following:

$$a \geq 0, a - \sqrt{\rho_0} \in L^2$$

and

$$||a^2 - a_v^2||_{1,2} < \varepsilon < 1$$

implies that there exists a constant $C > 0$, such that

$$||a - a_v||_{1,2} < C\varepsilon \ .$$

In deed we have

$$\begin{aligned} \int (a^2 - a_v^2) &\geq \int (a - a_v)^2 \inf(a + a_v) \\ &\geq a_v(0) \int (a - a_v)^2 \end{aligned}$$

and

$$\begin{aligned} \int (a^2 - a_v^2)_x^2 &= \int \left((a - a_v)_x(a + a_v) + (a - a_v)(a + a_v)_x\right)^2 \\ &\geq \int \frac{1}{2}(a - a_v)_x^2(a + a_v)^2 - (a - a_v)^2(a + a_v)_x^2 \\ &\geq \frac{1}{2}a_v(0)^2 \int (a - a_v)_x^2 - C_1\varepsilon^2. \end{aligned}$$

In the last inequality, we used the fact $H^1(\mathbf{R}^1) \hookrightarrow L^\infty(\mathbf{R}^1)$. So we have

$$\frac{1}{2}a_v(0)^2 \int (a - a_v)_x^2 \leq (1 + C_1)\varepsilon^2,$$

and the conclusion follows.

Thus we can replace the norm in definition 3.2 by

$$\inf_{s \in \mathbf{R}} ||\tau_s a - a_v||_{1,2} + ||\tau_s \theta_x - \theta_{v_x}||_2$$

This is norm used in the definition of the stability in Zhidkov [21].

Now what is left is to prove Assumption 3. In the following, we will write $\psi_v = (r_v, u_v)$ simply as $\psi = (r, u)$, H_v as H in the case of no confusion.
Then for

$$\delta\psi = (\delta r, \delta u) \in X \ ,$$

expand

$$(E(\psi + \delta\psi) - vP(\psi + \delta\psi)) - (E(\psi) - vP(\psi))$$

to the second order and we get

$$
\begin{aligned}
\langle H\delta\psi, \delta\psi\rangle &= \int \left(\frac{r_x^2}{4(\rho_0 - r)^3} + \frac{1}{2} U_1''(r) \right) \delta r^2 + \frac{r_x \delta r_x \delta r}{2(\rho_0 - r)^2} \\
&\quad + \frac{\delta r_x^2}{4(\rho_0 - r)} + (\rho_0 - r)\delta u^2 + (v - 2u)\delta u \delta r \\
&= \int \left(\frac{r_x^2}{4(\rho_0 - r)^3} - \frac{\partial}{\partial x}\left(\frac{r_x}{4(\rho_0 - r)^2} \right) - \frac{1}{2} F'(\rho_0 - r) \right) \delta r^2 \\
&\quad + \frac{\delta r_x^2}{4(\rho_0 - r)} + (v - 2u)\delta u \delta r + (\rho_0 - r)\delta u^2 .
\end{aligned}
$$

Notice that

$$
v - 2u = \frac{v\rho_0}{\rho_0 - r}
$$

(see section 2). We have

$$
\langle H\delta\psi, \delta\psi\rangle = (L\delta r, \delta r) + \int (\rho_0 - r) \left(\delta u + \frac{v\rho_0}{2(\rho_0 - r)^2} \delta r \right)^2 \tag{4.2}
$$

Here

$$
\begin{aligned}
L = &-\frac{\partial}{\partial x}\left(\frac{1}{4(\rho_0 - r)} \frac{\partial}{\partial x} \right) + \frac{r_x^2}{4(\rho_0 - r)^3} - \frac{\partial}{\partial x}\left(\frac{r_x}{4(\rho_0 - r)^2} \right) \\
&- \frac{1}{2} F'(\rho_0 - r) - \frac{v^2 \rho_0^2}{4(\rho_0 - r)^3} \\
= &-\frac{\partial}{\partial x}\left(\frac{1}{4(\rho_0 - r)} \frac{\partial}{\partial x} \right) + q(x),
\end{aligned}
$$

where

$$
q(x) = \frac{r_x^2}{4(\rho_0 - r)^3} - \frac{\partial}{\partial x}\left(\frac{r_x}{4(\rho_0 - r)^2} \right) - \frac{1}{2} F'(\rho_0 - r) - \frac{v^2 \rho_0^2}{4(\rho_0 - r)^3} \ .
$$

So

$$
q(x) \to -\frac{1}{2} F'(\rho_0) - \frac{v^2}{4\rho_0} = \lambda > 0 \ ,
$$

as $x \to \infty$, because

$$
v < v_c = \sqrt{-2\rho_0 F'(\rho_0)} \ .
$$

As

$$
\rho_0 - r \geq \rho_0 - r(0) > 0 \ ,
$$

we have

$$
\sigma_{ess}\left(-\frac{\partial}{\partial x}\left(\frac{1}{4(\rho_0 - r)} \frac{\partial}{\partial x} \right) \right) = [0, +\infty) \ .
$$

So by the Weyl essential spectrum theorem we conclude that

$$
\sigma_{ess}(L) = [\lambda, +\infty) \ .
$$

From the equation(2.3) satisfied by a,we deduce that $r = \rho_0 - a^2$ satisfies

$$
\frac{1}{2(\rho_0 - r)} r_{xx} + \frac{1}{4(\rho_0 - r)^2} r_x^2 + \left(\frac{v^2 \rho_0^2}{4(\rho_0 - r)^2} - \frac{v^2}{4} - F(\rho_0 - r) \right) = 0 \tag{4.3}
$$

Then we take the derivative in x on the above equation to get

$$Lr_x = 0$$

by simple computation. As r_x has only one zero at $x = 0$, by the Sturm-Liouville oscillation theorem, L has one and only one negative eigenvalue λ_1, with a positive eigenfunction χ_1. From the above discussion, we have the following spectral properties of L.

Lemma 4.1. *For any $z \in H^1(\mathbf{R})$ satisfying*

$$(z, \chi_1) = (z, r_x) = 0 \ ,$$

there exists $\delta > 0$, such that

$$(Lz, z) \geq \delta ||z||_{H^1}^2 \ .$$

Proof. First, from the above discussion, it follows that

$$(Lz, z) \geq \delta_1 ||z||_2^2 \ ,$$

for some $\delta_1 > 0$. Then

$$\begin{aligned}
(Lz, z) &= \int \left(\frac{1}{4(\rho_0 - r)} z_x^2 + q(x) z^2 \right) dx \\
&= \frac{1}{k+1} \int \left(\frac{1}{4(\rho_0 - r)} z_x^2 + q(x) z^2 \right) dx + \frac{k}{k+1} \int q(x) z^2 dx \\
&\quad + \frac{k}{k+1} \int \frac{1}{4(\rho_0 - r)} z_x^2 \\
&\geq \left(\frac{1}{k+1} \delta_1 - \frac{kM}{k+1} \right) ||z||_2^2 + \frac{k}{k+1} \frac{1}{4(\rho_0 - r(0))} \int z_x^2 dx,
\end{aligned}$$

here

$$M = \sup_{x \in \mathbf{R}} |q(x)|.$$

Choose k such that $kM = \frac{1}{2}\delta_1$,then for some $\delta > 0$, we have

$$(Lz, z) \geq \delta ||z||_{H^1}^2 \ .$$

□

Now if we set

$$\chi_- = (\chi_1, -\frac{v\rho_0}{2(\rho_0 - r)^2} \chi_1) \ ,$$

then by (4.2),we have

$$\langle H\chi_-, \chi_- \rangle = (L\chi_1, \chi_1) < 0 \ .$$

Define

$$P = \{ p \in X, p = (p_1, p_2) \mid (p_1, \chi_1) = (p_1, r_{v_x}) = 0 \}.$$

Then for

$$\psi = (r, u) \in X,$$

we have the decomposition

$$\psi = a\chi_- + b\partial_x \psi_v + p \ ,$$

where

$$a = (r, \chi_1), b = \frac{(r, \partial_x r_v)}{||\partial_x r_v||_2^2}, p \in P.$$

It is easy to see that this decomposition is unique. So far, we have verified (1) and (3) of assumption 3.

To complete the verification of Assumption 3, it remains only to check condition (2). It is contained in the following lemma.

Lemma 4.2. *For any $p = (r, u) \in P$, there exists a constant $C > 0$, such that*

$$\langle Hp, p \rangle \geq C||p||_X^2 \ .$$

Proof. In fact, it follows from the argument in [14]. We repeat it here for completeness. We divide the proof into two cases.

(1) If $||u||_2^2 \geq \frac{v^2}{\rho_0^2}||r||_2^2$, then

$$\begin{aligned}\int (\rho_0 - r_v)\left(u + \frac{v\rho_0}{2(\rho_0 - r_v)^2}r\right)^2 &\geq (\rho_0 - r_v(0))\int\left(\frac{1}{2}u^2 - \frac{v^2}{4\rho_0^2}r^2\right)\\ &\geq \frac{\rho_0 - r_v(0)}{4}\int u^2\end{aligned}$$

So recalling (4.2), we have

$$\begin{aligned}\langle Hp, p \rangle &\geq \delta||r||_{H^1(\mathbf{R})}^2 + \frac{\rho_0 - r_v(0)}{4}\int u^2\\ &\geq C||p||_X^2\end{aligned}$$

(2) If $||u||_2^2 < \frac{v^2}{\rho_0^2}||r||_2^2$, then

$$\begin{aligned}\langle Hp, p \rangle &\geq \delta||r||_{H^1(\mathbf{R})}^2\\ &\geq \frac{\delta}{2}||r||_{H^1(\mathbf{R})}^2 + \frac{\delta\rho_0^2}{v^2}||u||_2^2\\ &\geq C||p||_X^2.\end{aligned}$$

□

Thus, we have proved assumption 3 and completed the proof of stability .

5. **Instability**

In this section, we prove that the travelling bubbles are unstable if

$$\frac{dP_v}{dv} > 0.$$

For that we still consider system (1.4). But as

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

is not onto, we cannot use the instability theorem in [13] directly. Our instability proof will follow the strategy in [9].

To begin with, we construct the Liapunov function following the line in [9]. It follows from a series of Lemmas. We only provide proofs when differences appear. For more details, see [9].

The following lemma shows that if $\frac{dP_v}{dv} > 0$, then ϕ_v is not an energy minimizer under the constraint $P \equiv P(\phi_v)$.

Lemma 5.1. *If $\frac{dP_v}{dv} > 0$, then there exists a smooth curve $\psi(s) : [v-\epsilon, v+\epsilon] \to X$, such that $P(\psi(s)) \equiv P_v$, and $\frac{d^2E(\psi(s))}{ds^2} |_{s=v} < 0$.*

Proof. Recall that

$$\chi_- = (\chi_1, -\frac{v\rho_0}{2(\rho_0 - r)^2}\chi_1)$$

is the negative vector of H_v. Define

$$\psi(s) = \psi_s + l(s)\chi_-,$$

with appropriate function $l(s)$. Here

$$\psi_s = \left(r_s, -\frac{1}{2}s\frac{r_s}{\rho_0 - r_s} \right)$$

is the travelling wave with speed s.

Notice that

$$\begin{aligned}
\frac{\partial}{\partial l} P(\psi_s + l\chi_-) |_{l=0,s=v} &= \langle B\psi_v, \chi_- \rangle \\
&= \int r_v \frac{v\rho_0}{2(\rho_0 - r_v)^2}\chi_1 + \frac{1}{2}v\frac{r_v}{\rho_0 - r_v}\chi_1 \\
&> 0.
\end{aligned}$$

So by the implicit function theorem, we can find a function $l(s)$ defined near v, such that $l(v) = 0$, $\psi(v) = \psi_v$ and $P(\psi(s)) \equiv P_v$. The rest of proof is the same as in [13].

□

Corollary 5.1. *There exists $y_0 = (r_0, u_0) \in X$ such that*

$$(1 + |x|)|r_0(x)|, (1 + |x|)|u_0(x)| \in L^1(\mathbf{R}),$$

and

$$\langle By_0, \psi_v \rangle = 0, \langle Hy_0, y_0 \rangle < 0.$$

The following Lemma is a key step in our proof, and we think it is of independent interest.

Lemma 5.2. *For all $r(x) \in L^2(\mathbf{R})$ and $c \in \mathbf{R}$, there exists a sequence $\{u_n\}$ in $H^1(\mathbf{R})$ such that*

$$(1 + |x|)u_n(x) \in L^1(\mathbf{R}), \int u_n dx = c,$$

$u_n \to 0$ in $H^1(\mathbf{R})$ and

$$(u_n, r)_{L^2} = 0,$$

where

$$(u_n, r)_{L^2} := \int u_n r dx.$$

Proof. (1) If $r(x) \neq 0$, we take $\varphi(x) \in H^1(\mathbf{R})$ such that

$$\int \varphi(x)dx = c, (1+|x|)\varphi \in L^1.$$

There exists $\psi(x) \in C_0^\infty(\mathbf{R})$ such that

$$\int r_x \psi(x) \neq 0.$$

For otherwise

$$\forall \psi(x) \in C_0^\infty(\mathbf{R}^1), \int r_x \psi(x) = 0.$$

This means $r_x = 0$ in the distribution sense, that is, $r \equiv$ constant. But $r \in L^2$, so $r \equiv 0$, a contradiction.

Then let

$$u_n = \frac{1}{n}\varphi(\frac{1}{n}x) + a_n \psi_x(x).$$

Now we take

$$a_n = \frac{\int \frac{1}{n}\varphi(\frac{1}{n}x)r(x)dx}{\int r_x \psi dx},$$

to make

$$(u_n, r)_{L^2} = 0.$$

Then

$$|a_n| \leq \frac{||\frac{1}{n}\varphi(\frac{1}{n}x)||_2 ||r||_2}{\int r_x \psi} \to 0,$$

as $n \to \infty$. So $u_n \to 0$ in $H^1(\mathbf{R})$ and

$$(1+|x|)u_n(x) \in L^1(\mathbf{R}), \int u_n(x)dx = \int \frac{1}{n}\varphi(\frac{1}{n}x)dx = \int \varphi(x)dx = c.$$

(2) If $r \equiv 0$, we can take any $\psi(x) \in C_0^\infty(\mathbf{R^1})$ in the above proof.

□

Lemma 5.3. *There exists $y = (r_1, u_1) \in X$, such that*

$$(1+|x|)r_1(x), (1+|x|)u_1(x) \in L^1,$$

and

$$\langle By, \psi_v \rangle = 0, \langle Hy, y \rangle < 0,$$
$$\int_{-\infty}^{+\infty} r_1 dx = \int_{-\infty}^{+\infty} u_1(x)dx = 0.$$

Proof. From the last lemma,we can find $\{r^n\}, \{u^n\}$, such that

$$r^n, u^n \to 0$$

in $H^1(\mathbf{R})$ as $n \to \infty$. And

$$(1+|x|)r^n, (1+|x|)u^n \in L^1,$$
$$\int r^n = -\int r_0, \int u^n = -\int u_0,$$

where $y_0 = (r_0, u_0)$ is as in Cor 5.1.

Set

$$y^n = y_0 + (r^n, u^n),$$

then

$$||y^n - y_0||_X \to 0,$$

as $n \to \infty$. Since $\langle Hy, y\rangle$ is continuous in X, we have

$$\langle Hy^n, y^n\rangle \to \langle Hy_0, y_0\rangle < 0,$$

when $n \to \infty$. So we can choose N, such that $\langle Hy^N, y^N\rangle < 0$. Let

$$y = (r_1, u_1) := y^N,$$

then y is exactly what we want.

□

Remark 5.1. Lemma 5.3 is the main point which distinguishes current proof from the others in previous papers([9], [19]). Here we make a small correction to the usual (energy)decreasing direction used in constructing the Liapunov functional. The new direction, still decreasing, has the additional property that its integral over $\mathbf{R}$ is zero. It is this property that enables us to avoid the difficult estimate in previous papers. We think the same idea can be applied to other problems. It also has the following physical interpretation: if ψ_v is not energy minimizer under the constraint of constant momentum, neither is it even under constrains of constant particle numbers , twist angle and momentum.

Lemma 5.4. *There exists an* $\varepsilon > 0$ *and a unique* C^1 *map* α: $U_\varepsilon \mapsto \mathbf{R}$*, where* U_ε *is defined in Definition* 3.2*, such that for all* $\psi = (r, u) \in U_\varepsilon$*,*

$$\begin{aligned}
&(1)(r(.+\alpha(\psi)), \partial_x r_v) = 0,\\
&(2)\forall s \in \mathbf{R}, \alpha(\psi(.+s)) = \alpha(\psi) - s,\\
&(3)\alpha'(\psi) = \left(\frac{\partial_x r_v(.-\alpha(\psi))}{\int_{-\infty}^{\infty} r\partial_x^2 r_v(x-\alpha(\psi))dx}, 0\right).
\end{aligned}$$

Proof. We only have to consider the map:

$$(\psi, \alpha) \mapsto \int r(x+\alpha)\partial_x r_v(x)dx$$

and using the implicit function theorem at $\alpha = 0$ and $\psi = \psi_v$.

□

Definition 5.1. For all $\psi \in U_\varepsilon$, define $V(\psi)$ by the formula

$$\begin{aligned}
V(\psi) &= y(.-\alpha(\psi)) + \langle By(.-\alpha(\psi)), \psi\rangle J\alpha'(\psi)\\
&= y(.-\alpha(\psi)) + \frac{\langle By(.-\alpha(\psi), \psi\rangle}{\int_{-\infty}^{+\infty} r\partial_x^2 r_v(x-\alpha(\psi))dx}\left(0, \partial_x^2 r_v(x-\alpha(\psi))\right)
\end{aligned}$$

where y is as in Lemma 5.3.

Lemma 5.5. *V is in C^1 for $U_\varepsilon \mapsto X$. Moreover, V commutes with translations and*

$$V(\psi_v) = y, \langle V(\psi), B\psi\rangle = 0.$$

Corollary 5.2. *The solution $\psi^\lambda = R(\lambda, \psi)$ of the initial-value problem*

$$\frac{d\psi^\lambda}{d\lambda} = V(\psi^\lambda), \psi^0 = \psi$$

has the following properties:
(i) R is a C^1 function on $\{\lambda \mid |\lambda| < \lambda_0(\psi)\}$ for any $\psi \in U_\varepsilon$.
(ii) R commutes with translations for each λ.
(iii) $P(R(\lambda, \psi))$ is independent of λ.
(iV) $\partial R/\partial\lambda(0, \psi_v) = y$.

Lemma 5.6. *There is a C^1 functional*

$$\Lambda : \{\psi \in U_\varepsilon \mid P(\psi) = P_v\} \mapsto \mathbf{R},$$

such that

$$E(R(\lambda(\psi), \psi)) > E(\psi_v)$$

for all $\psi \in U_\varepsilon$ which are not translates of ψ_v and are such that $P(\psi) = P_v$.

Proof. Let $G(\psi) = \psi(. - \alpha(\psi))$ and $\pi : \psi = (r, u) \mapsto r$. Solve the equation

$$(\pi(G(R(\lambda, \psi)) - \psi_v), \chi_1) = 0 \tag{5.1}$$

locally near $(\lambda, \psi) = (0, \psi_v)$ by implicit function theorem. Then $\pi(G(\psi^\lambda) - \psi_v)$ is perpendicular to both χ_1 and $\partial_x r_v$, so $G(\psi^\lambda) - \psi_v)$ belongs to the positive subspace of H_v. Expand $E(\psi^\lambda)$ near ψ_v to have

$$\begin{aligned} E(\psi^\lambda) &= E(G(\psi^\lambda)) \\ &= E(\psi_v) + \frac{1}{2}\langle H_v(G(\psi^\lambda) - \psi_v), G(\psi^\lambda) - \psi_v\rangle + o(||G(\psi^\lambda) - \psi_v||^2) \end{aligned}$$

So, for λ small enough $E(\psi^\lambda) > E(\psi_v)$, unless $G(\psi^\lambda) = \psi_v$ or ψ is translate of ψ_v. □

Lemma 5.7. *Let $\psi \in U_\varepsilon$ be such that $P(\psi) = P(\psi_v)$ and ψ is not a translate of ψ_v. Then we have*

$$E(\psi_v) < E(\psi) + \Lambda(\psi)\langle E'(\psi), V(\psi)\rangle.$$

Lemma 5.8. *The curve $\psi(s)$ constructed in Lemma 5.1 satisfies*

$$E(\psi(s)) < E(\psi_v)$$

for $s \neq v$ and $P(\psi(s)) = P_v$. Furthermore

$$\langle E'(\psi(s)), V(\psi(s))\rangle$$

changes sign as s passes through v.

Now we are in a position to prove the instability .

Theorem 5.1. *The travelling bubbles are unstable if*

$$\frac{dP_v}{dv} > 0.$$

Proof. Let $\varepsilon > 0$ and U_ε be the neighborhood of ψ_v defined in section 3. Choose ε small so that Lemma 5.4 can be applied within U_ε. By Lemma 5.8, we can take $\psi_0 = (r_0, u_0) \in X$ arbitrarily close to ψ_v that are not translates of ψ_v and

$$P(\psi_0) = P(\psi_v), E(\psi_0) < E(\psi_v), \langle E'(\psi_0), V(\psi_0)\rangle > 0.$$

We define the Liapunov function A as follows. Let

$$\beta(t) = \alpha(\psi(t))$$

and

$$Y(x) = (\tilde{u_1}, \tilde{r_1}) := J^{-1}y = \left(\int_{-\infty}^{x} u_1 ds, \int_{-\infty}^{x} r_1 ds\right).$$

Here

$$\psi(t) = (r(t), u(t))$$

is the solution of (1.4) with initial ψ_0 and $y = (r_1, u_1)$ is as in Lemma 5.3. Let T_1 be the maximal time such that $\psi(t)$ stays in U_ε.

Define

$$\begin{aligned} A(t) &= -\left(Y(x - \beta(t)), \psi(x,t)\right) \\ &= -\int_{-\infty}^{+\infty} \tilde{u_1}\ (x - \beta(t)) r(x,t) + \tilde{r_1}\ (x - \beta(t)) u(x,t) dx \end{aligned}$$

As

$$\int_{-\infty}^{+\infty} r_1 dx = \int_{-\infty}^{+\infty} u_1 dx = 0$$

and

$$(1 + |x|) r_1(x), (1 + |x|) u_1(x) \in L^1,$$

we get

$$\tilde{u_1}, \tilde{r_1} \in L^2$$

(see [9] for details of proof).
So

$$|A(t)| \le ||\ \tilde{u_1}\ ||_2 ||r(t)||_2 + ||\ \tilde{r_1}\ ||_2 ||u(t)||_2 < C$$

for all $t \in [0, T_1)$. But

$$\begin{aligned} \frac{dA}{dt} &= -\beta'(t)\langle By(. - \beta(t)), \psi(t)\rangle - \langle Y(x - \beta(t)), \frac{\partial \psi}{\partial t}\rangle \\ &= \langle \langle By(. - \beta(t)), \psi(t)\rangle \alpha'(\psi) - Y(x - \beta(t)), JE'(\psi)\rangle \\ &= \langle V(\psi(t)), E'(\psi(t))\rangle \end{aligned}$$

As

$$0 < E(\psi_v) - E(\psi_0) = E(\psi_v) - E(\psi(t)),$$

Lemma 5.7 implies that

$$0 < \Lambda(\psi(t))\langle V(\psi(t)), E'(\psi(t))\rangle\ .$$

Because $\psi(t) \in U_\varepsilon(0 < t < T_1)$ and $\Lambda(\psi_v) = 0$,we can assume $\Lambda(\psi(t)) < 1$ by choosing ε smaller if necessary. So for all $t \in [0, T_1)$,

$$\langle V(\psi(t)), E'(\psi(t))\rangle \geq E(\psi_v) - E(\psi_0) > 0.$$

Hence

$$\frac{dA}{dt} \geq E(\psi_v) - E(\psi_0) > 0$$

for all $t \in [0, T_1)$, and so we conclude that $T_1 < \infty$. This shows that $\psi(t)$ eventually leaves the bubble tube U_ε, which implies instability.

□

Remark 5.2. In both [13] and [9] the Liapunov function method was used and they constructed the Liapunov function $A(u)$ by almost the same argument. The only difference between [13] and [9] is that: in [13] J is onto, and so the bound of $A(u(t))$ follows easily. But the mapping J is not onto for some important equations, for example, KDV and KP equations(see e.g.,[9], [12] and [19]). To overcome this difficulty, in [9] Bona, Souganidis and Strauss had to prove an estimate such as

$$|A(u(t))| \leq C(1 + t^\eta),$$

for some $\eta < 1$, which required tricky calculations. But in our proof we do not need such an estimate for the Liapunov function, since our proof is based on the different choice of a decreasing direction of the energy functional (see lemma 5.3). We think the same idea may be applicable to other dispersive equations.

Remark 5.3. For model (1.2), P_v is a single-humped function by explicit computation. So there exists a critical speed v_{cr} such that bubbles are unstable when $v < v_{cr}$ and stable when $v > v_{cr}$. See [3] for some physical explanation of this phenomenon.

We show that above remark also holds for more general nonlinearity.

Proposition 5.1. *For potential*

$$U(r) = (r - \rho_0)^2 \overline{U}(r),$$

we define

$$f(r) = 4r\overline{U}(r).$$

If $f(r)$ satisfies:

$$f'(r) > 0, f''(r) \geq 0, \forall r \in (\eta_0, \rho_0),$$

then P_v is a single-humped function. Here η_0 is defined in (F.1).

Proof. From the graph of P_v, it is easy to see that it suffices to show

$$\frac{P_v}{v}$$

is decreasing.
Recall

$$P_v = -\int_{-\infty}^{+\infty} r_v u_v dx$$

and

$$r_v = \rho_0 - a_v^2, u_v = \frac{1}{2}v\left(1 - \frac{\rho_0}{a_v^2}\right),$$

we have

$$\frac{P_v}{v} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(\rho_0 - a_v^2)^2}{a_v^2}dx = \int_0^{+\infty} \frac{(\rho_0 - a_v^2)^2}{a_v^2}dx.$$

As

$$-a_{v_{xx}} = F_v(a_v^2)a_v,$$

we have

$$a_{v_x}^2 = U_v(a_v^2),$$

which implies

$$\begin{aligned}
\frac{P_v}{v} &= \int_0^{\infty} \frac{(\rho_0 - a_v^2)^2}{a_v^2} \frac{1}{2a_v\sqrt{U_v(a_v^2)}}d(a_v^2) \\
&= \frac{1}{2}\int_{a_v^2(0)}^{\rho_0} \frac{(\rho_0 - x)^2}{x\sqrt{xU_v(x)}}dx \\
&= \int_{a_v^2(0)}^{\rho_0} \frac{\rho_0 - x}{x\sqrt{f(x) - v^2}}dx \\
&= 2\int_0^{\sqrt{v_c^2 - v^2}} \frac{\rho_0 - x_v(y)}{x_v(y)f'(x_v(y)}dy.
\end{aligned}$$

In the last equality, we use new variable

$$y = \sqrt{f(x) - v^2}, x_v(y) = f^{-1}(y^2 + v^2).$$

Denote

$$h(x) = \frac{\rho_0 - x}{xf'(x)},$$

then

$$h'(x) = -\frac{\rho_0 f'(x) + xf''(x)(\rho_0 - x)}{(xf'(x))^2} < 0$$

and

$$\frac{dx_v(y)}{dv} > 0,$$

according to the condition on f.
Accordingly

$$\frac{d}{dv}(\frac{P_v}{v}) = 2\int_0^{\sqrt{v_c^2 - v^2}} h'(x_v(y))\frac{dx_v(y)}{dv}dy < 0.$$

□

Remark 5.4. Notice that for (1.2), the potential

$$U(r) = (r - \rho_0)^2(r - A)$$

satisfies the condition in the above proposition. For the nonlinearity satisfying (F.1) and (F.2), P_v is increasing near 0 and decreasing near v_c. So a bubble is unstable when v is near 0, and stable when v is near v_c.

Remark 5.5. After this paper was completed, the author found that in [6], Barashenkov considered the proof of the same stability criterion. He proved stability by constructing a Liapunov functional directly. And his instability proof considered the case when the speed is very near the critical speed using the method of matching expansion.

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References

[1] I. V. Barashenkov and E. Y. Panova, *Stability and evolution of the quiesent and travelling solitonic bubbles*, Physica D **69**(1993),114-134.

[2] ______, A. D. Gocheva, V. G. Makhankov, and I. V. Puzynin, *Stability of the soliton-like "bubbles"*, Physica D **34**(1989),240-254.

[3] ______,I. V. Puzynin, T. Zhanlav and T. L. Boyadjiev, *Stability of the moving "bubbles" in a Bose condensate*, in Proc.Workshop on Solitons and Applications, V. G. Makhankov, V. K. Fedyanin, O. K. Pashaev, eds., World Scientific, Dubna(1990),281-297

[4] ______ and V. G. Makhankov, *Soliton-like "bubbles" in a system of interacting Bosons*, Phys.Lett.A.**128**(1988),52-56 .

[5] ______, T. L. Boyadjiev, I. V. Puzynin and T. Zhanlav, *Stability of moving bubbles in a system of interacting bosons*, Phys.Lett.A. **135**(1989),125-128.

[6] I. V. Barashenkov, *Stability criterion for dark solitons*, Phys.Rev.Lett.,**77**(1996),1193-1197.

[7] H. Berestychi and P. L. Lions, *Nonlinear scalar field equations,* Arch. Rational Mech. Anal., 82(1983), 313-376.

[8] F. Bethuel and J. C.Saut, *Travelling waves for the Gross-Pitaevskii equation. I*, Ann. Inst. H. Poincar Phys. Thor. **70**(1999), no.2,147–238.

[9] J. L. Bona, P. E. Souganidis and W. A. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc.R.Soc.Lond.A **411**(1987),395-412.

[10] Anne de Bouard, *Stability and instability of some nonlinear diapersive solitary waves in higher dimension*, Proc.R.Soc.Edinburgh **126A**(1996),89-112.

[11] ______, *Instability of stationary bubbles*, Siam.J.Math.Anal. Vol**26**(1995), 566-582.

[12] ______ and J.C.Saut, *Remarks on the stability of generalized KP solitary waves*, in Mathematical Problems in the Theory of water waves, AMS contemporary Mathematics, Vol **200**(1996),75-84.

[13] M. Grillakis, J. Shatah and W. A. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J.Funct.Anal.,**74**(1987),160-197.

[14] Guo BoLing and Wu YaPing, *Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation*, J.Diff.Equations **123**(1995),35-55

[15] C. A. Jones and P. H. Roberts, *Motions in a Bose condensate IV:Axisymmetric solitary waves*, J.Phy.A.Math.Gen. **15**(1982),2599-2569.

[16] ______ and S. J. Putterman and P. H. Roberts, *Motions in a Bose condensate V: Stability of solitary wave solutions of nonlinear Schrödinger equations in two and three dimensions*, J.Phy.A.Math.Gen. **19**(1986), 2591-3011.

[17] E. A. Kuznetsov and J. J. Rasmussen, *Self-focusing instability of two-dimensional solitons and vortices*, Phys.Rev.E,**51,5**(1995),4479-4484.

[18] Zhiwu Lin, *Slow travelling bubbles in two- and three-dimension*, preprint (1999).

[19] P. E. Souganidis and W. A. Strauss, *Instability of a class of dispersive solitary waves*, Proc.R.Soc.Edinburgh **114A**(1990),195-212.

[20] P. E. Zhidkov, *Some aspects of qualitative theory of nonlinear Korteweg-de Vries and Schrödinger equations*, preprint .

[21] ———, *Existence of solutions to the cauchy problem and stability of kink-solutions of the nonlinear Schrödinger equation*, Siberian.Math.J **33**(2)(1992), 239-246.

Department of Mathematics, Brown University Providence,RI 02912
E-mail address: linzw@math.brown.edu