UNSTABLE MANIFOLDS OF EULER EQUATIONS

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Abstract. We consider a steady state v_0 of the Euler equation in a fixed bounded domain in $\mathbf{R}^n$. Suppose the linearized Euler equation has an exponential dichotomy of unstable and center-stable subspaces. By rewriting the Euler equation as an ODE on an infinite dimensional manifold of volume preserving maps in $W^{k,q}$, $(k > 1 + \frac{n}{q})$, the unstable (and stable) manifolds of v_0 are constructed under certain spectral gap condition which is verified for both 2D and 3D examples. In particular, when the unstable subspace is finite dimensional, this implies the nonlinear instability of v_0 in the sense that arbitrarily small $W^{k,q}$ perturbations can lead to L^2 growth of the nonlinear solutions.

1. Introduction

We consider the incompressible Euler equation on a smooth bounded domain $\Omega \subset\subset \mathbb{R}^n$, $n \geq 2$, under the slip (or periodic in certain directions) boundary condition

$$\begin{cases} v_t + (v\cdot\nabla)v = -\nabla p \quad \text{and} \quad \nabla\cdot v = 0 & x\in\Omega \\ v\cdot N = 0, & x\in\partial\Omega \end{cases} \tag{E}$$

where $v = (v^1, \ldots, v^n)^T$ is the velocity field and N is the unit outward normal vector of Ω. We take

$$W^{k,q}_{Euler} \triangleq \{v \in W^{k,q}(\Omega, \mathbb{R}^n) \mid \nabla\cdot v = 0 \text{ in } \Omega,\ v\cdot N = 0 \text{ on } \partial\Omega\}, \qquad q > 1, \quad k > 1 + \frac{n}{q} \tag{1.1}$$

as the phase space. It is well known that (E) is well posed in these spaces, globally if $n = 2$ and locally if $n \geq 3$. As shown below, the pressure p can be written in terms of v through a quadratic mapping.

Let v_0 be a steady solution of (E). Linearize the equation at (E) and we obtain

$$v_t = -(v_0\cdot\nabla)v - (v\cdot\nabla)v_0 - \nabla p \triangleq Lv \tag{1.2}$$

where the operator L can be defined as acting only on v since the linearized pressure p can be determined by v linearly, though non-locally. To study the dynamics near v_0, the first step is to understand linear instability, that is, the spectrum of the operator L. The problem of linear instability of inviscid flows has a long history dated back to Rayleigh and Kelvin in 19th century. But even until now, very few sufficient conditions for the existence of unstable eigenvalues are known and most of the investigations had been restricted to shear flows and rotating flows. See [DR81],[DH66] and the references therein. Some recent results on instability conditions can be found in [L03] [L05] for shear flows and rotating flows, and in [L04a] for general 2D flows. Besides the discrete unstable spectrum, the linearized Euler operator may also have non-empty unstable essential spectrum due to nontrivial Lyapunov exponents of the steady flow v_0 ([FV91] [LM91] [SL09] [LS03] [V96]). Indeed, growth of

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linearized solutions can be seen in H^s $(s>1)$ norm near any nontrivial steady flows, due to the stretching of the steady fluid trajectory. One also notes that the choice of the Sobolev space (norm) actually affects the essential spectrum which corresponds to small spatial scales, but not discrete spectrum corresponding to large scales.

Consider a linearly unstable steady flow v_0, that is, the linearized Euler operator L has an unstable discrete eigenvalue. To discuss the nonlinear instability, it is important to specify the norms. On the one hand, certain regularity is necessary in the local well-posedness of classical solutions. On the other hand, as mentioned in the above, the choice of the norm already affects the essential spectrum at the linear level. Moreover, even near steady states without unstable eigenvalues, solutions are expected to grow in H^s norm with $s>1$. Therefore the growth in the energy norm L^2 of nonlinear solutions is more a nonlinear reflection of the linear instability from the discrete spectrum, which also corresponds to the instability in the large scale spatial scale (see [L04b, Section 6.2] for detailed discussions). Naturally, the ideal nonlinear instability result would be to obtain order $O(1)$ growth in L^2 distance (weaker energy norm) from the steady state v_0 of solutions starting with arbitrary small initial perturbation from v_0 in H^s norm (stronger norm), where $s>0$ is determined by the regularity of unstable eigenfunctions. Such nonlinear instability result (i.e. $H^s \to L^2$) is not only mathematically stronger, but also physically interesting due to the discussions in the above.

The proof of the nonlinear instability based on unstable eigenvalues is nontrivial for several reasons. The main difficulty is that the nonlinear term $v \cdot \nabla v$ contains a loss of derivative. Moreover, the norm-dependent unstable essential spectrum corresponds to growth in small spatial scales. It may interact with discrete unstable modes and then cause complications in proving nonlinear instability. In the last decade, there appeared several proofs of nonlinear instability for Euler equations ([BGS02] [Gre00] [VF03] [FSV97] [L04b]). In [FSV97], nonlinear instability in H^s $\left(s>1+\frac{n}{2}\right)$ norm was proven for $n-$dimensional Euler equations, under a spectral gap condition which was verified for 2D shear flows. All other papers proved nonlinear instability only for the 2D case, in the more desirable H^1 or L^2 norms and under different spectral assumptions. Particularly, in [L04b] nonlinear instability in L^2 norm was proved for general linearly unstable flows of 2D Euler, without additional assumption on the growth rate which was made in ([BGS02] [Gre00] [VF03]).

When there exist a collection σ_u of unstable eigenvalues of the linearized Euler operator at the steady flow v_0 with strictly larger real parts than the rest of the spectrum, it is very natural from a dynamical system point of view to ask if a locally invariant unstable manifold tangent to the eigenspace of σ_u exists. The answer to this question would provide a better picture of the nonlinear instability, including the dimensions and the directions of the unstable solutions with certain minimal growth rate. More importantly, such locally invariant manifolds provide more precise characterization of the local dynamical pictures near an unstable equilibria, and are also basic tools for constructing globally invariant structures such as heteroclinic and homoclinic orbits. These dynamical structures are important in understanding the turbulent fluid behaviors. The major obstacle to the construction of local invariant manifolds is again the loss of derivative due to the derivative nonlinearity $v \cdot \nabla v$. For dissipative models such as reaction-diffusion equations ([H81]) and Navier-Stokes equations ([Yu89] [Li05]), it is rather standard to construct invariant manifolds since the dissipation terms provide strong smoothing effect to overcome the loss of derivatives in the nonlinear terms. However, for non-dissipative continuum models including Euler equations, the linearized operators have no smoothing effect to help overcome the loss of derivative in the nonlinear terms. So it has been largely open to construct invariant manifolds for conservative continuum models such as Euler equations. In the proof of nonlinear instability for 2D Euler, one takes initial data perturbed along the direction of unstable eigenfunctions and then uses special properties of nonlinear solutions of 2D Euler to overcome the loss of derivatives, such as the bootstrap arguments in ([BGS02] [L04b] [VF03]) and the nonlinear energy estimates in ([Gre00]). However, these techniques can not be used for constructing invariant manifolds, since we do not know beforehand the initial conditions for solutions on unstable or stable manifolds.

In this paper, we obtain the first result on stable and unstable manifolds of Euler equations in any dimensions. To state the precise results, we formulate the following assumptions:

(A1) $v_0 \in W^{k+r,q}_{Euler}$, where $r \geq 4$.

(A2) $\exists\ \lambda_u > \lambda_{cs} > 0$, and closed subspaces X_u and X_{cs} of $W^{k,q}_{Euler}$ such that they satisfy $L\big(X_{u,cs} \cap domain(L)\big) \subset X_{u,cs}$ respectively and $W^{k,q}_{Euler} = X_u \oplus X_{cs}$. Moreover, let $L_u = L|_{X_u}$ and $L_{cs} = L|_{X_{cs}}$, then for some $M > 0$, they satisfy

$$|e^{tL_{cs}}| \leq Me^{\lambda_{cs} t}, \quad \forall\, t \geq 0 \quad \text{and} \quad |e^{tL_u}| \leq Me^{\lambda_u t}, \quad \forall\, t \leq 0.$$

(A3) The largest Lyapunov exponent μ_0 (in both forward and back time) of the linearized equations

$$y_t = Dv_0\big(x(t)\big)y \qquad y_t = -\Big(Dv_0\big(x(t)\big)\Big)^* y,$$

along any integral curve $x(t)$ of $x_t = v_0(x)$, satisfies

$$\lambda_u - \lambda_{cs} > K_0 \mu_0$$

where K_0 is a constant depending only on r and k.

Now we give our main theorem

Theorem 1.1. *Under above assumptions (A1)-(A3), there exists a unique $C^{r-3,1}$ local unstable manifold W^u of v_0 in $W^{k,q}_{Euler}$ which satisfies*

1. *It is tangent to X_u at v_0.*
2. *It can be written as the graph of a $C^{r-3,1}$ mapping from a neighborhood of v_0 in X_u to X_{cs}.*
3. *It is locally invariant under the flow of the Euler equation (E), i.e. solutions starting on W^u can only leave W^u through its boundary.*
4. *Solutions starting on W^u converges to v_0 at the rate $e^{\lambda t}$ as $t \to -\infty$ for any $\lambda < \lambda_u - K_0\mu_0$.*

The same results hold for local stable manifold of v_0 as the Euler equation (E) is time-reversible.

Remark. *In Sections 3 and 4, the assumptions (A2)-(A3) are verified for linearly unstable 2D shear flows and rotating flows, as well as 3D shear flows. For these flows, $\mu_0 = 0$ and thus (A3) is automatically satisfied if an unstable eigenvalue exists.*

Remark. *Suppose $v_0 \in W^{k_1+r,q_1} \cap W^{k_2+r,q_2}$ and the invariant decompositions $W^{k_i,q_i} = X^i_u \oplus X^i_{cs}$ along with the same exponents $\lambda_{u,cs}$ satisfy (A2) – (A3) for $i = 1, 2$. One may construct the local unstable manifolds $W^u_i \subset W^{k_i,q_i}$, $i = 1, 2$ from Theorem 1.1. Assume $X^1_u = X^2_u$, we claim $W^u_1 = W^u_2$ on an open neighborhood of v_0. In fact, let $k = \max\{k_1, k_2\}$ and*

$q = \max\{q_1, q_2\}$ and W^u be the unstable manifold of v_0 in $W^{k,q} \subset W^{k_i,q_i}$, $i = 1, 2$. Clearly $W^u \subset W_i^u$, $i = 1, 2$, with the same tangent spaces and thus the above claim follows.

When X_u is finite dimensional, which in particular is always true in 2D under assumption the $\lambda_u > \lambda_{cs} > (k-1)\,\mu_0$ as proved in Section 3, then the $W^{k,q}$ topology and L^2 topology are equivalent on W^u. Then an immediate consequence of the above theorem is the nonlinear instability in L^2 norm with initial data slightly perturbed from v_0 in $W^{k,q}$ norm.

Corollary 1. *Suppose (A1) –(A3) are satisfied and X_u is finite dimensional, then there exists $\delta > 0$ such that there exist a solution $v(t)$ such that $|v(0) - v_0|_{L^2} \geq \delta$ and $|v(t) - v_0|_{W^{k,q}} \to 0$ as $t \to -\infty$ exponentially.*

This is a stronger statement than the usual exponential nonlinear instability which by definition means that there exists $\delta > 0$ such that for any $\epsilon > 0$, there exists a solution $v(t)$ satisfying both $|v(0) - v_0| < \epsilon$ and $\sup_{0 \leq t \leq O(-\log \epsilon)} |v(t) - v_0| \geq \delta$. One notes that the perturbed solution $v(t)$ is allowed to depend on ϵ and no condition is imposed on the asymptotic behavior of $v(t)$ as $t \to -\infty$. In the contrast, any solution $v(t)$ in Corollary 1 satisfies, in addition to the requirements in the nonlinear instability definition for all $\epsilon > 0$, that it starts at v_0 when $t = -\infty$ and get out of the δ-neighborhood of v_0 in L^2 norm.

In the previous works (see references above) on nonlinear instability of the Euler equation, growing solutions have usually been found in the most unstable direction of the linearized equation (1.2) with roughly the maximal exponential growth rate. The above unstable manifold theorem actually provides solutions growing in other relatively weaker unstable directions. Though not necessary, it is easier to see this when X_u is finite dimensional. In fact, on the finite dimensional locally invariant manifold W^u, the Euler equation (E) becomes a smooth ODE and v_0 is a hyperbolic unstable node. When $L|_{X_u}$ has eigenvalues with different real parts, one may split X_u into strongly unstable subspace X_{uu} and weaker unstable subspace X_{wu} such that $X_u = X_{uu} \oplus X_{wu}$. The standard invariant manifold theory implies there exist the locally invariant weakly unstable manifold W^{wu} tangent to X_{wu} and the locally invariant strongly unstable fibers W_v^{su} with base point $v \in W^{wu}$ and extend in the direction of X_{uu}. Those solutions in W^{wu} grow in the directions of X_{wu} at a slower exponential rate. Moreover, the Hartman-Grobman theorem implies that a Hölder homeomorphism on X^u may transform the Euler equation restricted on W^u into a linear ODE system.

In Section 4, we construct linearly unstable 3D steady flow satisfying the assumptions in Theorem 1.1. By Corollary 1, this implies nonlinear exponential instability in L^2 norm. To our knowledge, this is the first proof of nonlinear instability of 3D Euler equation. We note that the methods for proving nonlinear instability of 2D Euler cannot be applied to prove nonlinear instability for 3D Euler. For example, the bootstrap arguments in ([BGS02] [L04b] [VF03]) strongly use the fact that vorticity is non-streching in 2D and therefore do not work in 3D due to the vorticity stretching effect.

Below, we sketch the main ideas in the proof of Theorem 1.1. The main difficulty in constructing the unstable manifolds for the Euler equation lies in the fact that a derivative loss occurs in the nonlinear terms while the linearized flow does not have the smoothing property. We will prove Theorem 1.1 mainly by considering the Euler equation (E) in the Lagrangian coordinates. In a seminal paper [Ar66], V. Arnold pointed out that the incompressible Euler equation can be viewed as the geodesic equation on the group of volume preserving diffeomorphisms. This point of view has been adopted and developed by several authors in their work on the Euler equations, such as [EM70, Sh85, Br99, SZ08a, SZ08b] to mention a few.

On the one hand, the main advantage of this approach is that (E) on a fixed domain as in this paper becomes a smooth infinite dimensional ODE [EM70] on the tangent bundle of the Lie group $\mathcal{G}$ of volume preserving diffeomorphisms of Ω and thus the difficulty arising from the loss of regularity disappears. A side remark is that this is in contrast with the Euler equation with free boundaries [SZ08a, SZ08b] where the Riemannian curvature of the infinite dimensional manifolds of volume preserving manifolds are unbounded operators and the Euler equation can not be considered as infinite dimensional ODEs. As a clarification, by saying that the Euler equation on fixed domains defines an ODE, we mean that it corresponds to a vector field which is everywhere defined and smooth on an infinite dimensional manifold. In this sense, evolutionary PDEs involving unbounded operators such as heat equation or wave equation do not define infinite dimensional ODEs.

On the other hand, in the Lagrangian coordinates, the steady state v_0 generates a special geodesic $u_0(t)$ which coincides with the integral curve starting at the identity map of the right invariant vector field on $\mathcal{G}$ generated by v_0. By carefully using local coordinates along this orbit generated by the group symmetry, we further transform the localized Euler equation into a weakly nonlinear non-autonomous ODE with linear exponential dichotomy. One notes that the multiplication on this Lie group $\mathcal{G}$ – the composition between $W^{k,q}$ volume preserving maps – is continuous, but not smooth, see Proposition 2.2. The assumption $v_0 \in W^{k+r,q}$ ensures that the localization we choose possesses certain smoothness. Moreover, compared to the usual exponential dichotomy enjoyed by many ODEs and even PDEs (see for example [CL88]), the exponential dichotomy here has the defect that the angle between the associated invariant subspaces may not have a uniform positive lower bound in time due to the possible growth of the linearized ODE flow of the vector field v_0. Moreover, this type of unwanted growth in t may also appear in the norms of the nonlinearities. Assumption (A3) is used to overcome this non-uniformity in t of the dichotomy in the construction of the unstable manifolds via the method based on the Lyapunov-Perron integral equations. Here we have to take the advantage of the fact that solutions on the unstable manifolds decay to v_0 exponentially as $t \to -\infty$. Since solutions on the center manifolds do not have similar decay properties, we still can not construct center manifolds of stead states.

Remark. *The assumption (A2) is the standard linear exponential dichotomy for constructing invariant manifolds. The extra gap assumption (A3) might be technical, but appears rather natural in our approach. In the proof, we change back and forth between Lagrangian and Eulerian coordinates, and each transformation induces a factor $e^{\mu_0 t}$ in the estimates. The extra gap $\lambda_u - \lambda_{cs} > K_0\mu_0$ guarantees that after all these transformations, the exponential dichotomy of unstable and central-stable parts still persists.*

The method introduced in this paper provides a general approach to construct unstable manifolds for many other continuum models in fluid and plasmas. In these models, the loss of derivative is also due to nonlinear terms from the material derivative. By working on Lagrangian coordinates, we can again overcome such a loss of derivative and the existence of unstable manifolds is conceivable with sufficient spectral gap. We are using this approach to construct unstable manifolds for density-dependent Euler equations and the Vlasov-Poisson system for collisionless plasmas.

2. Proof of Theorem 1.1

Lagrangian coordinates is a standard tool in studying the Euler equation. In Subsection 2.1 and 2.2, we will present the manifold structure of the set $\mathcal{G}$ of the Lagrangian maps and the ODE nature of (E) on $T\mathcal{G}$. These general results have actually been proved even for Euler equations defined on Riemannian manifolds in [EM70] in a rather geometric language. However, we need to establish a more concrete framework along with more detailed estimates to be used in the construction of local invariant manifolds in Subsections 2.3 – 2.6, which is done in a more directly equation based manner in Subsections 2.1 and 2.2. In Subsection 2.3, we rewrite in the Lagrangian coordinates the localized Euler equation in a neighborhood of the solution curve generated by v_0 and in Subsection 2.4, the linear exponential dichotomy is given. The unstable integral manifold corresponding the linear exponential dichotomy is constructed in Subsection 2.5 and then finally the unstable manifold in the Eulerian coordinates is obtained in Subsection 2.6.

Throughout this section, we will use $K > 0$ as a generic constant depending only on r and k and $C > 0$ only on n, r, k, q, v_0. Both K and C may change from line to line. We will use D or ∇ to denote the differentiation with respect to physical variables in Ω and $\mathcal{D}$ the Fréchet differentiations in function spaces.

2.1. **Lagrangian coordinates and the Lie group of volume preserving maps.** Let $u(t, \cdot) : \Omega \to \Omega$ be the Lagrangian coordinate map defined by

$$u(0, y) = y \quad \text{ and } u_t(t, y) = v(t, u(t, y)). \tag{2.1}$$

In particular, let $u_0(t, y)$ be the Lagrangian map of the steady vector field $v_0(x)$. Throughout the paper, we fix a constant $\mu > \mu_0 \geq 0$ such that

$$\lambda_u - \lambda_{cs} > K_0 \mu \tag{2.2}$$

where μ_0 is the Lyapunov exponent of v_0. From the definition of the Lyapunov exponent, we have

$$|u_0(t, \cdot)|_{C^l} + |\big(u_0(t, \cdot)\big)^{-1}|_{C^l} \leq C e^{l\mu|t|}, \quad t \in \mathbb{R},\ 0 \leq l \leq k + r \tag{2.3}$$

for some $C > 0$ independent of t. This possible exponential growth of the norm of u_0 makes the problem much more subtle than the unusual constructions of the local invariant manifolds in differential equations.

Since the flow is incompressible and $k > 1 + \frac{n}{q}$, we have for any $t \in \mathbb{R}$,

$$u(t, \cdot) \in \mathcal{G} \triangleq \{\phi \in W^{k,q}(\Omega, \mathbb{R}^n) \mid \phi \text{ is a diffeomorphism, } \det(D\phi) \equiv 1, \phi(\partial\Omega) = \partial\Omega\}. \tag{2.4}$$

Clearly the composition makes $\mathcal{G}$ a group. We will show that $\mathcal{G}$ is an infinite dimensional submanifold of $W^{k,q}(\Omega, \mathbb{R}^n)$. This will be our configuration space when the Euler equation is written in the Lagrangian coordinates. We will work with local coordinates on $\mathcal{G}$.

Formally, the tangent space of $\mathcal{G}$ is given by

$$T_{id}\mathcal{G} = W^{k,q}_{Euler}, \qquad T_\phi \mathcal{G} = \{w \mid w \circ \phi^{-1} \in W^{k,q}_{Euler}\} \quad \forall \phi \in \mathcal{G}, \tag{2.5}$$

where $W^{k,q}_{Euler}$ defined in (1.1) is the phase space of the velocity fields of (E). From the Hodge decomposition, a complementary space of $W^{k,q}_{Euler}$ in $W^{k,q}(\Omega, \mathbb{R}^n)$ is given by

$$(W^{k,q}_{Euler})^\perp = \{\nabla h \mid h \in W^{k+1,q}(\Omega, \mathbb{R})\}. \tag{2.6}$$

Here the orthogonality is in the sense that

$$\int_\Omega w \cdot \nabla h dx = 0, \qquad \forall w \in W^{k,q}_{Euler},\ h \in W^{k+1,q}(\Omega, \mathbb{R}).$$

It is clear that both $W^{k,q}_{Euler}$ and $(W^{k,q}_{Euler})^\perp$ are closed subspaces of $W^{k,q}(\Omega, \mathbb{R}^n)$ and

$$W^{k,q}(\Omega, \mathbb{R}^n) = W^{k,q}_{Euler} \oplus (W^{k,q}_{Euler})^\perp.$$

In fact, given any $X \in W^{k,q}(\Omega, \mathbb{R}^n)$, let

$$w = X - \nabla h \tag{2.7}$$

where h is the solution of

$$\Delta h = \nabla \cdot X \quad \text{in } \Omega \qquad \nabla_N h = X \cdot N \quad \text{on } \partial\Omega, \tag{2.8}$$

then obviously $X = w + \nabla h$ with $w \in W^{k,q}_{Euler}$ and this verifies the direct sum. Locally near the identity map id, we will write $\mathcal{G}$ as the graph of a smooth mapping from $W^{k,q}_{Euler}$ to $(W^{k,q}_{Euler})^\perp$ and thus $\mathcal{G}$ is rigorously a smooth manifold. Let $B_\delta(\cdot)$ denote the ball of radius δ centered at 0 in the corresponding Banach space.

Proposition 2.1. *There exists $\delta_0 > 0$ and a smooth mapping $\Psi : B_{\delta_0}(W^{k,q}_{Euler}) \to W^{k,q}(\Omega, \mathbb{R}^n)$, such that $\Psi(0) = id$, $\mathcal{D}\Psi(0) = I$, $\Psi(w) - w \in (W^{k,q}_{Euler})^\perp$ for any $w \in B_{\delta_0}(W^{k,q}_{Euler})$, and*

$$\Big(id + B_{\frac{\delta_0}{2}}\big(W^{k,q}(\Omega, \mathbb{R}^n)\big)\Big) \cap \mathcal{G} \subset \{\Psi(w) \mid w \in B_{\delta_0}(W^{k,q}_{Euler})\} \subset \Big(id + B_{2\delta_0}\big(W^{k,q}(\Omega, \mathbb{R}^n)\big)\Big) \cap \mathcal{G}.$$

Proof. Since $\partial\Omega$ is a smooth compact hypersurface in $\mathbb{R}^n$, the distance function to $\partial\Omega$ is smooth in a neighborhood of $\partial\Omega$. Let $d : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support such that it coincides with this distance function in a neighborhood of $\partial\Omega$. Consider the mapping

$$G:\ W^{k,q}(\Omega, \mathbb{R}^n) \to Y \triangleq \{(f, g) \in W^{k-1,q}(\Omega, \mathbb{R}) \times W^{k-\frac{1}{q},q}(\partial\Omega, \mathbb{R}) \mid \int_{\partial\Omega} g\, dS = 0\}$$

defined as

$$G(\phi) = \big(\det(D\phi),\ (d \circ \phi)|_{\partial\Omega} - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} d \circ \phi\, dS\big)$$

where $|\partial\Omega|$ denotes the area of $\partial\Omega$. Obviously, G is a smooth mapping and $G|_{\mathcal{G}} \equiv (1, 0)$. Moreover, suppose $\phi \in U$ and $G(\phi) = (1, 0)$ where U is a neighborhood of the identity map id in $W^{k,q}(\Omega, \mathbb{R}^n)$. Then ϕ is a diffeomorphism from Ω to its image and

$$|\phi(\Omega)| = |\Omega| \quad \text{ and } \quad d \circ \phi = a \triangleq \frac{1}{|\partial\Omega|}\int_{\partial\Omega} d \circ \phi\, dS$$

imply that $d \circ \phi \equiv a = 0$. Otherwise $a \neq 0$ would imply that $\phi(\Omega)$ either strictly covers Ω or is strictly contained in Ω, either of which contradicts with the first identity above. Therefore

$$U \cap \mathcal{G} = U \cap G^{-1}\{(1, 0)\}.$$

It is easy to compute

$$\mathcal{D}G(id)X = (\nabla \cdot X,\ X \cdot N - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} X \cdot N\, dS) \in Y, \quad \forall\, X \in W^{k,q}(\Omega, \mathbb{R}^n).$$

From the standard theory of elliptic problems with Neumann boundary conditions,

$$DG(id)\nabla h = (\Delta h,\, \nabla_N h - \frac{1}{|\partial\Omega|}\int_\Omega \Delta h\, dx)$$

is an isomorphism from $(W^{k,q}_{Euler})^\perp$ to Y. Therefore the proposition follows from the Implicit Function Theorem. $\square$

Near any $\phi_0 \in \mathcal{G}$, the local coordinate map Ψ composed with the right translation, $\Psi(\cdot)\circ\phi_0$, gives a smooth local coordinate map near ϕ_0. Therefore $\mathcal{G}$ is a Banach submanifold with the model space $W^{k,q}_{Euler}$. It is well-known [EM70] that $\mathcal{G}$ is not such a standard Lie group as one needs to be careful with the smoothness of the left translations. Let $\mathcal{C}(\phi_1,\phi_2) = \phi_1\circ\phi_2$ and it is straightforward to verify

Proposition 2.2. *$\mathcal{C} \in C^l\Big(\big(\mathcal{G}\cap W^{k+l,q}(\Omega,\mathbb{R}^n)\big)\times\mathcal{G},\mathcal{G}\Big)$ and, for any $\phi_2\in\mathcal{G}$, the right translation $\mathcal{C}(\cdot,\phi_2)\in C^\infty(\mathcal{G})$.*

2.2. **Euler equation as an ODE on $T\mathcal{G}$.** It is well-known that the pressure p can be represented in terms of the velocity field v. In fact, by taking the inner product of v_t and N on $\partial\Omega$ and the divergence of the v_t and using $\nabla\cdot v = 0$ in Ω and $N\cdot V = 0$ on $\partial\Omega$, we obtain

$$\begin{cases} -\Delta p = \Sigma_{i,j=1}^n \partial_i(v^j\partial_j v^i) = \Sigma_{i,j=1}^n \partial_i v^j\partial_j v^i = \mathrm{tr}(Dv)^2 & \text{in } \Omega \\ \nabla_N p = -N\cdot(v\cdot\nabla)v = -\nabla_v(v\cdot N) + \nabla_v N\cdot v = v\cdot\Pi(v) & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

where the symmetric operator $\Pi\in C^\infty\big(\Omega, L(T\partial\Omega)\big)$ is the second fundamental form of $\partial\Omega$ defines as $\Pi(x)(\tau) = \nabla_{\tau^\top}N$ with $\tau^\top\in T_x\partial\Omega$ being the tangential component of τ.

Based on the form of the pressure, we define the symmetric bounded bilinear mapping $B : (W^{k,q}(\Omega,\mathbb{R}^n))^2 \to W^{k,q}(\Omega,\mathbb{R}^n)$ as

$$\mathcal{B}(X_1,X_2) = \nabla\gamma$$

where

$$\begin{cases} -\Delta\gamma = \mathrm{tr}(DX_1DX_2) = \Sigma_{i,j=1}^n \partial_i X_1^j\partial_j X_2^i & \text{in } \Omega \\ \nabla_N\gamma = X_1\cdot\Pi(X_2) & \text{on } \partial\Omega. \end{cases}$$

The boundedness of $\mathcal{B}$ is clear from the standard elliptic theory. Note that here we do not assume $\nabla\cdot X_{1,2} = 0$ in Ω or $N\cdot X_{1,2}$ on $\partial\Omega$ in the definition of $\mathcal{B}$. According to the Hodge decomposition given through (2.8) and a similar calculation as in (2.9), it holds that

$$DX_1(X_2) + \mathcal{B}(X_1,X_2) = (X_2\cdot\nabla)X_1 + \mathcal{B}(X_1,X_2) \in W^{k,q}_{Euler}, \qquad \forall X_1,\, X_2\in W^{k,q}_{Euler}. \tag{2.10}$$

As a side remark, it indicates that, when embedded in $L^2(\Omega,\mathbb{R}^n)$, $\mathcal{B}$ is the second fundamental form of $\mathcal{G}$ at id which can be rigorously verified through a standard procedure.

Define the mapping $\mathcal{P} : \mathcal{G}\times\big(W^{k,q}(\Omega,\mathbb{R}^n)\big)^2 \to W^{k,q}(\Omega,\mathbb{R}^n)$ as

$$\mathcal{P}(\phi, X_1, X_2) = \mathcal{B}(X_1\circ\phi^{-1}, X_2\circ\phi^{-1})\circ\phi. \tag{2.11}$$

We also define the projection $Q : \mathcal{G}\times W^{k,q}(\Omega,\mathbb{R}^n) \to T\mathcal{G}$ as

$$Q(\phi, X) = (X\circ\phi^{-1} - \nabla h)\circ\phi \in T_\phi\mathcal{G} \tag{2.12}$$

where ∇h for $X\circ\phi^{-1}$ is defined in (2.7). Obviously, $\mathcal{P}$ is symmetrically bilinear in X_1 and X_2. As in (2.10), it holds

$$\Big(\big(D(X_1\circ\phi^{-1})\big)\circ\phi\Big)(X_2) + \mathcal{P}(\phi, X_1, X_2) \in T_\phi\mathcal{G} = \{w : \Omega\to\mathbb{R}^n \mid w\circ\phi^{-1}\in W^{k,q}_{Euler}\}. \tag{2.13}$$

In fact, (2.10) also leads to that, when embedded in L^2, $\mathcal{P}(\phi, \cdot, \cdot)$ is the second fundamental form of $\mathcal{G}$ at ϕ. Euler equation (E) and (2.9) imply that the Euler equation (E) takes the form in the Lagrangian coordinates

$$u_{tt} + \mathcal{P}(u, u_t, u_t) = 0, \quad u(t) \in \mathcal{G}. \tag{2.14}$$

Moreover, for any $\phi_0 \in \mathcal{G}$, we have

$$\mathcal{P}(\phi \circ \phi_0, X_1 \circ \phi_0, X_2 \circ \phi_0) = \mathcal{P}(\phi, X_1, X_2) \circ \phi_0, \tag{2.15}$$

i.e. $\mathcal{P}$ is invariant under the right translation. Therefore,

$$u(t) \circ \phi_0 \text{ is also a solution for any solution } u(t) \text{ of (2.14)}. \tag{2.16}$$

The following proposition states that $\mathcal{P}$ is a smooth mapping and thus has no regularity loss which is far from trivial even though $\mathcal{B}$ is a bounded bilinear operator. To see this, one note that even the dependence of the term $X_1 \circ \phi \in W^{k,q}$ on $\phi \in W^{k,q}$ is not smooth unless X_1 belongs to a space of better regularity. The proof of the proposition is essentially a careful analysis of the commutator between the operations of $\mathcal{B}$ and the composition by $\phi \in \mathcal{G}$.

Proposition 2.3. *$\mathcal{P} : \mathcal{G} \times \big(W^{k,q}(\Omega, \mathbb{R}^n)\big)^2 \to W^{k,q}(\Omega, \mathbb{R}^n)$ and $Q : \mathcal{G} \to L\big(W^{k,q}(\Omega, \mathbb{R}^n)\big)$ are C^∞ and, for any $m > 0$, there exist $C_0, K > 0$ depending only on m and k such that*

$$|(\mathcal{D}_\phi)^m \mathcal{P}(\phi, X_1, X_2)|_{W^{k,q}} \le C_0 |D\phi|^K_{W^{k-1,q}} |X_1|_{W^{k,q}} |X_2|_{W^{k,q}} \tag{2.17}$$

$$|(\mathcal{D}_\phi)^m Q(\phi, X)|_{W^{k,q}} \le C_0 |D\phi|^K_{W^{k-1,q}} |X|_{W^{k,q}}. \tag{2.18}$$

Here $(\mathcal{D}_\phi)^m \mathcal{P}$ should be considered as a multilinear operator from $T_\phi \mathcal{G}$ to $W^{k,q}(\Omega)$. Since $\mathcal{P}$ is bilinear in $X_{1,2}$, the bounds on the derivatives of $\mathcal{P}$ with respect to $X_{1,2}$ follow from (2.17) for $m = 0$. Also, we note that $|D\phi|_{W^{k-1,q}} \ge 1$ since $k - 1 > \frac{n}{q}$ and $det D\phi \equiv 1$.

Proof. We will present only the proof for $\mathcal{P}$ as the one for Q follows through almost exactly the same (or even slightly simpler) procedure. Due to the invariance (2.15) of $\mathcal{P}$ under the right translation, we only need to show its smoothness near $\phi = id$. As ϕ belongs to the manifold $\mathcal{G}$, the smoothness of $\mathcal{P}$ is equivalent to the smoothness of $\mathcal{P}\big(\Phi(w), X_1, X_2\big)$ with respect to $w \in W^{k,q}_{Euler}$ and $X_{1,2} \in W^{k,q}(\Omega, \mathbb{R}^n)$ for any smooth local coordinate map $\Phi : B_\delta(W^{k,q}_{Euler}) \to \mathcal{G}$. Moreover, to prove the Fréchet smoothness of $\mathcal{P}$, it suffices to show the Gâteaux differentiability of $\mathcal{P}$ up to the m-th order for any $m > 0$, which would imply that Gâteaux derivative $\mathcal{D}^{m-1}\mathcal{P}$ is continuous and thus it is also the $(m-1)$-th Fréchet derivative. To show the Gâteaux differentiability up to the m-th order, it suffices to prove the smoothness of

$$\mathcal{P}\big(\phi(s_1, \ldots, s_m), X_1(s_1, \ldots, s_m), X_2(s_1, \ldots, s_m)\big) \in W^{k,q}(\Omega, \mathbb{R}^n)$$

for any $\phi(s_1, \ldots, s_m) \in \mathcal{G} \subset W^{k,q}(\Omega, \mathbb{R}^n)$ and $X_{1,2}(s_1, \ldots, s_m) \in W^{k,q}(\Omega, \mathbb{R}^n)$ with smooth parameters $(s_1, \ldots, s_m) \in U \subset \mathbb{R}^m$. We will show by induction

$$Z \triangleq \partial_{s_1} \ldots \partial_{s_m} \mathcal{P}(\phi, X_1, X_2)|_{s_1 = \ldots = s_m = 0} \in W^{k,q}(\Omega, \mathbb{R}^n) \tag{ML}$$

and obtain its bound in the form of (2.17).

The boundedness of the bilinear transformation $\mathcal{B}$ implies (2.17) and (ML) for $m = 0$. Assume (ML) and (2.17) hold for $0 \le m < m_0$ and we will prove it for $m = m_0$. Let $\mathcal{P}(\phi, X_1, X_2) = (\nabla \gamma) \circ \phi$, where γ, defined as in the definition of $\mathcal{B}$, depends on $s_1, \ldots, s_{m_0}$

through ϕ, X_1, and X_2. In the following, since it will be much easier to carry out some calculations in the Eulerian coordinates, let

$$\tilde{X}_j = X_j \circ \phi^{-1}, \quad j = 1, 2; \qquad \tau_i = (\partial_{s_i}\phi) \circ \phi^{-1}, \quad \boldsymbol{D}_{s_i} = \partial_{s_i} + \nabla_{\tau_i}, \qquad i = 1, \dots, m_0$$

and

$$\tilde{Z} = \boldsymbol{D}_{s_1} \dots \boldsymbol{D}_{s_{m_0}} \nabla\gamma = Z \circ \phi^{-1}.$$

Clearly, it is sufficient to show $\tilde{Z} \in W^{k,q}$, which will be achieved by studying its normal component on $\partial\Omega$, divergence, and curl.

To start, for any vector field $W \in W^{k-1+m_0,q}(\Omega, \mathbb{R}^n)$, let $Y(y) = \big(D\phi(y)\big)^{-1} W\big(\phi(y)\big)$, or equivalently $D\phi(Y) = W \circ \phi$, then

$$D\phi(\partial_{s_i} Y) = -D\phi_{s_i}(Y) + \big((DW) \circ \phi\big)\phi_{s_i} \implies \partial_{s_i} Y \in W^{k-1,q}(\Omega, \mathbb{R}^n).$$

Differentiating the above identity one more time implies

$$D\phi(\partial_{s_j s_i} Y) = -D\phi_{s_j}(\partial_{s_i} Y) - D\phi_{s_i}(\partial_{s_j} Y) - D\phi_{s_j s_i}(Y) + \big((DW)\circ\phi\big)\phi_{s_j s_i} + \big((D^2 W)\circ\phi\big)(\phi_{s_j}, \phi_{s_i})$$

and thus the smoothness of ϕ and W yield

$$\partial_{s_j s_i} Y \in W^{k-1,q}(\Omega, \mathbb{R}^n) \quad \text{ if } m_0 \geq 2.$$

Repeating this procedure we obtain $\partial_{s_{i_1} \dots s_{i_{m_0}}} Y \in W^{k-1,q}(\Omega, \mathbb{R}^n)$ inductively. Changing from the Eulerian coordinates to the Lagrangian coordinates, we have

$$\big(\nabla_W \tilde{Z} - \boldsymbol{D}_{s_1} \dots \boldsymbol{D}_{s_{m_0}} (D^2\gamma(W))\big) \circ \phi = \nabla_Y Z - \partial_{s_1} \dots \partial_{s_{m_0}} \nabla_Y \big((\nabla\gamma) \circ \phi\big).$$

Applying the induction assumption to the last term above and using the commutator formula $[\partial_{s_i}, \nabla_Y] = \nabla_{\partial_{s_i} Y}$ to move the ∇_Y to the outside to produce $\nabla_Y Z$, we obtain

$$\big(\nabla_W \tilde{Z} - \boldsymbol{D}_{s_1} \dots \boldsymbol{D}_{s_{m_0}} (D^2\gamma(W))\big) \circ \phi = \nabla_Y Z - \partial_{s_1} \dots \partial_{s_{m_0}} \nabla_Y \big((\nabla\gamma) \circ \phi\big) \in W^{k-1,q}. \tag{2.19}$$

Taking $W = e_1, \dots, e_n$ of the standard basis of $\mathbb{R}^n$, (2.19) and the definition of $\mathcal{P}$ imply the curl $\nabla \times \tilde{Z}$ contains only the commutators terms and thus satisfies

$$\nabla \times \tilde{Z} \in W^{k-1,q}.$$

Similarly the divergence satisfies

$$\nabla \cdot \tilde{Z} + \boldsymbol{D}_{s_1} \dots \boldsymbol{D}_{s_{m_0}} (\mathrm{tr}(D\tilde{X}_1)(D\tilde{X}_2)) \in W^{k-1,q}.$$

Expanding the above last term using the product rule on $\boldsymbol{D}_{s_i}$, it consists of terms in the form of

$$(\boldsymbol{D}_{s_{i_1}} \dots \boldsymbol{D}_{s_{i_m}} \partial_{l_1} \tilde{X}_1^{l_2})(\boldsymbol{D}_{s_{j_1}} \dots \boldsymbol{D}_{s_{j_{m_0-m}}} \partial_{l_2} \tilde{X}_2^{l_1}), \quad \{i_1, \dots, i_m, j_1, \dots j_{m_0-m}\} = \{1, \dots, m_0\}.$$

Move ∂_{l_1} and ∂_{l_2} to the outside in the same fashion as in the derivation of (2.19) (replacing W by $e_{l_{1,2}}$ and $\nabla\gamma$ by $\tilde{X}_{1,2}^{l_{1,2}}$) and using the smoothness of $X_{1,2}$ in $s_1, \dots, s_{m_0}$, it is easy to obtain $\boldsymbol{D}_{s_1} \dots \boldsymbol{D}_{s_{m_0}} (\mathrm{tr}(D\tilde{X}_1)(D\tilde{X}_2)) \in W^{k-1,q}$ and thus

$$\nabla \cdot \tilde{Z} \in W^{k-1,q}.$$

Finally, using

$$\boldsymbol{D}_{s_i} N = \Pi(\tau_i), \quad \boldsymbol{D}_{s_j} \boldsymbol{D}_{s_i} N = (\nabla_{\tau_j} \Pi)(\tau_i) + \Pi(\phi_{s_j s_i} \circ \phi^{-1}), \quad \dots$$

the induction assumption, and the assumption $N \cdot \nabla\gamma = 0$ on $\partial\Omega$ in the definition of $\mathcal{P}$ it is straight forward to obtain

$$\tilde{Z} \cdot N = N \cdot \boldsymbol{D}_{s_1} \ldots \boldsymbol{D}_{s_{m_0}} \nabla\gamma \in W^{k-\frac{1}{q},q}(\partial\Omega, \mathbb{R})$$

in the same fashion. Therefore, the standard estimates in elliptic theory implies that (ML) holds for $m = m_0$. Moreover, inequality (2.17) follows from the observation that the composition by ϕ or ϕ^{-1} only produces terms like $|\phi|^l_{W_{k,q}}$ in the estimates of the $W^{k,q}$ norms and thus the proof of the proposition is complete. □

Proposition 2.3 provides the key element for us to prove that (2.14) is a smooth second order ODE on the infinite dimensional configuration manifold $\mathcal{G}$.

Proposition 2.4. *For any $u_0 \in \mathcal{G}$ and $w_0 \in T_{u_0}\mathcal{G}$, the initial value problem of the Euler equation* (2.14) *has a unique solution $\big(u(t), u_t(t)\big) \in T\mathcal{G}$, locally in time, depending on (t, u_0, w_0) smoothly.*

Proof. Due to the right translation of (2.14) given in (2.16), we may assume u_0 belongs to a small neighborhood of id. From Proposition 2.1,

$$(w, \nabla h) \to \Psi(w) + \nabla h$$

is a local diffeomorphism from $W^{k,q}_{Euler} \times (W^{k,q}_{Euler})^\perp$ to $W^{k,q}(\Omega, \mathbb{R}^n)$. Taking the $\Psi(w)$ component, we obtain a smooth $\Phi : id + B_\delta\big(W^{k,q}(\Omega, \mathbb{R}^n)\big) \to \mathcal{G}$ such that

$$u - \Phi(u) \in (W^{k,q}_{Euler})^\perp, \ \forall u \in id + B_\delta\big(W^{k,q}(\Omega, \mathbb{R}^n)\big) \tag{2.20}$$

and

$$\Phi(u) = u, \ \forall\, u \in \mathcal{G} \cap \Big(id + B_\delta\big(W^{k,q}(\Omega, \mathbb{R}^n)\big)\Big).$$

Consider a modification of (2.14)

$$u_{tt} + \mathcal{P}\big(\Phi(u), u_t, \mathcal{D}\Phi(u)u_t\big) = 0, \quad u \in B_\delta\big(W^{k,q}(\Omega, \mathbb{R}^n)\big)\ u_t \in B_\delta\big(W^{k,q}(\Omega, \mathbb{R}^n)\big). \tag{2.21}$$

From the smoothness of Φ and $\mathcal{P}$, equation (2.21) is a smooth ODE defined on an open subset of an infinite dimensional Banach space $\big(W^{k,q}(\Omega, \mathbb{R}^n)\big)^2$ and thus is locally well-posed with smooth dependence on the initial value. Moreover, any solution $u(t)$ of (2.21) satisfying $u(t) \in \mathcal{G}$ for all t also solves (2.14). Therefore, to complete the proof, we only need to show that, if the initial data is given on $T\mathcal{G}$, then the solution of (2.21) also stays on $T\mathcal{G}$, i. e. $u(t) \in \mathcal{G}$ and $u_t(t) \in T_{u(t)}\mathcal{G}$. Let $v = u_t \circ \big(\Phi(u)\big)^{-1}$. Equation (2.21) yields

$$v_t \circ \Phi(u) + \big((Dv) \circ \Phi(u)\big)\big(\mathcal{D}\Phi(u)u_t\big) = u_{tt} = -\mathcal{P}\big(\Phi(u), u_t, \mathcal{D}\Phi(u)u_t\big)$$

and thus (2.13) implies $v_t \circ \Phi(u) \in T_{\Phi(u)}\mathcal{G}$ or equivalently $v_t \in T_{id}\mathcal{G} = W^{k,q}_{Euler}$. Therefore $v(t) \in W^{k,q}_{Euler}$, and thus $u_t = v \circ \Phi(u) \in T_{\Phi(u)}\mathcal{G}$, for all t follows from the initial assumption. Consequently

$$\big(u - \Phi(u)\big)_t = \big(I - \mathcal{D}\Phi(u)\big)u_t = 0$$

where we also used that (2.20) implies $\mathcal{D}\Phi(u)X = X$ for any $X \in T_{\Phi(u)}\mathcal{G}$. Therefore $u = \Phi(u) \in \mathcal{G}$ and the proof completes. □

According to (2.16), the second order ODE (2.14) defined on the Lie group $\mathcal{G}$ is invariant under the right translation. The standard procedure of taking $v = u_t \circ u^{-1}$ reduces it to a first order equation on the corresponding Lie algebra $W^{k,q}_{Euler} = T_{id}\mathcal{G}$, which turns out to be the usual form (E) of the Euler equation in the Eulerian coordinates. However, according to Proposition 2.2, the composition as the multiplication operation on the group $\mathcal{G}$ is not smooth, this procedure induces the loss of one order of spatial derivative and make (E) into a PDE, i. e. the right side of (E) does not define a smooth vector field on $W^{k,q}_{Euler}$.

2.3. **Euler equation near v_0 as a non-autonomous ODE on $T\mathcal{G}$.** Even though Euler equation is equivalent to an infinite dimensional ODE on $T\mathcal{G}$, the Lagrangian frame work also brings two complications:

- $\mathcal{G}$ is not flat, which means that we may have to carry out the analysis in local coordinates on $\mathcal{G}$ and
- the steady velocity field $v_0(x)$ of the Euler equation (E) corresponds to a dynamic solution $\Big(u_0(t,y), u_{0t}(t,y) = v_0\big(u(t,y)\big)\Big)$ of (2.14).

It is natural to look for ways to take the advantage of the group structure of $\mathcal{G}$ to reduce (2.14), localized near $u_0(t)$, to a non-autonomous second order ODE defined in local coordinate neighborhood of id. Based on the comments at the end of Subsection 2.2, taking $w = u\circ u_0^{-1}$ for $u(t)$ near $u_0(t)$ would result in loss of regularity, which can also be seen explicitly through the simple calculation $u_t = w_t\circ u_0 + (Dw\circ u_0)(v_0\circ u_0)$ leading to $w_t \notin W^{k,q}$ due to the presence of Dw. Instead, for any solution $\big(u(t), u_t(t) = v(t)\circ u(t)\big)\in T\mathcal{G}$ of the Euler equation (2.14) with $u(t)$ close to $u_0(t)$, let

$$u = \Phi(t,w) \triangleq u_0(t)\circ\Psi\big(w(t)\big), \quad w(t)\in B_{\delta_0}(W^{k,q}_{Euler}), \tag{2.22}$$

where Ψ is given in Proposition 2.1 which also implies $\Phi(t,\cdot)$, for any $t\in\mathbb{R}$, is a local diffeomorphism from $W^{k,q}_{Euler}$ to $\mathcal{G}$. Therefore, the linearizations

$$\begin{aligned}
&\tilde X = \mathcal{D}\Phi(t,w)X = \big(Du_0\circ\Psi(w)\big)\mathcal{D}\Psi(w)X, && X\in W^{k,q}_{Euler}\\
&X = \big(\mathcal{D}\Phi(t,w)\big)^{-1}\tilde X = \big(\mathcal{D}\Psi(w)\big)^{-1}\big((Du_0)^{-1}\circ\Psi(w)\big)\tilde X, && \tilde X\in T_u\mathcal{G}
\end{aligned}$$

are isomorphisms from between $W^{k,q}_{Euler}$ and $T_u\mathcal{G}$. Substitute (2.22) in to (2.14), one may compute

$$u_t = \Phi_t(t,w) + \mathcal{D}\Phi(t,w)w_t = u_{0t}\circ\Psi(w) + (Du_0\circ\Psi(w))\mathcal{D}\Psi(w)w_t \tag{2.23}$$

$$v = u_t\circ u^{-1} = v_0 + \big((Du_0)\circ u_0^{-1}\big)\big((\mathcal{D}\Psi(w)w_t)\circ\Psi(w)^{-1}\circ u_0^{-1}\big) \tag{2.24}$$

and

$$\begin{aligned}
u_{tt} =&\Phi_{tt}(t,w) + 2\mathcal{D}\Phi_t(t,w)w_t + \mathcal{D}\Phi(t,w)w_{tt} + \mathcal{D}^2\Phi(t,w)(w_t,w_t)\\
=&u_{0tt}\circ\Psi(w) + 2\left(Du_{0t}\circ\Psi(w)\right)\mathcal{D}\Psi(w)w_t + \big(D^2u_0\circ\Psi(w)\big)\big(\mathcal{D}\Psi(w)w_t, \mathcal{D}\Psi(w)w_t\big)\\
&+ \left(Du_0\circ\Psi(w)\right)\big(\mathcal{D}^2\Psi(w)(w_t,w_t) + \mathcal{D}\Psi(w)w_{tt}\big).
\end{aligned}$$

Therefore, for $u(t)$ close to $u_0(t)$, the Euler equation (2.14) is rewritten as

$$w_{tt} + \mathcal{F}(t,w,w_t) = 0, \quad w\in B_{\delta_0}(W^{k,q}_{Euler}),\ w_t\in W^{k,q}_{Euler}, \tag{2.25}$$

where, for $w\in B_{\delta_0}(W^{k,q}_{Euler})$ and $X\in W^{k,q}_{Euler}$, the term $\mathcal{F}(t,w,X)$ is arranged into the linear and quadratic parts in X

$$\mathcal{F}(t,w,X) = A(t,w)X + B(t,w)(X,X)\in W^{k,q}_{Euler} \tag{2.26}$$

with the linear and bilinear (in X) operators $A(t,w)$ and $B(t,w)$ are given by

$$\begin{aligned} A(t,w)X =&2\big(\mathcal{D}\Phi(t,w)\big)^{-1}\big(\mathcal{D}\Phi_t(t,w)X + \mathcal{P}(u, v_0\circ u, \tilde{X})\big) \\ =&2\big(\mathcal{D}\Psi(w)\big)^{-1}\big((Du_0)^{-1}\circ\Psi(w)\big)\big((Dv_0\circ u)\tilde{X} + \mathcal{P}(u, v_0\circ u, \tilde{X})\big) \end{aligned} \tag{2.27}$$

$$\begin{aligned} B(t,w)(X,X) =&\big(\mathcal{D}\Phi(t,w)\big)^{-1}\big(\mathcal{D}^2\Phi(t,w)(X,X) + \mathcal{P}(u,\tilde{X},\tilde{X})\big) \\ =&\big(\mathcal{D}\Psi(w)\big)^{-1}\big((Du_0)^{-1}\circ\Psi(w)\big)\Big(\big(Du_0\circ\Psi(w)\big)\mathcal{D}^2\Psi(w)(X,X) \\ &+\big(D^2u_0\circ\Psi(w)\big)\big(\mathcal{D}\Psi(w)X, \mathcal{D}\Psi(w)X\big) + \mathcal{P}(u,\tilde{X},\tilde{X})\Big), \end{aligned} \tag{2.28}$$

where

$$u=\Phi(t,w)=u_0\circ\Psi(w) \qquad \tilde{X}=\mathcal{D}\Phi(t,w)X=\big(Du_0\circ\Psi(w)\big)\mathcal{D}\Psi(w)X.$$

In the above calculation, the invariance of $\mathcal{P}$ under the right translation (2.15) and equation (2.14) were used in handling both u_{tt} and u_{0tt}. Here even though the linear operator $\big(\mathcal{D}\Phi(t,w)\big)^{-1}$ acts only on the subspace $T_u\mathcal{G}$, the terms $A(t,w)X$ and $B(t,w)(X,X)$ are well-defined and thus $\mathcal{F}(t,w,X)\in W^{k,q}_{Euler}$. In fact, (2.13) implies

$$\mathcal{D}\Phi_t(t,w)X + \mathcal{P}(u,v_0\circ u,\tilde{X}) = (Dv_0\circ u)\tilde{X} + \mathcal{P}(u,v_0\circ u,\tilde{X}) \in T_u\mathcal{G}$$

and thus $A(t,w)$ is well-defined. To see that $B(t,w)$ is well-defined, we note the second linearization of Φ along a line $w+sX$, $s\in\mathbb{R}$, at $s=0$, is given by

$$u_{ss}=\mathcal{D}^2\Phi(t,w)(X,X)=\big(Du_0\circ\Psi(w)\big)\mathcal{D}^2\Psi(w)(X,X)+\big(D^2u_0\circ\Psi(w)\big)\big(\mathcal{D}\Psi(w)X,\mathcal{D}\Psi(w)X\big).$$

Let

$$Y=u_s\circ u^{-1}=\tilde{X}\circ u^{-1}\in W^{k,q}_{Euler},$$

then

$$u_{ss}=Y_s\circ u + DY(\tilde{X}).$$

Since $Y_s\in W^{k,q}_{Euler}$ and (2.13) implies

$$DY(\tilde{X})+\mathcal{P}(u,\tilde{X},\tilde{X})\in T_u\mathcal{G},$$

we obtain

$$\mathcal{D}^2\Phi(t,w)(X,X)++\mathcal{P}(u,\tilde{X},\tilde{X}) = u_{ss}+\mathcal{P}(u,\tilde{u}_s,\tilde{u}_s)\in T_u\mathcal{G}$$

and thus

$$B(t,w)(X,X)=\big(\mathcal{D}\Phi(t,w)\big)^{-1}\big(\mathcal{D}^2\Phi(t,w)(X,X)+\mathcal{P}(u,\tilde{X},\tilde{X})\big)\in W^{k,q}_{Euler}$$

is well-defined.

Remark. *An alternative way to rewrite the Euler's equation to derive the above form is to follow the Lagrangian variational principle. One may first express the action* $\int\int_\Omega \frac{|u_t|^2}{2}dydt$*, defined on* $T\mathcal{G}$*, using* (2.22) *and* (2.23)*. The Euler's equation in terms of* w *follows from the the variation of the action.*

Recall we assumed in (A1) that $v_0\in W^{k+r,q}$ with $r\geq 4$ and $\mu>\mu_0\geq 0$ is a constant fixed before (2.3).

Lemma 2.5. *The nonlinear mapping $\mathcal{F}$ satisfies*

$$\mathcal{F} \in C^{r-4}\big(\mathbb{R} \times B_{\delta_0}(W^{k,q}_{Euler}) \times W^{k,q}_{Euler}, W^{k,q}_{Euler}\big), \quad \mathcal{F}(t,w,0) \equiv 0.$$

Moreover, there exist $K > 0$ depending only on r and k and $C > 0$ depending only on r, n, k, q, v_0, such that, for $t \in \mathbb{R}$, $w, X \in W^{k,q}_{Euler}$ and $|w|_{W^{k,q}} < \delta_0$, we have

$$|A(t,\cdot)|_{C^{r-2}\big(B_{\delta_0}(W^{k,q}_{Euler}), L(W^{k,q}_{Euler})\big)} \leq Ce^{K\mu|t|},$$
$$|B(t,\cdot)|_{C^{r-2}\big(B_{\delta_0}(W^{k,q}_{Euler}), L(W^{k,q}_{Euler}\otimes W^{k,q}_{Euler}, W^{k,q}_{Euler})\big)} \leq Ce^{K\mu|t|}.$$

Here the highest order derivative is in the Gâteaux sense which is sufficient to yield the $C^{r-3,1}$ bounds.

Proof. The smoothness of $\mathcal{F}$ follows directly from its expression (2.26) – (2.28) and Propositions 2.1 – 2.4. The property $\mathcal{F}(t,w,0) \equiv 0$ follows directly from (2.26) which is actually a consequence of the right translation invariance of the Euler equation. To demonstrate the latter, we notice that $(u_0 \circ \phi, u_{0t} \circ \phi)$ is a solution of (2.14) for any $\phi \in \mathcal{G}$. Then (2.22) implies that, for any $w \in B_{\delta_0}(W^{k,q}_{Euler})$, $(w,0)$ is a time independent solution of (2.25) and thus $\mathcal{F}(t,w,0) = 0$. The derivation of the estimates is tedious, but straightforwardly from Proposition 2.1 – 2.3, (2.17), and (2.26) – (2.28).

□

Remark. *As proved in Proposition 2.3,* (2.14) *is a smooth infinite dimensional ODE. However, the local coordinate systems based on the composition would always cause loss of derivatives due to Proposition 2.2. Here our local coordinate mapping $\Phi(t,\cdot)$ allows us to obtain some limited smoothness due to assumption (A1) of the extra regularity of v_0.*

2.4. **Linear exponential dichotomy in Lagrangian coordinates.** Since $\mathcal{F}(t,w,0) = 0$ for any small $w \in T_{id}\mathcal{G}$, we have $\mathcal{D}_w F(t,w,0) = 0$. We can rewrite (2.25) as

$$z_t = A_0(t)z + F(t,z), \qquad z = (z_1, z_2)^T \in B_{\delta_0}(W^{k,q}_{Euler}) \times W^{k,q}_{Euler} \tag{2.29}$$

where

$$A_0(t) = \begin{pmatrix} 0 & I \\ 0 & -A(t,0) \end{pmatrix}, \qquad F(t,z) = \begin{pmatrix} 0 \\ A(t,0)z_2 - \mathcal{F}(t,z_1,z_2) \end{pmatrix}.$$

From (2.27), the explicit form of $A(t,0)$ is given by

$$A(t,0)X = 2\big(Du_0(t)\big)^{-1}\Big(\big(Dv_0 \circ u_0(t)\big)Du_0(t)X + \mathcal{P}\big(u_0, v_0 \circ u_0(t), Du_0(t)X\big)\Big)$$

and Lemma 2.5 implies that there exists $C > 0$ independent of δ_0 such that

$$F(t,0) = 0 = \mathcal{D}_z F(t,0), \quad |\mathcal{D}^2_z F(t,\cdot)|_{C^{r-4}\big(B_{\delta_0}(W^{k,q}_{Euler})^2, W^{k,q}_{Euler}\big)} \leq Ce^{K\mu|t|}, \tag{2.30}$$

where again the highest order derivative is in the Gâteaux sense which is sufficient to yield the $C^{r-3,1}$ bounds. The linearization of (2.25) takes the form of

$$w_{tt} + \mathcal{D}_X\mathcal{F}(t,0,0)w_t = 0 \Leftrightarrow z_t = A_0(t)z \tag{2.31}$$

whose well-posedness is guaranteed by Lemma 2.5. Let $T(t,t_0)$ be the solution operator of (2.31) with initial time t_0 and terminal time t.

On the one hand, for $w \in W^{k,q}_{Euler}$, $z = (w,0)^T$ is a solution of (2.31). On the other hand, linearizing (2.24) at the steady solution $z = (w_0,0)^T$ one may compute $z = (w(t), w_t(t))^T$ is a solution of the linearization of (2.29) at $z = (w_0,0)^T$, where

$$w_t(t) = \big(\mathcal{D}\Psi(w_0)\big)^{-1}\big((Du_0(t))^{-1} \circ \Psi(w_0)\big)\big(v(t) \circ u_0(t) \circ \Psi(w_0)\big)$$

and $v = v(t)$ is a solution of (1.2). In particular, taking $w_0 = 0$, we have that $z = (w(t), w_t(t))^T$ is a solution solution of (2.31) where

$$w_t(t) = \big(Du_0(t)\big)^{-1}\big(v(t) \circ u_0(t)\big). \tag{2.32}$$

This correspondence between the linearized solutions of (1.2) and (2.31) and assumption (A2) would yield the (t-dependent) exponential dichotomy $(W^{k,q}_{Euler})^2 = Y_u(t) \oplus Y_{cs}(t)$ of (2.31) such that $T(t,t_0)Y_{u,cs}(t_0) = Y_{u,cs}(t)$ and the exponential decay rate of $T(t,t_0)$ as $t \to -\infty$ (or $+\infty$) in $Y_u(t)$ (or $Y_{cs}(t)$) is bounded roughly by λ_u (or λ_{cs}). To define $Y_u(t)$, it is natural from (2.32) that, for $(w, w_t) \in Y_u(t)$, w_t takes the form of $\big(Du_0(t)\big)^{-1}\big(e^{tL}v \circ u_0(t)\big)$ with $v \in X_u$, where L (as well as $L_{u,cs}$) is the linear operator defined in the linearized Euler equation (1.2). Also the decay of w as $t \to -\infty$ requires it take the form of $w(t) = \int_{-\infty}^t w_t dt'$. Therefore, for any $t \in \mathbb{R}$, let

$$Y_u(t) = \{\Big(\int_{-\infty}^t \big(Du_0(\tau)\big)^{-1}\big((e^{\tau L}v) \circ u_0(\tau)\big)d\tau,\ \big(Du_0(t)\big)^{-1}\big(e^{tL}v \circ u_0(t)\big)\Big)^T \mid v \in X_u\}.$$

The convergence of the above infinite integral follows directly from assumptions (A2) and (A3) with K sufficient large depending only on k. Similarly, for $(w, w_t) \in Y_{cs}(t)$, w_t takes the form of $\big(Du_0(t)\big)^{-1}\big(e^{tL}v \circ u_0(t)\big)$ with $v \in X_{cs}$, and w should take the form of

$$w_0 + \int_0^t \big(Du_0(\tau)\big)^{-1}\big((e^{tL}v) \circ u_0(\tau)\big)d\tau.$$

However, $w_0 \in W^{k,q}_{Euler}$ is arbitrary and thus we can absorb the integral term into w_0. Define

$$Y_{cs}(t) = \{\Big(w,\ \big(Du_0(t)\big)^{-1}\big(e^{tL}v \circ u_0(t)\big)\Big)^T \mid w \in W^{k,q}_{Euler},\ v \in X_{cs}\}.$$

Lemma 2.6. *It holds $(W^{k,q}_{Euler})^2 = Y_{cs}(t) \oplus Y_u(t)$. Moreover, let $P_{u,cs}(t) \in L\big((W^{k,q}_{Euler})^2\big)$ be the projections associate to this decomposition and*

$$T_{u,cs}(t,t_0) = T(t,t_0)|_{Y_{u,cs}(t_0)}.$$

Then for any $t, t_0 \le 0$, we have $T_{u,cs}(t,t_0)Y_{u,cs}(t_0) = Y_{u,cs}(t)$ and there exist constants $K > 0$ depending only on k and $C_0 \ge 1$ depending only on k, n, q, v_0 such that if $\lambda_u > K\mu$, we have

$$\begin{aligned}
&|P_{u,cs}(t)|_{L\left((W^{k,q}_{Euler})^2\right)} \le C_0 e^{-K\mu t}, && \forall\, t \le 0\\
&|T_{cs}(t,t_0)| \le C_0 e^{\lambda_{cs}(t-t_0) - K\mu t_0}, && \forall\, 0 \ge t \ge t_0\\
&|T_u(t,t_0)| \le C_0 e^{(\lambda_u - K\mu)(t-t_0) - K\mu t_0}, && \forall\, t \le t_0 \le 0.
\end{aligned}$$

Remark. *This lemma shows that unlike the traditional exponential dichotomy, the norms of the projections in the invariant splitting here is not uniformly bounded in t and may approach ∞ as $t \to -\infty$. This means that the angles between the unstable and center-stable subspaces $Y_{u,cs}(t)$ of (2.31) may not have a uniform positive lower bound as $t \to -\infty$.*

Proof. We first show the invariance of $Y_{cs,u}(t)$ under $T(t,t_0)$. In fact, for any

$$Y_u(t_0) \ni z = (w, w_1)^T = \Big(\int_{-\infty}^{t_0} \big(Du_0(\tau)\big)^{-1} \big((e^{\tau L} v) \circ u_0(\tau)\big) d\tau,\ \big(Du_0(t_0)\big)^{-1} \big(e^{t_0 L} v \circ u_0(t_0)\big)\Big)^T,$$

where $v \in X_u$, let

$$v(t) = e^{tL} v \in X_u, \qquad \tilde{w} = \int_{-\infty}^{0} \big(Du_0(\tau)\big)^{-1} \big((e^{\tau L} v) \circ u_0(\tau)\big) d\tau.$$

Clearly,

$$\begin{aligned} Y_u(t) \ni z(t) =& \big(w(t), w_1(t)\big)^T \triangleq \Big(\int_{-\infty}^{t} \big(Du_0(\tau)\big)^{-1} \big((e^{\tau L} v) \circ u_0(\tau)\big) d\tau,\ \big(Du_0(t)\big)^{-1} \big(e^{tL} v \circ u_0(t)\big)\Big)^T \\ =& (\tilde{w}, 0)^T + \Big(\int_{0}^{t} \big(Du_0(\tau)\big)^{-1} \big(v(\tau) \circ u_0(\tau)\big) d\tau,\ \big(Du_0(t)\big)^{-1} \big(v(t) \circ u_0(t)\big)\Big)^T \end{aligned}$$

is the solution of (2.31) with initial data $(\tilde{w}, v)$ at $t = 0$. Moreover, it satisfies $z(t_0) = z$ and thus $Y_u(t) \ni z(t) = T(t,t_0)z$ which implies the invariance of $Y_u(t)$.

The above arguments also leads to the decay estimate of $T_u(t,t_0)$. In fact,one may compute

$$w_1(t) = \big(Du_0(t)\big)^{-1} \Big(\big(e^{(t-t_0)L} ((Du_0(t_0) w_1) \circ (u_0(t_0))^{-1})\big) \circ u_0(t) \Big).$$

Since $v \in X_u$, we obtain from (2.3) and assumptions (A2) and (A3)

$$|w_1(t)|_{W^{k,q}} \le C e^{\lambda_u (t-t_0) - K\mu t} |w_1|_{W^{k,q}}, \qquad \forall\, t \le t_0 \le 0.$$

Finally from $w(t) = \int_{-\infty}^{t} w_1(\tau) d\tau$, we obtain the estimate for $T_u(t,t_0)$. The proof of the invariance of $Y_{cs}(t)$ and the estimate for $T_{cs}(t,t_0)$ are similar.

To prove the direct sum and obtain the bounds on the projection operators, let $P^0_{cs,u} \in L(W^{k,q}_{Euler})$ be the projections given by the decomposition $W^{k,q}_{Euler} = X_{cs} \oplus X_u$ assumed in hypothesis (A2). Given any $z = (w, w_1)^T \in (W^{k,q}_{Euler})^2$ and $t \le 0$, let

$$\begin{aligned} v_{cs,u}(t) &= P^0_{cs,u} e^{-tL} \Big(\big(Du_0(t) w_1\big) \circ u_0(t)^{-1} \Big) \in X_{cs,u} \\ w_0(t) &= w - \int_{-\infty}^{t} \big(Du_0(\tau)\big)^{-1} \big((e^{\tau L} v_u(t)) \circ u_0(\tau)\big) d\tau \end{aligned}$$

From (2.3) and assumptions (A2) and (A3), we have

$$|e^{\tau L} v_u(t)|_{W^{k,q}} \le C e^{\lambda_u (\tau - t) - K\mu t} |w_1|_{W^{k,q}}, \quad \forall\, \tau \le t.$$

Let

$$\begin{aligned} z_u =& \Big(\int_{-\infty}^{t} \big(Du_0(\tau)\big)^{-1} \big((e^{\tau L} v_u(t)) \circ u_0(\tau)\big) d\tau,\ \big(Du_0(t)\big)^{-1} \big(e^{tL} v_u(t) \circ u_0(t)\big)\Big)^T \in Y_u(t) \\ z_{cs} =& \Big(w_0(t),\ \big(Du_0(t)\big)^{-1} \big(e^{tL} v_{cs}(t) \circ u_0(t)\big)\Big)^T \in Y_{cs}(t). \end{aligned}$$

Obvious $z = z_u + z_{cs}$ and this splitting is unique. Therefore $(W^{k,q}_{Euler})^2 = Y_{cs}(t) \oplus Y_u(t)$ and $P_{cs,u}(t) z = z_{cs,u}$. It is straight forward to first obtain the estimates on $P_u(t)$ based on the above inequalities and the bound on $P_{cs}(t) = I - P_u(t)$ also follows. $\square$

Let

$$F_{u,cs}(t,z) = P_{u,cs}(t)F(t,z).$$

Then Lemma 2.6 and (2.30) imply that there exist $K > 0$ and $C_1 > 0$ such that for any $t \le 0$, it holds

$$F_{cs,u}(t,0) = 0 = \mathcal{D}_z F_{cs,u}(t,0), \quad |\mathcal{D}_z^2 F_{u,cs}(t,z)|_{C^{r-4}(B_{\delta_0}(W^{k,q}_{Euler})^2, W^{k,q}_{Euler})} \le C_1 e^{-K\mu t}, \tag{2.33}$$

where again the highest order derivative is in the Gâteaux sense which is sufficient to yield the $C^{r-3,1}$ bounds.

2.5. **Integral unstable manifolds of** (2.25)**.** We will follow the Lyapunov-Perron integral equation method to construct the integral unstable manifolds. In the standard construction of invariant manifolds, where the bounds on the invariant splitting and the nonlinear terms are uniform in t, small Lipschitz constant of $F_{cs,u}$ is sufficient in the construction of local invariant manifolds. As we do not have t-uniform estimates here, we repeatedly used the property that $F(t,z) = O(|z|^2)$ to yield an extra $|z(t)| = O(e^{\lambda t})$ with $\lambda t < 0$. This quadratic nature of F combined with the exponential gap condition (2.36) allows us to complete the proof of Proposition 2.7 in the below. However, similar construction would not work for the construction of the center-stable manifold since solutions $z(t)$ on the center-stable manifold do not satisfy $z(t) \to 0$ as $t \to \pm\infty$.

For $\lambda \in (\lambda_{cs}, \lambda_u)$ and $\delta_1 \le \delta_0$ to be determined, let

$$\Gamma = \{z = (z_u, z_{cs}) \in C^0\big((-\infty, 0], \overline{B_{\delta_1}(W^{k,q}_{Euler})^2}\big) \mid |z|_\lambda \le \delta_1\}$$

where $z_{u,cs}(t) \in Y_{u,cs}(t)$ and

$$|z|_\lambda \triangleq \sup_{t \le 0} e^{-\lambda t} |z(t)|_{W^{k,q}}.$$

This Γ is the set of functions with the desired backward in time decay expected to be satisfied by the solutions on the unstable manifolds. For any $z \in \Gamma$ and

$$z_{u0} \in Y_u(0), \qquad |z_{u0}|_{W^{k,q}} \le \frac{\delta_1}{2C_0} \tag{2.34}$$

where C_0 is given in Lemma 2.6, define $\mathcal{L}(\cdot, z_{u0})$, where $\tilde z = (\tilde z_u, \tilde z_{cs}) = \mathcal{L}(z, z_{u0})$ is defined as, for any $t \le 0$,

$$\begin{cases} \tilde z_u(t) = T_u(t,0) z_{u0} + \int_0^t T_u(t,\tau) F_u\big(\tau, z(\tau)\big) d\tau \\ \tilde z_{cs}(t) = \int_{-\infty}^t T_{cs}(t,\tau) F_{cs}\big(\tau, z(\tau)\big) d\tau. \end{cases} \tag{2.35}$$

Proposition 2.7. *There exists $K > 0$ depending only on r and k such that if λ and δ_1 satisfy*

$$\lambda \in \big(\lambda_{cs} + K\mu,\ \lambda_u - K\mu\big) \tag{2.36}$$

$$C_0 C_1 \delta_1 \Big(\frac{1}{\lambda - \lambda_{cs}} + \frac{1}{\lambda_u - K\mu - \lambda}\Big) < \frac{1}{2} \tag{2.37}$$

where C_0 and C_1 are from Lemma 2.6 and (2.33)*, respectively, then $\mathcal{L}(\cdot, z_{u0})$ is a contraction on Γ with the Lipschitz constant $\frac{1}{2}$ for any z_{u0} satisfying* (2.34)*. Moreover $|\mathcal{L}|_{C^{r-3,1}} \le C$ for some $C > 0$ depending only on $r, k, q, n,$ and v_0.*

Remark. *Assumption (A3) with a reasonably large K, depending only on r and k, guarantees the existence of λ and δ_1 satisfying the above inequalities.*

Remark. *It is standard in the invariant manifold theory (see, for example,* [CL88]*) to prove that $z \in \Gamma$ solves* (2.29) *with $z_u(0) = z_{u0}$ if and only if, z is the fixed point of $\mathcal{L}(\cdot, z_{u0})$. Therefore, Proposition 2.7 shows that, for any given z_{u0} satisfying* (2.34)*, there exists a unique solution of* (2.29) *satisfying the exponential decay as $t \to -\infty$ with the decay rate at least λ.*

Proof. In the proof the generic constant K may change from line to line, but always depends only on k and r. From the definition of $\mathcal{L}$, Lemma 2.6, assumptions (A3), (2.36), and (2.34), and the second order Taylor expansion of F based on (2.33), (instead of the usual small Lipschitz estimates of F), we obtain for $t \leq 0$

$$\begin{aligned} e^{-\lambda t}|\tilde{z}_u(t)|_{W^{k,q}} \leq & C_0|z_{u0}|_{W^{k,q}} + \int_t^0 \frac{1}{2}C_0C_1 e^{-\lambda t+(\lambda_u - K\mu)(t-\tau)-K\mu\tau+2\lambda\tau}d\tau|z|_\lambda^2 \\ \leq & C_0|z_{u0}|_{W^{k,q}} + \frac{1}{2}C_0C_1|z|_\lambda^2\int_t^0 e^{(\lambda_u-K\mu-\lambda)(t-\tau)}d\tau \\ \leq & C_0|z_{u0}|_{W^{k,q}} + \frac{C_0C_1\delta_1}{2(\lambda_u - K\mu - \lambda)}|z|_\lambda \end{aligned}$$

and

$$e^{-\lambda t}|\tilde{z}_{cs}(t)|_{W^{k,q}} \leq \int_{-\infty}^t \frac{1}{2}C_0C_1e^{-\lambda t+\lambda_{cs}(t-\tau)-K\mu\tau+2\lambda\tau}d\tau|z|_\lambda^2 \leq \frac{C_0C_1\delta_1|z|_\lambda}{2(\lambda-\lambda_{cs})}.$$

Therefore (2.37) implies that

$$|\mathcal{L}(z, z_{u0})|_\lambda \leq C_0|z_{u0}|_{W^{k,q}} + \frac{1}{4}|z|_\lambda \leq \frac{3}{4}\delta_1 \tag{2.38}$$

and thus $\mathcal{L}(\cdot, z_{u0})$ maps Γ into itself.

To prove $\mathcal{L}(\cdot, z_{u0})$ is a contraction on Γ, we note that for any $z^{1,2} \in \Gamma$, (2.33) implies

$$|F_{u,cs}(t, z^2(t)) - F_{u,cs}(t, z^1(t))| \leq C_1\delta_1 e^{(2\lambda-K\mu)t}|z^2 - z^1|_\lambda, \quad \forall t \leq 0$$

Therefore, we have

$$\begin{aligned} e^{-\lambda t}|\tilde{z}_u^2(t) - \tilde{z}_u^1(t)| \leq & C_0C_1\delta_1\int_t^0 e^{(\lambda_u-K\mu-\lambda)(t-\tau)}d\tau|z^2-z^1|_\lambda \leq \frac{C_0C_1\delta_1|z^2-z^1|_\lambda}{\lambda_u - K\mu-\lambda} \\ e^{-\lambda t}|\tilde{z}_{cs}^2(t) - \tilde{z}_{cs}^1(t)| \leq & C_0C_1\delta_1\int_{-\infty}^t e^{(\lambda_{cs}-\lambda)(t-\tau)}d\tau|z^2-z^1|_\lambda \leq \frac{C_0C_1\delta_1|z^2-z^1|_\lambda}{\lambda-\lambda_{cs}} \end{aligned}$$

and thus (2.37) implies that $\mathcal{L}(\cdot, z_{u0})$ is a contraction.

Since $\mathcal{L}$ is linear in z_{u0}, we only need to prove its smoothness in $z \in \Gamma$. For any $z \in \Gamma$, formally the linearization of $\mathcal{L}$ is given by

$$\mathcal{D}_z\mathcal{L}(z, z_{u0})z_1 = \bar{z}_1 = (\bar{z}_{1u}, \bar{z}_{1cs})$$

where for $t \leq 0$,

$$\begin{cases} \bar{z}_{1u}(t) = \int_0^t T_u(t,\tau)\mathcal{D}_zF_u\big(\tau, z(\tau)\big)z_1(\tau)d\tau \\ \bar{z}_{1cs}(t) = \int_{-\infty}^t T_{cs}(t,\tau)\mathcal{D}_zF_{cs}\big(\tau, z(\tau)\big)z_1(\tau)d\tau. \end{cases} \tag{2.39}$$

The same procedure as in the above shows that $|\mathcal{D}_z\mathcal{L}|_\lambda \leq \frac{1}{2}$. To show it is indeed the derivative of $\mathcal{L}$, take $z_{1,2} \in \Gamma$, let

$$\tilde{z}_{1,2} = (\tilde{z}_{1,2u}, \tilde{z}_{1,2cs}) = \mathcal{L}(z_{1,2}, z_{u0}), \qquad \bar{z} = (\bar{z}_u, \bar{z}_{cs}) = \mathcal{D}_z\mathcal{L}(z_1, z_{u0})(z_2 - z_1).$$

It is straight forward to compute from (2.33) and Lemma 2.6, that for $t \le 0$,

$$\begin{aligned}&e^{-\lambda t}|\tilde z_{2u}(t) - \tilde z_{1u}(t) - \bar z_u(t)|\\ =&e^{-\lambda t}|\int_0^t T_u(t,\tau)\Big(F_u\big(\tau, z_2(\tau)\big) - F_u\big(\tau, z_1(\tau)\big) - \mathcal{D}F_u\big(\tau, z_1(\tau)\big)\big(z_2(\tau) - z_1(\tau)\big)\Big)d\tau|\\ \le&C_0C_1|z_2 - z_1|_\lambda^2 \int_t^0 e^{-\lambda t + (\lambda_u - K\mu)(t-\tau) - K\mu\tau + 2\lambda\tau} d\tau \le \frac{C_0C_1|z^2 - z^1|_\lambda^2}{\lambda_u - K\mu - \lambda}.\end{aligned}$$

The estimate for the center-stable component is very similar and this proves that $\mathcal{D}_z\mathcal{L}$ is indeed the derivative of $\mathcal{L}$.

Finally, we will show that $\mathcal{D}_z\mathcal{L}$ is Lipschitz. In fact, let

$$\bar z_{1,2} = (\bar z_{12u}, \bar z_{12cs}) = \mathcal{D}_z\mathcal{L}(z_{1,2}, z_{u0})z.$$

Then for any $t \le 0$,

$$\begin{aligned}&e^{-\lambda t}|\bar z_{2cs}(t) - \bar z_{1cs}(t)| = e^{-\lambda t}|\int_{-\infty}^t T_{cs}(t,\tau)\big(\mathcal{D}_zF_{cs}(\tau, z_2(\tau)) - \mathcal{D}_zF(\tau, z_1(\tau))\big)z(\tau)d\tau|\\ \le&C_0C_1|z_2 - z_1|_\lambda|z|_\lambda\int_{-\infty}^t e^{(\lambda_{cs}-\lambda)(t-\tau)+(\lambda - K\mu)\tau}d\tau \le \frac{C_0C_1}{\lambda_{cs} - \lambda}|z_2 - z_1|_\lambda|z|_\lambda.\end{aligned}$$

The estimates for the unstable component is the same and the proof of the higher order smoothness is similar. □

From the Contraction Mapping Theorem and Proposition 2.7, the mapping $\mathcal{L}$ has a unique fixed point

$$z_*(t, z_{u0}) = \big(z_{*u}(t, z_{u0}), z_{*cs}(t, z_{u0})\big)$$

which is $C^{r-3,1}$ in z_{u0} and satisfies $z_{*u}(0, z_{u0}) = z_{u0}$. Moreover, (2.38) implies

$$|z_*(u_0)|_\lambda \le 2C_0|z_{u0}|_{W^{k,q}}. \tag{2.40}$$

Like in the standard invariant manifold theory, these are solutions on the invariant integral unstable manifold of the non-autonomous system (2.25). Define

$$h_L(z_{u0}) = z_{*cs}(0, z_{u0}) \in X_{cs}$$

for all z_{u0} satisfying (2.34), then the $C^{r-3,1}$ graph

$$W_L^u \triangleq graph(h_L)$$

defines the slice of the unstable integral manifold of (2.25) for $t_0 = 0$. Obviously the uniqueness of the fixed point implies that $z_*(\cdot, 0) = 0$ and thus $h_L(0) = 0$ and $0 \in W_L^u$. Differentiating the fixed point equation we obtain

$$\mathcal{D}_{z_{u0}}z_*(z_{u0}) = \mathcal{D}_{z_{u0}}\mathcal{L}\big(z_*(z_{u0}), z_{u0}\big) + \mathcal{D}_z\mathcal{L}\big(z_*(z_{u0}), z_{u0}\big)\mathcal{D}_{z_{u0}}z_*(z_{u0})$$

Clearly, (2.39) implies that $\mathcal{D}_z\mathcal{L}(0, z_{u0}) = 0$ and thus

$$\mathcal{D}_{z_{u0}}z_*(\cdot, 0) = \mathcal{D}_{z_{u0}}\mathcal{L}(0,0) = \big(T_u(\cdot, 0), 0\big)$$

which does not have the center-stable component. Therefore, we obtain that

$$\mathcal{D}_{z_{u0}}h_L(0) = 0$$

which means that, at 0, the tangent space of the unstable integral manifold W_L^u

$$T_0W_L^u = Y_u(0) = \{(U(v),\, v)^T \mid v \in X_u\},$$

where

$$U(v) \triangleq \int_{-\infty}^{0} \big(Du_0(\tau)\big)^{-1}\big((e^{\tau L}v)\circ u_0(\tau)\big)d\tau, \quad |U|_{L(X_u, W^{k,q}_{Euler})} \leq \infty. \tag{2.41}$$

Here the boundedness of U follows from (2.3), (2.36) and assumptions (A2) and (A3).

2.6. **Unstable manifold in the Eulerian coordinates.** From the unstable integral manifold (at $t=0$) W^u_L constructed in the Lagrangian coordinates and the corresponding relationship given in (2.22) and (2.24), we obtain the $C^{r-3,1}$ invariant unstable manifold in the Eulerian coordinates

$$W^u \triangleq \{v = v_0 + \big(\mathcal{D}\Psi(w)w_1\big)\circ \Psi(w)^{-1} \mid (w, w_1) \in W^u_L\}.$$

The above expression was derived by substituting $t=0$ into (2.24).

Lemma 2.8. *There exist $K>0$ (depending only on r and k), $\delta_2,\ C>0$ depending only on $r,k,q,n,$ and v_0 such that*

(1) *There exists $H \in C^{r-3,1}\big(B_{3\delta_2}(X_u), X_{cs}\big)$ satisfying $|H|_{C^{r-3,1}} \leq C$, $H(0)=0$, $DH(0)=0$, and*

$$\{v_0+v_1+H(v_1) \mid v_1 \in B_{\delta_2}(X_u)\} \subset W^u \cap \big(v_0 + B_{2\delta_2}(W^{k,q}_{Euler})\big) \subset \{v_0+v_1+H(v_1) \mid v_1 \in B_{3\delta_2}(X_u)\}.$$

(2) *For any $v_\# \in \big(v_0 + B_{2\delta_2}(W^{k,q}_{Euler})\big)\cap W^u$ the solution $v(t)$ of the Euler equation (E) with the initial value $v(0)=v_\#$ satisfies*

$$v(t) \in W^u, \quad |v(t)-v_0|_{W^{k,q}} \leq C|v_\# - v_0|_{W^{k,q}} e^{(\lambda - K\mu)t}, \qquad \forall\, t \leq 0.$$

Proof. (1) For any $v_1 \in X_u$ with

$$|v_1|_{W^{k,q}} < \frac{\delta_1}{2C_0\big(1+|U|_{L(X_u, W^{k,q}_{Euler})}\big)},$$

let $z_{u0} = (Uv_1, v_1) \in Y_u(0)$, where U is defined in (2.41) and it implies z_{u0} satisfies (2.34), and

$$G(v_1) = \big(\mathcal{D}\Psi(w)w_1\big)\circ\Psi(w)^{-1} \quad \text{ where } (w, w_1) = z_*(0, z_{u0}) = z_{u0} + h_L(z_{u0}) \in W^u_L.$$

Clearly the definition of W^u_L and W^u and the properties of h_L imply $W^u = \{v_0 + G(v_1)\}$ and

$$G(0)=0, \quad DG(0)v_1 = w_1 = v_1,$$

and $G \in C^{r-3,1}$ with bounds depending only on $r,k,q,n,$ and v_0. Therefore, the existence and properties of H follows from the Implicit Function Theorem immediately.

(2) For any

$$v_\# = v_0 + \big(\mathcal{D}\Psi(w_\#)w_{1\#}\big)\circ\Psi(w_\#) \in \big(v_0 + B_{2\delta_2}(W^{k,q}_{Euler})\big)\cap W^u, \text{ with } (w_\#, w_{1\#}) \in W^u_L,$$

let $v(t)$ be the solution of (E) with $v(0)=v_\#$ and $z(t) = \big(w(t), w_t(t)\big)$ the solution of (2.25) with the initial value $z(0) = (w_\#, w_{1\#})$. Since $z(0) = (w_\#, w_{1\#}) \in W^u_L$ and thus $z \in \Gamma$ and (2.40) implies

$$|w(t)|_{W^{k,q}} + |w_t(t)|_{W^{k,q}} \leq e^{\lambda t}|z|_\lambda \leq 2C_0 e^{\lambda t}|z_{u0}|_{W^{k,q}}, \qquad t \leq 0.$$

Therefore (2.24) implies the desired decay estimate.

$$|v(t)-v_0|_{W^{k,q}} \leq Ce^{(\lambda-K\mu)t} \to 0 \text{ as } t \to -\infty.$$

To see the local invariance of W^u, fixed $T \geq 0$, for $t_0 \in (-\infty, T]$, let

$$\tilde{w}(t) = \Psi^{-1}\Big(u_0(t_0) \circ \Psi\big(w(t+t_0)\big) \circ u_0(t_0)^{-1}\Big), \quad t \leq 0.$$

Since the right translation invariance and (2.22) imply

$$\tilde{u}(t) = u_0(t) \circ \Psi\big(\tilde{w}(t)\big) = u_0(t+t_0) \circ \Psi\big(w(t+t_0)\big) \circ u_0(t_0)^{-1} = u(t+t_0) \circ u_0(t_0)^{-1}$$

is a solution of (2.14), we have $(\tilde{w}, \tilde{w}_t)$ is a solution of (2.25) which clearly corresponds to the solution $\tilde{v}(t) = v(t+t_0)$ of the Euler equation (E). Moreover, Proposition 2.1 implies

$$\begin{aligned}
|\tilde{w}(t)|_{W^{k,q}} + |\tilde{w}_t(t)|_{W^{k,q}} &\leq C\big(|w(t+t_0)|_{W^{k,q}} + |w_t(t+t_0)|_{W^{k,q}}\big) \leq Ce^{\lambda(t+t_0)}|z_{u0}|_{W^{k,q}} \\
&\leq Ce^{\lambda T}\big(1 + |U|_{L(X_u, W^{k,q}_{Euler})}\big)\, |v_\# - v_0|_{W^{k,q}}\, e^{\lambda t} \\
&\leq 2\delta_2 Ce^{\lambda T}\big(1 + |U|_{L(X_u, W^{k,q}_{Euler})}\big)e^{\lambda t}.
\end{aligned}$$

By choosing

$$\delta_2 \leq \frac{\delta_1}{2Ce^{\lambda T}\big(1 + |U|_{L(X_u, W^{k,q}_{Euler})}\big)},$$

we obtain that the solution $\tilde{z}(t) = \big(\tilde{w}(t), \tilde{w}_t(t)\big) \in \mathit{\Gamma}$. Therefore $v(t_0) = \tilde{v}(0) \in W^u$ which implies the invariance. $\square$

The property $DH(0) = 0$ immediately implies, as expected, the tangent space at the steady state v_0 is given by

$$T_{v_0}W^u = X^u$$

and the proof of Theorem 1.1 is complete.

3. Two-dimensional Euler equations

In this and the next sections, we will illustrate how assumptions (A1) – (A3) can be satisfied for certain steady states of the Euler equation (E). In this section, we consider the case of $\Omega = S^1 \times (-y_0, y_0)$, that is, $2\pi-$periodic in x and with rigid walls on $\{y = \pm y_0\}$.

Let $v = (v_1, v_2)^T : \Omega \to \mathbb{R}^2$ satisfy $\nabla \cdot v = 0$ in Ω and $v \cdot N = 0$ on $\partial\Omega$. On the one hand, let

$$\omega = \partial_x v_2 - \partial_y v_1, \qquad s = \frac{1}{|\Omega|}\int_\Omega v_1 dxdy$$

be the curl and the average horizontal momentum, respectively. We note that s is an invariant of (E) due to the translation symmetry in x. On the other hand, v is uniquely determined by ω and s through

$$v = J\nabla\Delta^{-1}\omega + se_1, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1 = (1,0)^T \tag{3.1}$$

where Δ^{-1} is the inverse of the Laplacian with zero Dirichlet boundary condition. It is clear that

$$\frac{1}{C}|v|_{W^{k,q}} \leq |\omega|_{W^{k-1,q}} + |s| \leq C|v|_{W^{k,q}}, \qquad k \geq 1,\ q > 1$$

for some $C > 0$. In the (ω, s) representation, (E) takes the form

$$\omega_t + v \cdot \nabla\omega = 0, \qquad s_t = 0 \tag{3.2}$$

where v is considered as determined by (ω, s) by (3.1).

Remark. *Due to the nontrivial first cohomology group of Ω, the vorticity alone does not determine a vector field in $W^{k,q}_{Euler}$ and thus the average horizontal momentum has to be included in the reformulation of the problem. If one considers $\Omega = T^2$, the 2D torus, then both momentum invariants each of which corresponds to a nontrivial element in the first cohomology groups should be included.*

Suppose $v_0 \in W^{k+4,q}$, $k > 1 + \frac{2}{q}$, is a steady state of (E), which corresponds to (ω_0, s_0) has Lyapunov exponent $\mu_0 \geq 0$ (both forward and backward in time). We linearize (3.2) at (ω_0, s_0) to obtain

$$\begin{pmatrix}\omega\\ s\end{pmatrix}_t = -\begin{pmatrix}v_0\cdot\nabla\omega\\ 0\end{pmatrix} - \begin{pmatrix}(J\nabla\Delta^{-1}\omega + se_1)\cdot\nabla\omega_0\\ 0\end{pmatrix} \triangleq L_0\begin{pmatrix}\omega\\ s\end{pmatrix} + L_1\begin{pmatrix}\omega\\ s\end{pmatrix}. \tag{3.3}$$

Assume that there exists an unstable eigenvalue λ_0 with $\operatorname{Re}\lambda_0 > (k-1)\mu_0$ of the linearized Euler operator L (defined in (1.2)) on L^q, and let $v \in L^q$ be the eigenfunction with corresponding ω and s. Then obviously $s = 0$ and

$$\lambda_0\omega + v_0 \cdot \bigtriangledown\omega = -v\cdot\nabla\omega_0.$$

An integration of above along the steady trajectory $\mathbf{X}_0(s)$ yields

$$\omega = \int_0^\infty e^{-\lambda_0 s} v\cdot\nabla\omega_0\left(\mathbf{X}_0(s)\right) ds.$$

By the standard bootstrap argument and the assumption $\operatorname{Re}\lambda_0 > (k-1)\mu_0$, we get $\omega \in W^{k-1,q}$ and $v \in W^{k,q}$.

For any $\lambda_- \in ((k-1)\mu_0, \operatorname{Re}\lambda_0)$ which does not equal the real part of any unstable eigenvalue , let

$$\tilde{X}_{cs} = \{(\omega, s)^T \mid \omega\in W^{k-1,q} \mid \limsup \frac{1}{t}\log|e^{t(L_0+L_1)}\omega|_{W^{k-1,q}} \leq \lambda_-\}$$

which is clearly a invariant subspace of $e^{L_0+L_1}$. Let

$$\sigma_{cs} = \sigma(e^{L_0+L_1}|_{\tilde{X}_{cs}}), \qquad \sigma_u = \sigma(e^{L_0+L_1}|_{W^{k-1,q}\times\mathbb{R}})\backslash\sigma_{cs}, \qquad \lambda_+ = \log(\inf\{|\lambda| \mid \lambda\in\sigma_u\}) \geq \lambda_-.$$

As the groups of bounded operators $e^{t(L_0+L_1)}$ and e^{tL} are conjugate through (3.1), we also have the invariance of $X_{cs,u}$ under e^{tL}. Since $e^{tL_0}\omega = \omega\circ u_0(t)^{-1}$, inequality (2.3) implies $|e^{tL_0}|_{L(W^{k-1,q})} \leq Ce^{(k-1)\mu|t|}$ for any $\mu > \mu_0$ and some $C > 0$ depending on μ. In particular, v_0 is divergence free, yields that e^{tL_0} is a group of isometries on any L^q space. Since L_1 is a compact operator acting on $(\omega, s)^T$, $e^{t(L_0+L_1)}$ is a compact perturbation to e^{tL_0} in the space $W^{k-1,q}$ and thus

- $\lambda_+ > \lambda_-$ and σ_u is an isolated compact subset of $\sigma(e^{L_0+L_1})$. Let $\tilde{X}_u$ be the eigenspace of $e^{L_0+L_1}$ corresponding to σ_u, and
$$X_{cs,u} = \{v\in W^{k,q}_{Euler} \mid (\omega, s)\in\tilde{X}_{cs,u}\}.$$
- $\tilde{X}_u$ and X_u are finite dimensional and
- (A2) is satisfied for any λ_{cs} and λ_u with $\lambda_- < \lambda_{cs} < \lambda_u < \lambda_+$.

Assumption (A3) depends on the Lyapunov exponents of v_0. In particular, if v_0 is a linearly unstable shear flow, $\mu_0 = 0$ and (A3) is also satisfied. An example is $v_0 = (\sin\beta y, 0)$. By [L03, Theorem 1.2], v_0 is linearly unstable when $\beta > 1$ and $\left(\frac{\pi}{2y_0}\right)^2 < \beta^2 - 1$.

Remark. *Consider rotating flows $v_0 = U(r)\vec{e}_\theta$ in an annulus $\Omega = \{a < r < b\}$. Then by similar arguments as above, assumptions (A2)-(A3) are satisfied as long as v_0 is linearly unstable.*

4. Three-dimensional Euler equations

In this section, we construct examples of 3D unstable steady flows for which Theorem 1.1 can be applied to get unstable (stable) manifolds. Consider $\Omega = T^3$ to be a 3D torus with periods L_x, L_y and L_z in x, y and z variables. For any profile $U(y,z)$, the 3D shear flow $\vec{u}_0 = (U(y,z), 0, 0)$ is a steady solution of 3D Euler equation. We construct unstable 3D shears satisfying assumptions (A1)-(A3) in several steps.

The linearized 3D Euler equation around a 3D shear $(U(y,z), 0, 0)$ is

$$\partial_t u + U u_x + v U_y + w U_z = -P_x, \tag{4.1}$$

$$\partial_t v + U v_x = -P_y,\ \ \partial_t w + U w_x = -P_z, \tag{4.2}$$

$$u_x + v_y + w_z = 0, \tag{4.3}$$

with periodic boundary conditions. There are almost no results about the the linear instability of general 3D shears. So we construct unstable 3D shears near unstable 2D shear flows $(U_0(y), 0, 0)$ where $U_0(y)$ is periodic with period L_y. First, we give a sufficient condition for linear instability of 2D periodic shears, which generalizes the result in [L03] for shear flows in a channel with rigid walls.

Lemma 4.1. *Consider a periodic shear profile $U(y) \in C^2(0, L_y)$ with only one inflection value U_s and*

$$K(y) = -\frac{U''(y)}{U(y) - U_s} > 0. \tag{4.4}$$

Let $-\alpha_{\max}^2$ be the lowest eigenvalue of the Sturm-Liouville operator

$$L\varphi = -\varphi'' - K(y)\varphi \tag{4.5}$$

with the periodic boundary conditions on $y \in [0, L_y]$. Then the Rayleigh equation

$$U''\phi - (U - c)\left(\phi'' - \alpha^2\phi\right) = 0, \tag{4.6}$$

with periodic boundary conditions on $y \in [0, L_y]$ has unstable eigenmodes ($\operatorname{Im} c > 0$) for any $\alpha \in (0, \alpha_{\max})$.

Remark. *Under the assumptions in the above Lemma, the lowest eigenvalue of L is negative, since $(L(1), 1) = -\int K(y)dy < 0$. A typical example satisfying (4.4) is $U_0(y) = \sin\left(\frac{2\pi}{L_y} y\right)$ for which $K(y) = \left(\frac{2\pi}{L_y}\right)^2$.*

Proof. The proof is similar to the case of rigid walls ([L03]), so we only point out some small modifications. Let ϕ_s be the eigenfunction of L corresponding to the lowest eigenvalue $-\alpha_{\max}^2$. Then $(\phi, c, \alpha) = (\phi_s, U_s, \alpha_{\max})$ is a neutral solution to the Rayleigh equation (4.6). By Sturm-Liouville theory, $-\alpha_{\max}^2$ is a simple eigenvalue and we can take $\phi_s > 0$. First, we study bifurcation of unstable modes near the neutral mode. Denote y_1 to be a minimum point of ϕ_s and let $y_2 = y_1 + L_y$. We normalize ϕ_s such that $\phi_s(y_1) = 1,\ \phi_s'(y_1) = 0$. Define $\phi_1(y;c,\varepsilon)$ and $\phi_2(y;c,\varepsilon)$ to be the solutions of

$$-\phi'' + \frac{U''}{U - U_s - c}\phi + \left(\alpha_{\max}^2 + \varepsilon\right)\phi = 0, \tag{4.7}$$

with $\phi_1(y_1) = 1,\ \phi_1'(y_1) = 0$ and $\phi_2(y_1) = 0, \phi_2'(y_1) = 1$. Here $\varepsilon < 0$ and $\operatorname{Im} c > 0$. Define

$$I(c,\varepsilon) = \phi_1(y_2;c,\varepsilon) + \phi_2'(y_2;c,\varepsilon) - 2,$$

then the existence of a solution to the Rayleigh equation (4.7) with periodic boundary conditions on $y \in [y_1, y_2]$ is equivalent to the existence of a root of I with $\operatorname{Im} c > 0$. When $c \to 0$, $\varepsilon \to 0-$ and $|\operatorname{Re} c| / \operatorname{Im} c$ remains bounded, as in [L03] we can show that $\phi_1(y;c,\varepsilon)$ $(\phi_2(y;c,\varepsilon))$ converges to $\phi_s(y)$ $(\phi_z(y))$ uniformly in $C^1[y_1,y_2]$. Here, $\phi_z(y) \in C^1[y_1,y_2]$ satisfies that $\phi_z'(y_2) = 1$ and $\phi_z(y_2) \neq 0$ since $\phi_z(y)$ can not be another eigenfunction associated with the simple eigenvalue $-\alpha_{\max}^2$. By similar calculations as in [L03], it can be shown that when $c \to 0$, $\varepsilon \to 0-$ and $|\operatorname{Re} c| / \operatorname{Im} c$ remains bounded,

$$\frac{\partial I}{\partial \varepsilon} \to \phi_z(y_2) \int_{y_1}^{y_2} \phi_s^2(y)\, dy,$$

and

$$\frac{\partial I}{\partial c} \to -\phi_z(y_2)\left(i\pi \sum_{k=1}^{l} \left(|U'|^{-1} K\phi_s^2\right)|_{y=a_k} + \mathcal{P}\int_{y_1}^{y_2} \left(K(y)\,\phi_s^2(y)\right) / \left(U(y) - U_s\right) dy\right).$$

Here, $a_1, \cdots, a_l$ are the inflection points such that $U(a_k) = U_s,\ k = 1, \cdots, l$ and $\mathcal{P}\int_{y_1}^{y_2}$ denotes the Cauchy principal part. Then by a variant of implicit function theorem as in [L03], there exists $\varepsilon_0 < 0$ such that for any $\varepsilon_0 < \varepsilon < 0$, there is an unstable solution ϕ_ε with $c = c(\varepsilon)$ to Rayleigh's equation (4.7). By the same arguments in [L03], such unstable modes can be continuated to all wave numbers $\alpha \in (0, \alpha_{\max})$. $\square$

Our second step is to show that 3D shears near an unstable 2D shear are also linearly unstable. More precisely, we have

Lemma 4.2. *Let $U_0(y) \in C^2(0, L_y)$ be such that the Rayleigh equation (4.6) has an unstable solution with (α_0, c_0) $(\alpha_0, \operatorname{Im} c_0 > 0)$. Fixed $L_z > 0$, consider $U(y,z) \in C^1((0,L_y) \times (0,L_z))$ which is L_y, L_z-periodic in y and z respectively. If $\|U(y,z) - U_0(y)\|_{W^{1,p}((0,L_y)\times(0,L_z))}$ $(p > 2)$ is small enough, then there exists an unstable solution $e^{i\alpha_0(x-ct)}(u,v,w,P)(y,z)$ to the linearized equation (4.1)-(4.3) with $|c - c_0|$ small. Moreover, if $U(y,z) \in C^\infty$, then $(u,v,w,P) \in C^\infty$.*

The proof of above lemma is almost the same as in the case of rigid walls ([LL11]), so we skip it here. By Lemmas 4.1 and 4.2, there exist linearly unstable 3D shears $\vec{u}_0 = (U(y,z), 0, 0)$. Below, we show that the assumption (A2) of linear exponential dichotomy is true in spaces $W_{Euler}^{m,2} = H^m$ ($m \geq 1$ is integer) for such unstable 3D shears. Then the assumption (A3) is automatic since $\mu_0 = 0$. Let $G_t = e^{Lt}$ be the linearized Euler semigroup near a steady flow $\vec{u}_0(\vec{x})$ and denote $r_{ess}(G_t; H^m)$ to be the essential spectrum radius of G_t in space H^m. By rather standard semigroup theory (see e.g. [Shi83, Section 1]), to get the linear exponential dichotomy (A2), it suffices to show that $r_{ess}(G_t; H^m) = 1$. The essential spectrum of linearized Euler operator had been studied a lot ([FV91] [LM91] [SL09] [V96]) by using the geometric optics method. We use the following characterization of $r_{ess}(G_t; H^m)$ in [SL09].

Lemma 4.3. [SL09] *Consider the following ODE system*

$$\begin{cases} \vec{x}_t = \vec{u}_0\left(\vec{x}\right) \\ \vec{\xi}_t = -\partial \vec{u}_0\left(\vec{x}\right)^T \ \vec{\xi} \\ \vec{b}_t = -\partial \vec{u}_0\left(\vec{x}\right)\vec{b} + 2\left(\partial \vec{u}_0\left(\vec{x}\right)\vec{b}, \vec{\xi}\right)\vec{\xi}\left|\vec{\xi}\right|^{-2}, \end{cases} \tag{4.8}$$

where $\vec{u}_0\left(\vec{x}\right)$ *is a steady flow of 3D Euler equation in* T^3 *and* $\vec{x} \in T^3, \vec{\xi}, \vec{b} \in \mathbf{R}^3$. *Denote*

$$\Lambda_m = \lim_{t\to\infty} \frac{1}{t} \ln \sup_{\substack{\vec{x}_0 \in T^3,\ \left|\vec{\xi}_0\right|=1 \\ \vec{b}_0 \perp \vec{\xi}_0,\ \left|\vec{b}_0\right|=1}} \left|\vec{b}\left(t\right)\right| \left|\vec{\xi}\left(t\right)\right|^m \tag{4.9}$$

where $\left(\vec{x}\left(t\right), \vec{b}\left(t\right), \vec{\xi}\left(t\right)\right)$ *is the solution of (4.8) with initial data* $\left(\vec{x}_0, \vec{\xi}_0, \vec{b}_0\right)$. *Then we have*

$$r_{ess}\left(G_t; H^m\right) = e^{t\Lambda_m}.$$

Lemma 4.4. *For* $\vec{u}_0 = \left(U\left(y,z\right), 0, 0\right)$ *in* T^3, *we have* $\Lambda_m = 0$.

Proof. Denote

$$\vec{x}\left(t\right) = \left(x\left(t\right), y\left(t\right), z\left(t\right)\right),\ \vec{\xi}\left(t\right) = \left(\xi_1\left(t\right), \xi_2\left(t\right), \xi_3\left(t\right)\right),\ \vec{b}\left(t\right) = \left(b_1\left(t\right), b_2\left(t\right), b_3\left(t\right)\right),$$

and

$$\vec{x}_0 = \left(x_0, y_0, z_0\right),\ \vec{\xi}_0 = \left(\xi_1^0, \xi_2^0, \xi_3^0\right),\ \vec{b}_0 = \left(b_1^0, b_2^0, b_3^0\right).$$

The solution of first two equations of (4.8) yield

$$x\left(t\right) = x_0 + U\left(y_0, z_0\right)t,\ y\left(t\right) = y_0,\ z\left(t\right) = z_0$$

and

$$\xi_1\left(t\right) = \xi_1^0, \quad \xi_2\left(t\right) = -U_y\xi_1^0 t + \xi_2^0,\ \xi_3\left(t\right) = -U_z\xi_1^0 t + \xi_3^0.$$

Plugging above forms into the equation of $\vec{b}\left(t\right)$, we have

$$\dot{b}_1 = -\left(U_y b_2 + U_z b_3\right) + \frac{2\left(\xi_1^0\right)^2\left(U_y b_2 + U_z b_3\right)}{\left|\vec{\xi}\left(t\right)\right|^2}, \tag{4.10}$$

$$\dot{b}_2 = \frac{2\xi_1^0\left(U_y b_2 + U_z b_3\right)\left(\xi_2^0 - U_y\xi_1^0 t\right)}{\left|\vec{\xi}\left(t\right)\right|^2}, \tag{4.11}$$

$$\dot{b}_3 = \frac{2\xi_1^0\left(U_y b_2 + U_z b_3\right)\left(\xi_3^0 - U_z\xi_1^0 t\right)}{\left|\vec{\xi}\left(t\right)\right|^2}. \tag{4.12}$$

To show that $\Lambda_m = 0$, it suffices to prove that $\left|\vec{b}(t)\right|$ only has polynomial growth, uniformly in $\left(\vec{x}_0, \vec{\xi}_0, \vec{b}_0\right)$. From equations (4.11) and (4.12), we have

$$\begin{aligned}
&\frac{d}{dt}\left(U_y b_2 + U_z b_3\right) \\
&= \frac{2\left(U_y b_2 + U_z b_3\right)\left[\xi_1^0\xi_2^0 U_y + \xi_1^0\xi_3^0 U_z - \left(\xi_1^0\right)^2\left(U_y^2 + U_z^2\right)t\right]}{\left|\vec{\xi}(t)\right|^2} \\
&= -\frac{\left(U_y b_2 + U_z b_3\right)}{\left|\vec{\xi}(t)\right|^2}\frac{d}{dt}\left|\vec{\xi}(t)\right|^2,
\end{aligned}$$

by noting that

$$\begin{aligned}
\left|\vec{\xi}(t)\right|^2 &= \left(\xi_1^0\right)^2 + \left(-U_y\xi_1^0 t + \xi_2^0\right)^2 + \left(-U_z\xi_1^0 t + \xi_3^0\right)^2 \\
&= 1 - 2\left(\xi_1^0\xi_2^0 U_y + \xi_1^0\xi_3^0 U_z\right)t + \left(\xi_1^0\right)^2\left(U_y^2 + U_z^2\right)t^2.
\end{aligned}$$

Thus

$$\frac{d}{dt}\left[\left(U_y b_2 + U_z b_3\right)\left|\vec{\xi}(t)\right|^2\right] = 0$$

and

$$\left(U_y b_2 + U_z b_3\right)(t) = \frac{U_y b_2^0 + U_z b_3^0}{\left|\vec{\xi}(t)\right|^2}. \tag{4.13}$$

For any fixed $t > 0$, we find a lower bound for $\left|\vec{\xi}(t)\right|^2$ by minimizing the function

$$f\left(\xi_1^0, \xi_2^0, \xi_3^0\right) = \left(\xi_1^0\right)^2 + \left(-U_y\xi_1^0 t + \xi_2^0\right)^2 + \left(-U_z\xi_1^0 t + \xi_3^0\right)^2$$

subject to the constraint $\left(\xi_1^0\right)^2 + \left(\xi_2^0\right)^2 + \left(\xi_3^0\right)^2 = 1$. By calculations of Lagrange multiplier, we get

$$\begin{aligned}
\min_{\left|\vec{\xi}_0\right|=1}\left|\vec{\xi}(t)\right|^2 &= \frac{2 + \left(U_y^2 + U_z^2\right)t^2 - \sqrt{\left(2 + \left(U_y^2 + U_z^2\right)t^2\right)^2 - 4}}{2} \\
&= \frac{2}{2 + \left(U_y^2 + U_z^2\right)t^2 + \sqrt{\left(2 + \left(U_y^2 + U_z^2\right)t^2\right)^2 - 4}}
\end{aligned}$$

So for $t > 0$, we get the estimate

$$\left|\vec{\xi}(t)\right|^2 \geq \frac{1}{2 + \left(U_y^2 + U_z^2\right)t^2}.$$

Thus by (4.13),

$$\left|\left(U_y b_2 + U_z b_3\right)(t)\right| \leq c_1 t^2 + c_2,$$

for $c_1, c_2 > 0$ independent of $\left(\vec{x}_0, \vec{\xi}_0, \vec{b}_0\right)$. By (4.10)-(4.12), we have

$$\left|\dot{b}_1\right| \leq 2\left|\left(U_y b_2 + U_z b_3\right)(t)\right|, \quad \left|\dot{b}_2\right|, \left|\dot{b}_3\right| \leq \left|\left(U_y b_2 + U_z b_3\right)(t)\right|,$$

and thus

$$\left|\vec{b}\,(t)\right| \leq c_3 t^3 + c_4$$

for some constants $c_3, c_4 > 0$. This finishes the proof of the lemma. $\square$

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