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A sharp stability criterion for the Vlasov–Maxwell system

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Abstract. We consider the linear stability problem for a 3D cylindrically symmetric equilibrium of the relativistic Vlasov–Maxwell system that describes a collisionless plasma. For an equilibrium whose distribution function decreases monotonically with the particle energy, we obtained a linear stability criterion in our previous paper [24]. Here we prove that this criterion is sharp; that is, there would otherwise be an exponentially growing solution to the linearized system. We also treat the considerably simpler periodic $1\frac{1}{2}$D case. The new formulation introduced here is applicable as well to the non-relativistic case, to other symmetries, and to general equilibria.

1. Introduction

We consider a plasma at such high temperature or low density that collisions can be ignored compared with the electromagnetic forces. Such a collisionless plasma is modeled by the relativistic Vlasov–Maxwell (RVM) system. We assume all physical constants like the speed of light c and the mass of particles m to be 1, for the sole purpose of simplifying our notation. All the results we obtain below can be modified straightforwardly to apply to the true physical situations with general masses, charges, etc. In the physical literature, the non-relativistic version of the Vlasov–Maxwell system is more commonly considered but our results easily extend to that case. Our notation is as follows. Let $f^{\pm}(t, x, v)$ be the ion and electron distribution functions, $\mathbf{E}(t, x)$ and $\mathbf{B}(t, x)$ be the electric and magnetic fields and $\mathbf{E}^{ext}, \mathbf{B}^{ext}$ be the external fields. Then the RVM system is

$$\text{(1a)}\qquad \partial_t f^{\pm} + \hat{v}\cdot\nabla_x f^{\pm} \pm (\mathbf{E}+\mathbf{E}^{ext}+\hat{v}\times(\mathbf{B}+\mathbf{B}^{ext}))\cdot\nabla_v f^{\pm} = 0,$$

$$\text{(1b)}\qquad \partial_t\mathbf{E} = \nabla\times\mathbf{B}-\mathbf{j},\quad \nabla\cdot\mathbf{E}=\rho,\quad \rho=\int (f^+ - f^-)dv,$$

$$\text{(1c)}\qquad \partial_t\mathbf{B} = -\nabla\times\mathbf{E},\quad \nabla\cdot\mathbf{B}=0,\quad \mathbf{j}=\int \hat{v}(f^+ - f^-)dv,$$

where $\hat{v} = v/\langle v\rangle$ and $\langle v\rangle = \sqrt{1+|v|^2}$. Alternatively, in many physical problems [3,4], a non-neutral plasma is also considered, where there is only a single species of particle.

One of the central problems in the theory of plasmas is to understand plasma stability and instability [26,30]. For example, to control the plasma instability in a fusion device is a key issue for the nuclear fusion program. Many other examples occur in astrophysical contexts. So far, most studies on plasma stability are based on macroscopic MHD models. For such fluid models, the famous *energy principle* was discovered by Bernstein, Frieman, Kruskal and Kulsrud [2] in the 1950s, first for static equilibria and later for symmetry-preserving perturbations of symmetric equilibria [27]. These energy principles allow one to reduce the study of linear stability to checking the positivity of a certain relatively simple quadratic energy form $W(\xi,\xi)$. They have been widely used in the plasma physics community [5,8] to study many types of important plasma instabilities. However, the collision-dominant assumption required in deriving these MHD models from kinetic models is seriously violated in many almost collisionless situations in nuclear fusion [5] and space plasmas [28]. This puts into question the applicability of such energy principles in physical situations where collisions are infrequent. While energy principles have been derived for some simple approximate models, such as collisionless MHD and guiding center models [19,20,9], there have been very few such studies on the more accurate but more complicated microscopic Vlasov–Maxwell models. A good understanding of stability for Vlasov systems could provide a theoretical basis to compare and check the validity of stability results for various approximate plasma models like MHD. Moreover, many plasma instability phenomena have an essentially microscopic nature, for which kinetic models like Vlasov–Maxwell are required [28].

Combining the results of this paper with [24], we have established an energy principle for a large class of symmetric equilibria of various Vlasov–Maxwell systems. More precisely, for a large class of equilibria that enjoy certain kinds of symmetry, the study of linear stability of symmetry-preserving perturbations is reduced to simply checking the positivity of a self-adjoint operator $\mathcal{L}^0$, or equivalently the positivity of the quadratic form $(\mathcal{L}^0\xi,\xi)$. Compared with the usual MHD energy principle, our energy principle has several new features and advantages. In the MHD case, the quadratic energy form $W(\xi,\xi)$ can be written as $(F\xi,\xi)$ where the force operator F has a complicated spectral structure such as gaps in its essential spectrum [8]. It is difficult to analyze, especially in higher dimensions and in non-trivial magnetic field geometries. Our operator $\mathcal{L}^0$ for RVM is essentially an elliptic operator plus a bounded non-local term and thus has a relatively simple spectral structure. This structure allows us to obtain important additional information about the linear instability. For example, we show that the maximal growth rate is controlled by the lowest negative eigenvalue of $\mathcal{L}^0$ and that the number of growing modes equals the number of negative eigenvalues of $\mathcal{L}^0$.

Linear stability under the condition $\mathcal{L}^0 \geq 0$ was proven in [24]. The main result of the present paper is to prove the converse; that is, to construct a growing mode if $\mathcal{L}^0 \ngeq 0$. As in [24] we specifically consider two RVM models, the simpler $1\frac{1}{2}$D periodic case with $x \in \mathbb{R}$, $v \in \mathbb{R}^2$, and the full 3D case in the whole space $\mathbb{R}^3$ with cylindrical symmetry. However, our methods are also applicable to Vlasov–Maxwell models with other symmetries, with boundary conditions, or in a non-relativistic setting, and will yield similar results.

Now we state our main result for the cylindrically symmetric 3D case. As remarked in [24], the existence of a plasma equilibrium of the 3D RVM model in the whole space requires an external field. To simplify notation we consider a 3D non-neutral *electron* plasma with an external field. This scenario does indeed occur in many physical situations [3]. So $f^+ = 0$, and instead of f^- we use the notation f for the electrons. Our equilibrium is cylindrically symmetric with electron distribution $f^0 = \mu(e, p)$, where

$$e = \sqrt{1 + |v|^2} - \phi^0(r, z) - \phi^{ext}(r, z),$$
$$p = r\big(v_\theta - A_\theta^0(r, z) - A_\theta^{ext}(r, z)\big)$$

and with equilibrium fields

$$\mathbf{E}^0 = -\partial_r \phi^0 \mathbf{e}_r - \partial_z \phi^0 \mathbf{e}_z, \quad \mathbf{B}^0 = -\partial_z A_\theta^0 \mathbf{e}_r + \frac{1}{r}\partial_r\big(r A_\theta^0\big)\mathbf{e}_z.$$

In order to be an equilibrium, (A_θ^0, ϕ^0) must satisfy the elliptic system

$$\Delta \phi^0 = \partial_{zz}\phi^0 + \partial_{rr}\phi^0 + \frac{1}{r}\partial_r \phi^0 = \int \mu dv \tag{2}$$

$$\left(\Delta - \frac{1}{r^2}\right) A_\theta^0 = \partial_{zz} A_\theta^0 + \partial_{rr} A_\theta^0 + \frac{1}{r}\partial_r A_\theta^0 - \frac{1}{r^2} A_\theta^0 = \int \hat{v}_\theta \mu dv. \tag{3}$$

Here we use cylindrical coordinates (r, θ, z) and denote by $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ the standard basis. We also assume axisymmetry of the external fields in the form

$$\mathbf{E}^{ext} = -\partial_r \phi^{ext}(r, z)\mathbf{e}_r - \partial_z \phi^{ext} \mathbf{e}_z,$$
$$\mathbf{B}^{ext} = -\partial_z A_\theta^{ext}(r, z)\mathbf{e}_r + \frac{1}{r}\partial_r\big(r A_\theta^{ext}\big)\mathbf{e}_z.$$

We assume that the equilibrium is confined, namely that f^0 has compact support S in phase space. Having compact support is a realistic assumption for a confined plasma. We make the further assumption that f^0 and $\mathbf{E}^0$, $\mathbf{B}^0$ are continuous everywhere, including on the boundary of the support. In [24], with properly chosen external fields, an example of a continuous non-neutral plasma equilibrium with support in a torus was constructed. We also assume that $\partial\mu/\partial e = \mu_e < 0$ inside S. This condition is widely believed to make the equilibrium more likely to be stable [3,9,29]. We study the stability of such an equilibrium under perturbations that preserve cylindrical symmetry.

In order to state our main results, we define certain linear operators acting on cylindrically symmetric scalar functions $h \in L^2(\mathbb{R}^3)$ by

$$\mathcal{A}_1^0 h = -\partial_{zz}h - \partial_{rr}h - \frac{1}{r}\partial_r h - \int \mu_e dv h + \int \mu_e \mathcal{P}(h) dv, \tag{4}$$

$$\mathcal{A}_2^0 h = -\partial_{zz}h - \partial_{rr}h - \frac{1}{r}\partial_r h + \frac{1}{r^2}h - \int \hat{v}_\theta \mu_p dv r h - \int \hat{v}_\theta \mu_e \mathcal{P}(\hat{v}_\theta h) dv, \tag{5}$$

$$\mathcal{B}^0 h = \int \mu_e \mathcal{P}(\hat{v}_\theta h) dv - \int \hat{v}_\theta \mu_e dv h, \tag{6}$$

and

$$\mathcal{L}^0 = (\mathcal{B}^0)^* \left(\mathcal{A}_1^0\right)^{-1} \mathcal{B}^0 + \mathcal{A}_2^0, \tag{7}$$

where $\mathcal{P}$ is the projection operator of $L^2_{|\mu_e|}$ onto $\ker D$. Here D denotes the transport operator associated with the steady fields, namely

$$D = \hat{v} \cdot \nabla_x + (\mathbf{E}^0 + \mathbf{E}^{ext} + \hat{v} \times (\mathbf{B}^0 + \mathbf{B}^{ext})) \cdot \nabla_v$$

and $L^2_{|\mu_e|}$ denotes the $|\mu_e|$-weighted $L^2_{x,v}$ space. It was proven in [24] that these operators are well-defined and that $\mathcal{L}^0$ is self-adjoint. First we recall our previous result in [24].

Theorem 1.1 ([24]). *Consider a non-negative axisymmetric equilibrium* $(f^0, \mathbf{E}^0, \mathbf{B}^0)$ *as above with compact support* S *in phase space. Assume* $\mu_e < 0$ *inside* S. *For axisymmetric perturbations, we have following results.*

(i) $\mathcal{L}^0 \geq 0$ *implies spectral stability. That is, if* $\mathcal{L}^0 \geq 0$ *then there does not exist a growing mode.*

(ii) Any growing mode must be purely growing. That is, if

$$[e^{\lambda t} f(x, v), e^{\lambda t}\mathbf{E}(x), e^{\lambda t}\mathbf{B}(x)] \quad (\mathrm{Re}\lambda > 0)$$

with $\mathbf{E}, \mathbf{B} \in L^2$, $f \in L^1 \cap L^\infty$ *is a solution of the linearized system, then* λ *is real.*

(iii) If $\mathcal{L}^0 \ngeq 0$ *and* $-\alpha^2$ *denotes the lowest negative eigenvalue of the operator* $\mathcal{L}^0$, *then the maximal growth rate* λ *cannot exceed* α.

Theorem 1.1 asserts the linear stability if $\mathcal{L}^0 \geq 0$ and it also estimates the maximal growth rate if $\mathcal{L}^0 \ngeq 0$. However, it leaves open the converse, namely the question of the existence of a growing mode when $\mathcal{L}^0 \ngeq 0$. In this paper, we fill this gap by showing that there indeed always exists a growing mode if $\mathcal{L}^0 \ngeq 0$. This is the main result of the present paper.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1,*

(i) if $\mathcal{L}^0 \ngeq 0$, there exists a growing mode; that is, an exponentially growing weak solution

$$[e^{\lambda t} f(x, v), e^{\lambda t}\mathbf{E}(x), e^{\lambda t}\mathbf{B}(x)] \quad (\lambda > 0)$$

of the linearized problem with $f \in L^1 \cap L^\infty$ and $\mathbf{E}, \mathbf{B} \in H^1$.

(ii) The dimension of the space of symmetry-preserving growing modes equals the dimension of the negative eigenspace of $\mathcal{L}^0$.

The combination of Theorems 1.1 and 1.2, establishes an "energy principle" for this class of equilibria, in terms of the relatively simple operator $\mathcal{L}^0$. Thus this operator $\mathcal{L}^0$ not only provides the sharp stability criterion, but also contains information about the number of unstable modes and their maximal growth rate. The projection $\mathcal{P}$ that occurs in the definition of $\mathcal{L}^0$ is a highly non-local operator since $\mathcal{P}h(x, v)$ turns out to be essentially the average of h in the phase space occupied by the particle trajectory with the steady field $(\mathbf{E}^0 + \mathbf{E}^{ext}, \mathbf{B}^0 + \mathbf{B}^{ext})$ starting at (x, v). So our sharp stability criterion $\mathcal{L}^0 \geq 0$ is also highly *non-local*, which reflects the collective nature of plasma stability. Because of the condition $\mu_e < 0$, it turns out that all the non-local terms are stabilizing.

In [11], Y. Guo investigated the stability of a two-species plasma satisfying 3D RVM without external fields, in a bounded domain with the perfectly conducting boundary condition. In a similar setting to ours, a sufficient condition for stability was obtained in [11] by the energy-Casimir method. Extending the calculations in [11] to the whole space case, we would obtain the stability condition that $L^0 > 0$, where L^0 is the differential operator

$$L^0 = -\partial_{zz} - \partial_{rr} - \frac{1}{r}\partial_r + \frac{1}{r^2} - r\int \hat{v}_\theta \mu_p dv, \tag{8}$$

the last two terms being multiplication operators. However, since $\mathcal{L}^0 > L^0$, the stability criterion $\mathcal{L}^0 \geq 0$ in our Theorem 1.1 is a significant improvement because of the additional stabilizing effects that come from the non-local terms in $\mathcal{L}^0$. More importantly, in the $1\frac{1}{2}$D case discussed below, we showed in [24] that these non-local stabilizing terms are indispensable to prove the stability of any equilibrium, even a homogeneous one. We believe that the non-local stabilizing terms must play an important role in plasma stability in the 3D case as well.

The simplest case that permits a magnetic field is the so-called $1\frac{1}{2}$ dimensional case. In this case, physical space is one-dimensional $x \in \mathbb{R}$

and momentum space is two-dimensional $v = (v_1, v_2) \in \mathbb{R}^2$. Moreover, $\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = (0, 0, B)$. We consider solutions that are periodic in x and we may assume that there is no external field. In Sect. 2, before going on to the proofs of Theorems 1.1 and 1.2 in 3D, we prove precise analogues of our theorems for this much simpler case.

Our condition that $\mathcal{L}^0 \geq 0$ or that $\mathcal{L}^0 \ngeq 0$ can be verified in several important cases. The simplest one is the purely 1D case for a homogeneous equilibrium (f^0 depending only on v), for which there is an explicit dispersion relation and the celebrated Penrose criterion. However, even for a homogeneous equilibrium for which magnetic effects are included, the problem becomes quite complicated. In fact, in [17] and in Sect. 4.1 of [24], the stability criterion is worked out explicitly in terms of inequalities on four weighted integrals of derivatives of μ.

In Sects. 4.2–4.4 of [24], we explicitly work out our stability criterion for a periodic purely magnetic equilibrium in $1\frac{1}{2}$D that is a small perturbation of a homogeneous equilibrium, with minimal period P. There we prove *instability* of any such periodic equilibrium under perturbations of period $2P$. Furthermore, we prove the *stability* of a particular class of equilibria under perturbations of period P. More specifically, we construct an example of the form $\mu^+ = \mu^- = \sigma(\langle v \rangle)(1 + p^2)$ that is unstable under perturbations of period $2P$ but stable under perturbations of period P. Thus perturbations of longer wavelength are more likely to induce instability. The non-local term in $\mathcal{L}^0$ plays a crucial role in these explicit results. In [25], we also prove the validity of these stability and instability results on the *non-linear* dynamical level.

Of course, in general an explicit verification of the location of the spectrum of a self-adjoint operator is not an easy task. Our result reduces a complicated eigenvalue problem in the 6D phase space to the study of a self-adjoint elliptic operator in the 3D physical space, and thus is much easier to implement in the numerical study of plasma stability. Moreover, our sharp criterion can be used to further derive simpler stability criteria in some important physical regimes, such as the practical case of very large external magnetic fields in the fusion reactors.

We now sketch the main ideas in the proofs of Theorems 1.2 and its $1\frac{1}{2}$ dimensional analogue, which are concerned with the construction of growing modes provided that $\mathcal{L}^0 \ngeq 0$. We begin with some brief historical comments on linear instability for Vlasov systems. One of the main difficulties in studying Vlasov instability is its collective and thus highly non-local nature. In the physics literature, most classical studies [26,30] treat homogeneous equilibria with vanishing electric and magnetic fields, in which case an explicit algebraic dispersion relation is usually available. However, any non-trivial electromagnetic field will make the dispersion relations much more difficult to analyze because they depend upon some complicated trajectory integrals. In [13] and later publications [14–16], Guo and Strauss developed a perturbation approach to prove the instability of some weakly *inhomogeneous* equilibria of Vlasov systems. They proved the instability of various electromagnetic structures that are close to an unstable homogeneous equilibrium. In [22] Lin developed a new non-perturbative approach to find purely growing modes for highly inhomogeneous equilibria of 1D Vlasov–Poisson. This approach has recently been used [12] as well for galaxy models satisfying 3D Vlasov–Poisson. There are two elements in this approach. The first is to formulate a family of dispersion operators for the electric potential, depending on a positive parameter λ. The second is to prove the existence of a purely growing mode by finding a parameter λ_0 for which the dispersion operator has a non-trivial kernel. The key observation is that these dispersion operators are self-adjoint due to the reversibility of the particle trajectories. A continuity argument is applied to find the parameter λ_0 corresponding to a growing mode, by comparing the spectra of the dispersion operators for very small and very large values of λ.

Let us explain the difficulties in extending this approach to the full *electromagnetic* case. We first recall the method in [24] for the periodic $1\frac{1}{2}$D case. Assuming that the growing mode has periodic electromagnetic potentials (ϕ, ψ) such that $E_1 = -\partial_x\phi$, $B = \partial_x\psi$, $E_2 = -\partial_t\psi$, we express f in term of them by integrating along the trajectories. Plugging f into the Maxwell system and using the condition $\mu_e < 0$ to eliminate ϕ, we get a self-adjoint dispersion operator for ψ alone. Then we apply the continuity argument as in [22]. The difficulty with this approach is that the equation

$$\partial_t E_1 = -j_1 \tag{9}$$

(the first current equation of Maxwell) does not follow from the dispersion operator. What we did in [24], under additional evenness assumptions in the variable x, was to prove by means of a parity argument that j_1 has zero mean. Then (9) does indeed follow from the Poisson equation

$$\partial_x E_1 = \rho$$

and we get a growing mode.

In order to make this construction in the $1\frac{1}{2}$D case without any evenness assumption, we need a new formulation. To do this, we express $E_1 = -\partial_x\phi - \lambda b$ where the scalar b is introduced to account for the possible non-zero spatial average of E_1 and λ is the exponential growth rate. Once again we express f in terms of (ϕ, ψ, b) by integration over the trajectories and plug it into the Maxwell system. Equation (9) can now be handled by means of this additional number b. Again we eliminate ϕ using the condition $\mu_e < 0$ and the resulting equations for ψ and b can be written in a self-adjoint matrix operator form. We then apply the continuity argument to this new dispersion matrix by keeping track of its negative spectrum.

In the axisymmetric 3D case the proof of Theorem 1.2 is much more subtle. We start with the electric potential ϕ and the magnetic potential $\mathbf{A} = (A_r, A_\theta, A_z)$. Of course we define $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Our strategy is to represent f in terms of $(\phi, \mathbf{A})$ and plug it into the Maxwell system to get a *self-adjoint* formulation for the electromagnetic potentials.

Furthermore, we have the same difficulty as mentioned above that the current equation

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{j}$$

does not follow from the construction. To surmount this difficulty as well as achieve the self-adjointness, we impose the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ and use the cylindrical symmetry to define a "super-potential" $\pi(r, z)$ such that $\nabla \times (\pi \mathbf{e}_\theta) = (A_r, 0, A_z)$. This super-potential is an essential part of our construction. The introduction of this "super-potential" π allows us to separate the θ and (r, z) components of the current equation (40). The resulting system for (ϕ, A_θ, π) indeed turns out to be self-adjoint.

We are looking for a growing mode solution $[e^{\lambda t} f(x, v), e^{\lambda t}\mathbf{E}(x), e^{\lambda t}\mathbf{B}(x)]$ for some $\lambda > 0$. We express f in terms of (ϕ, A_θ, π) by integrating along the trajectories (characteristics) of the equilibrium to get formula (43) in Sect. 3. This formula involves the non-local operator

$$(\mathcal{Q}^\lambda k)(x, v) = \int_{-\infty}^{0} \lambda e^{\lambda s} k(X(s; x, v), V(s; x, v)) ds$$

where (X, V) denotes the trajectories. Plugging the formula for f into the Maxwell system, we get the system (45), (47), (48) of three equations for the unknowns ϕ, A_θ and π. They are conveniently written in terms of several linear operators $\mathcal{A}_1^\lambda$, $\mathcal{A}_2^\lambda$, $\mathcal{A}_3^\lambda$, $\mathcal{B}^\lambda$, $\mathcal{C}^\lambda$, $\mathcal{D}^\lambda$ which involve $\mathcal{Q}^\lambda$ and are defined in Sect. 3. We then eliminate ϕ using the the condition $\mu_e < 0$ and (45) to get a 2×2 self-adjoint matrix operator $\mathcal{M}^\lambda$ for (A_θ, π), depending on a positive parameter λ:

$$\mathcal{M}^\lambda \begin{pmatrix} A_\theta \\ \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathcal{M}^\lambda = \begin{pmatrix} \mathcal{L}^\lambda & (\mathcal{F}^\lambda)^* \\ \mathcal{F}^\lambda & -\mathcal{A}_4^\lambda \end{pmatrix},$$

where the operators $\mathcal{L}^\lambda$, $\mathcal{F}^\lambda$ and $\mathcal{A}_4^\lambda$ are certain combinations of the other operators.

The continuity argument consists of showing that a portion of the spectrum of $\mathcal{M}^\lambda$ moves from negative to positive values as λ increases from 0 to ∞ and therefore there exists a $\lambda > 0$ such that $\mathcal{M}^\lambda$ has a non-trivial kernel. To accomplish this, we count the number of its negative eigenvalues. However, this matrix operator is bounded neither from below nor from above, so the continuity argument cannot be applied directly.

In fact, $-\mathcal{A}_4^\lambda$ in the lower right corner of $\mathcal{M}^\lambda$ has an infinite-dimensional negative spectrum. We perform an n-dimensional truncation using a projection operator P_n; see Sect. 5. The truncated operator $\mathcal{M}_n^\lambda$ has entries that are high-order integro-differential operators. It is bounded from below by a bound depending on n. We prove that it has at least $n + 1$ negative eigenvalues for some small $\lambda > 0$ and has at most n negative eigenvalues for some large $\lambda < \infty$. In order to accomplish the former statement (λ small), we have to take the limit as $\lambda \searrow 0$. In the upper left corner of $\mathcal{M}^\lambda$ we have $\mathcal{L}^\lambda$, whose formal limit is what we have called $\mathcal{L}^0$. The main assumption is that $\mathcal{L}^0$ has a negative eigenvalue. This involves the limit $\mathcal{Q}^\lambda \to \mathcal{P}$, where the operator $\mathcal{P}$ takes the phase-space average over trajectories. (See Lemma 4.1(b) and the remark that follows it.) Thus by continuity $\mathcal{M}_n^{\lambda_n}$ has a non-trivial kernel for some λ_n in between (see Sect. 6). This provides an approximate growing mode.

In Sect. 7 we let $n \to \infty$. The limit of the approximate growing mode is shown to satisfy the original linearized Vlasov–Maxwell system weakly. However, it is still very subtle to show that this limit indeed gives us a true growing mode. There are two issues to clarify. The first is to show that the limit does not vanish, for which we need a uniform bound of the approximate growing modes. The second issue is to show that the growth rate λ_n does not tend to zero as n goes to infinity. For this, we need to get uniform control of the spectrum of $\mathcal{M}_n^\lambda$ for small λ and large n. This turns out to be quite delicate since the operators involved merely converge to their limits weakly as $\lambda \searrow 0$. In our proof the compactness of the support of the confined plasma equilibria plays a crucial role, allowing us to get some compactness of the operators.

As for Theorem 1.2(ii), the lower bound on the number of growing modes is a corollary of the continuity argument. To get the upper bound, the key observation is that any two growing modes are orthogonal in some sense due to a certain invariance property proven in [24]. We note that such counting formulae are unknown for the standard energy principles [2,20] for approximate plasma models like MHD. In our case the simple spectral structure of the operator $\mathcal{L}^0$ is essential.

The new formulation and techniques developed in this paper can also be used to detect linear instability of general Vlasov–Maxwell equilibria without the monotone assumption $\mu_e < 0$. The idea is to formulate the growing mode problem as a 3×3 indefinite matrix dispersion operator including ϕ and then to use the truncation and continuity arguments to study it. In this way we find a sufficient instability criterion by utilizing the difference of the signatures of the matrix operators at small and large parameters. We illustrate this idea in Sect. 9 by getting a instability criterion in $1\frac{1}{2}$D purely magnetic case that generalizes the sharp criterion in the monotone case.

The methods of this paper and of [24] can also be used for *non-relativistic* Vlasov–Maxwell systems and also for other symmetries, for example, the $2\frac{1}{2}$D Vlasov–Maxwell system with its z-symmetry [7]. For such cases, but still assuming that the distribution function depends monotonically on the particle energy, we can establish similar energy principles in terms of a certain self-adjoint operator $\mathcal{L}^0$. For the non-relativistic case the operator $\mathcal{L}^0$ is formally obtained from its relativistic version by dropping the hat in $\hat{v}$. Since the results and the proofs are similar to the cases we treat here, we do not elaborate any further.

The paper is organized as follows. In Sect. 2, we treat the easier $1\frac{1}{2}$D case. The proof for the 3D case is split into six sections. In Sect. 3, we formulate the problem using (ϕ, A_θ, π) and derive the dispersion matrix operator $\mathcal{M}^\lambda$ for (A_θ, π). In Sect. 4, we present the key mapping and spectral properties of the operators appearing in the formulation. In Sect. 5, we study their behavior for small λ and introduce the finite-dimensional truncation. In Sect. 6, we find the approximate growing mode for each n. In Sect. 7, we take the limit of the approximate growing modes. In Sect. 8, we check that this limit is indeed a true growing mode. In Sect. 9, we extend our formulation to equilibria that are not monotone in the energy e.

2. $1\frac{1}{2}$ dimensional case

In this case, the physical space is one-dimensional $x \in \mathbb{R}$ and the momentum space is two-dimensional $v = (v_1, v_2) \in \mathbb{R}^2$. Moreover, $\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = (0, 0, B)$. Assuming no external fields and setting all physical constants to be 1, system (1) reduces to the following $1\frac{1}{2}$D RVM system

$$\partial_t f^\pm + \hat{v}_1 \partial_x f^\pm \pm (E_1 + \hat{v}_2 B)\partial_{v_1} f^\pm \pm (E_2 - \hat{v}_1 B)\partial_{v_2} f^\pm = 0 \tag{10a}$$

$$\partial_t E_1 = -j_1, \quad \partial_t E_2 + \partial_x B = -j_2 \tag{10b}$$

$$\partial_t B = -\partial_x E_2, \quad \partial_x E_1 = \rho \tag{10c}$$

with

$$\rho = \int (f^+ - f^-)dv, \quad j_i = \int \hat{v}_i (f^+ - f^-)dv \quad (i = 1, 2).$$

The main reason to consider $1\frac{1}{2}$D RVM is its simplicity and yet it preserves many of the essential features of 3D RVM. We refer to [28] for astrophysical applications of this model and to [6] for a proof of global well-posedness. We will consider solutions of the system (10) that are periodic in the variable x with a given period P.

First we take a P-periodic equilibrium

$$f^{0,\pm} = \mu^\pm(e^\pm, p^\pm) = \mu^\pm\big(\langle v\rangle \pm \phi^0(x), v_2 \pm \psi^0(x)\big), \qquad E_1^0 = -\partial_x\phi^0, \quad E_2^0 = 0, \quad B^0 = \partial_x\psi^0, \tag{11}$$

where the pair (ϕ^0, ψ^0) satisfies the ODE system

$$\partial_x^2\phi^0 = -\rho^0 = -\int (f^{0,+} - f^{0,-})dv, \qquad \partial_x^2\psi^0 = -j_2^0 = -\int \hat{v}_2(f^{0,+} - f^{0,-})dv. \tag{12}$$

We assume that

$$\mu^\pm \geq 0, \quad \mu^\pm \in C^1, \quad \mu_e^\pm < 0, \quad \left|\mu_e^\pm\right| + \left|\mu_p^\pm\right| \leq c(1 + |e|)^{-\alpha} \tag{13}$$

for some $\alpha > 2$. In [24] we proved that there exist infinitely many periodic electromagnetic equilibria of the above form. Now we denote

$$D^{\pm} = \hat{v}_1 \partial_x \pm \left(E_1^0 + \hat{v}_2 B^0\right)\partial_{v_1} \mp \hat{v}_1 B^0 \partial_{v_2},$$

$$L^2_{|\mu_e^{\pm}|} = \left\{ f \mid f \text{ periodic in } x,\ \|f\|_{\pm}^2 \equiv \int_0^P \int_{-\infty}^{\infty} |f|^2 \left|\mu_e^{\pm}\right| dv dx < \infty \right\},$$

and $\mathcal{P}^{\pm}$ = the projection operator of $L^2_{|\mu_e^{\pm}|}$ onto $\ker D^{\pm}$. We define the following linear operators acting on $L^2_P(\mathbb{R})$, where the subscript P refers to the periodicity.

(14) $$\mathcal{A}_1^0 h = -\partial_x^2 h - \Big(\sum_{\pm} \int \mu_e dv\Big) h + \sum_{\pm} \int \mu_e^{\pm} \mathcal{P}^{\pm} h dv.$$

(15) $$\mathcal{A}_2^0 h = -\partial_x^2 h - \Big(\sum_{\pm} \int \hat{v}_2 \mu_p^{\pm} dv\Big) h - \sum_{\pm} \int \mu_e^{\pm} \hat{v}_2\, \mathcal{P}^{\pm}(\hat{v}_2 h) dv.$$

(16) $$\mathcal{B}^0 h = \Big(\sum_{\pm} \int \mu_p^{\pm} dv\Big) h + \sum_{\pm} \int \mu_e^{\pm} \mathcal{P}^{\pm}(\hat{v}_2 h) dv$$

and

(17) $$\mathcal{L}^0 = (\mathcal{B}^0)^* \left(\mathcal{A}_1^0\right)^{-1} \mathcal{B}^0 + \mathcal{A}_2^0.$$

Similarly to the 3D case, we proved in [24] the following theorem.

Theorem 2.1. *Consider periodic perturbations of any equilibrium satisfying the conditions given above. Then*

(i) $\mathcal{L}^0 \geq 0$ *implies spectral stability.*
(ii) Any growing mode must be purely growing.
(iii) If $-\alpha^2$ *denotes the lowest eigenvalue of the operator* $\mathcal{L}^0$*, then the maximal growth rate cannot exceed* α*.*

Moreover, it was shown in [24] that if ψ^0, ϕ^0 are even functions of x around $x = P/2$ and if $\mathcal{L}^0$ has an *even* eigenfunction corresponding to a negative eigenvalue, then there exists a growing mode. In the following theorem proven in this section, we assert that $\mathcal{L}^0 \ngeq 0$ always implies the existence of a growing mode, without any additional evenness assumptions.

Theorem 2.2. *Under the same assumptions,*

(i) If $\mathcal{L}^0 \ngeq 0$*, then there exists a real periodic growing mode* $[e^{\lambda t} f(x, v), e^{\lambda t} E(x), e^{\lambda t} B(x)]$ *with* $f, E, B \in W_P^{1,1}$ *and* $\lambda > 0$*, where* $E = (E_1, E_2)$*.*
(ii) The dimension of the space of growing modes equals the dimension of the negative eigenspace of $\mathcal{L}^0$*.*

The combination of Theorems 2.1 and 2.2 provides an energy principle for the $1\frac{1}{2}$D case, in terms of the operator $\mathcal{L}^0$.

With the sole purpose of simplifying our notation, we present the proof in the case of a constant ion background n_0. (For the more general two-species case, the proofs remain almost the same except for the more cumbersome notation.) Then the $1\frac{1}{2}$D RVM for one species becomes

$$\partial_t f + \hat{v}_1 \partial_x f - (E_1 + \hat{v}_2 B)\partial_{v_1} f - (E_2 - \hat{v}_1 B)\partial_{v_2} f = 0 \tag{18a}$$

$$\partial_t E_1 = -j_1 = \int \hat{v}_1 f dv, \quad \partial_t B = -\partial_x E_2 \tag{18b}$$

$$\partial_t E_2 + \partial_x B = -j_2 = \int \hat{v}_2 f dv \tag{18c}$$

with the constraint

$$\partial_x E_1 = n_0 - \int f dv. \tag{19}$$

Fixing any such equilibrium with a period P, we will consider the system (21) with periodic boundary conditions of the same period P.

The equilibrium is assumed to have the form $f^0 = \mu(e, p)$, $E_1^0 = -\partial_x \phi^0$, $E_2^0 = 0$, $B^0 = \partial_x \psi^0$, where the electromagnetic potentials (ϕ^0, ψ^0) satisfy the ODE system

$$\partial_x^2 \phi^0 = n_0 - \int \mu(e, p) dv, \quad \partial_x^2 \psi^0 = \int \hat{v}_2 \mu(e, p) dv$$

with the electron energy and the "angular momentum" defined by

$$e = \langle v \rangle - \phi^0(x), \quad p = v_2 - \psi^0(x). \tag{20}$$

(The e is distinguished from the exponential e in context.) The only assumptions we make on μ are

$$\mu \geq 0, \quad \mu \in C^1, \quad \mu_e \equiv \frac{\partial \mu}{\partial e} < 0 \tag{21}$$

and, in order for $\int (|\mu_e| + |\mu_p|) dv$ to be finite,

$$(|\mu_e| + |\mu_p|)(e, p) \leq c(1 + |e|)^{-\alpha} \text{ for some } \alpha > 2. \tag{22}$$

Hence the linearized evolution equations are

$$(\partial_t + D) f = \mu_e \hat{v}_1 E_1 - \mu_p \hat{v}_1 B + (\mu_e \hat{v}_2 + \mu_p) E_2, \tag{23}$$

where D is the transport operator associated with the steady fields,

$$D = \hat{v}_1 \partial_x - \left(E_1^0 + \hat{v}_2 B^0\right)\partial_{v_1} + \hat{v}_1 B^0 \partial_{v_2}$$

together with

$$(24)\qquad \begin{aligned} \partial_x E_1 = -\int f dv, \quad \partial_t E_1 = \int \hat{v}_1 f dv, \\ \partial_t E_2 + \partial_x B = \int \hat{v}_2 f dv, \quad \partial_t B + \partial_x E_2 = 0. \end{aligned}$$

We define the Hilbert space

$$L^2_{|\mu_e|} = \left\{ f(x, v) \;\middle|\; f \ P\text{-periodic in } x, \ \|f\|^2_{|\mu_e|} \equiv \int_0^P \int_{\mathbb{R}^2} |f|^2 |\mu_e| dv dx < \infty \right\}$$

and denote its inner product by $(\cdot, \cdot)_{|\mu_e|}$. Let $\mathcal{P}$ be the projection operator of $L^2_{|\mu_e|}$ onto the kernel of D. We also denote by $L^p_P(H^2_P)$ the space of P-periodic $L^p_x(H^2_x)$ functions for $p \geq 1$.

Similarly to the two-species case, we define the following four operators, each of which acts from H^2_P to L^2_P,

$$\begin{aligned} \mathcal{A}_1^0 h &= -\partial_x^2 h - \Big(\int \mu_e dv \Big) h + \int \mu_e \mathcal{P} h dv, \\ \mathcal{A}_2^0 h &= -\partial_x^2 h - \Big(\int \hat{v}_2 \mu_p dv \Big) h - \int \mu_e \hat{v}_2 \mathcal{P}(\hat{v}_2 h) dv, \\ \mathcal{B}^0 h &= \Big(\int \mu_p dv \Big) h + \int \mu_e \mathcal{P}(\hat{v}_2 h) dv \end{aligned}$$

and

$$\mathcal{L}^0 = (\mathcal{B}^0)^* \big(\mathcal{A}_1^0\big)^{-1} \mathcal{B}^0 + \mathcal{A}_2^0.$$

In these definitions one should keep in mind that $\mu \geq 0$ is a function of x and v, that $\mu_e = \partial\mu/\partial e < 0$ and that $\mu_p = \partial\mu/\partial p$. It was shown in [24] that $\mathcal{A}_1^0$ is invertible on the range of $\mathcal{B}^0$ so that $\mathcal{L}^0$ is well-defined. The following is the analogue of Theorem 2.2.

Theorem 2.3. *Assume* (21) *and* (22).

(i) If $\mathcal{L}^0 \not\geq 0$, then there exists a real growing mode $[e^{\lambda t} f(x, v), e^{\lambda t} E(x), e^{\lambda t} B(x)]$ with $f, E, B \in W^{1,1}$ and $\lambda > 0$.
(ii) The dimension of the space of growing modes equals the dimension of the negative eigenspace of $\mathcal{L}^0$.

For the proof of this theorem we introduce the particle paths $(X(t; x, v), V(t; x, v))$, which are the characteristics of D. They are defined as the solutions of

$$(25)\qquad \dot{X} = \hat{V}_1, \quad \dot{V}_1 = \partial_x \phi^0(X) - \hat{V}_2 \partial_x \psi^0(X), \quad \dot{V}_2 = \hat{V}_1 \partial_x \psi^0(X)$$

with the initial conditions $X(0) = x$, $V(0) = v$. Using the particle paths, the next three operators depending on a parameter $\lambda > 0$ were already introduced in [24]

$$\mathcal{A}_1^\lambda h = -\partial_x^2 h - \Big(\int \mu_e dv\Big)h + \int \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} h(X(s))dsdv,$$

$$\mathcal{A}_2^\lambda h = -\partial_x^2 h + \lambda^2 h - \Big(\int \hat{v}_2 \mu_p dv\Big)h - \int \hat{v}_2 \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} \hat{V}_2(s) h(X(s))dsdv,$$

$$\mathcal{B}^\lambda h = \Big(\int \mu_p dv\Big)h + \int \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} \hat{V}_2(s) h(X(s))dsdv.$$

The following lemma in [24] shows that $\mathcal{A}_1^\lambda$ is invertible on the range of $\mathcal{B}^\lambda$, so that the operator

$$\mathcal{L}^\lambda = (\mathcal{B}^\lambda)^* \big(\mathcal{A}_1^\lambda\big)^{-1} \mathcal{B}^\lambda + \mathcal{A}_2^\lambda.$$

is also well-defined.

Lemma 2.4 **([24]).** *Assume* $\lambda \geq 0$.

(i) *The operators* $\mathcal{A}_j^\lambda$, $\mathcal{L}^\lambda$ $(j = 1, 2)$ *are self-adjoint on* L_P^2 *with the common domain* H_P^2. *Their spectra are discrete.*
(ii) $\mathcal{A}_1^\lambda \geq 0$.
(iii) *The null-space* $N(\mathcal{A}_1^\lambda)$ *consists of the constant functions. The inverse* $(\mathcal{A}_1^\lambda)^{-1}$ *is bounded from* $\{h \in L_P^2 \mid \int_0^P h dx = 0\} = N(\mathcal{A}_1^\lambda)^\perp \supset R(\mathcal{B}^\lambda)$ *into* H_P^2.

We also introduce the following three functions that depend on $\lambda > 0$.

$$b^\lambda(x) = \int \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} \hat{V}_1(s) dsdv,$$

$$c^\lambda(x) = \int \hat{v}_2 \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} \hat{V}_1(s) dsdv,$$

$$d^\lambda = (\mathcal{B}^\lambda)^* \big(\mathcal{A}_1^\lambda\big)^{-1} b^\lambda - c^\lambda$$

and three constants

$$l^\lambda = \frac{1}{P} \int_0^P \int \hat{v}_1 \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} \hat{V}_1(s) dsdvdx,$$

$$m^\lambda = \frac{1}{P}\Big(\big(\mathcal{A}_1^\lambda\big)^{-1} b^\lambda, b^\lambda\Big), \quad k^\lambda = P(\lambda^2 - l^\lambda - m^\lambda).$$

Define $\mathcal{F}^\lambda$ to be the operator from $\mathbb{R}$ to L_P^2 by $\mathcal{F}^\lambda(b) = bd^\lambda$. Its adjoint $(\mathcal{F}^\lambda)^*$ mapping L_P^2 to $\mathbb{R}$ is defined by $\mathcal{F}^\lambda(\psi) = (\psi, d^\lambda)$. We define the matrix operator $\mathcal{M}^\lambda$ from $H_P^2 \times \mathbb{R}$ to $L_P^2 \times \mathbb{R}$ by

$$\mathcal{M}^\lambda \begin{pmatrix} \psi \\ b \end{pmatrix} = \begin{pmatrix} \mathcal{L}^\lambda \psi + bd^\lambda \\ -bk^\lambda + (\psi, d^\lambda) \end{pmatrix} = \begin{pmatrix} \mathcal{L}^\lambda & \mathcal{F}^\lambda \\ (\mathcal{F}^\lambda)^* & -k^\lambda \end{pmatrix} \begin{pmatrix} \psi \\ b \end{pmatrix}.$$

By Lemma 2.4, it is obvious that $\mathcal{M}^\lambda$ is self-adjoint and has only discrete spectrum. The following lemma explains the purpose of $\mathcal{M}^\lambda$.

Lemma 2.5. *If $\mathcal{M}^\lambda$ has a non-trivial nullspace of even functions for some $\lambda > 0$, then there exists a purely growing mode in $W^{1,1}$ of* (23), (24).

To clarify the ideas, below we present our original derivation of the matrix operator $\mathcal{M}^\lambda$ from the equations satisfied by a growing mode. The proof of Lemma 2.5 is almost the reverse process of this derivation, as in the proof of Lemma 2.5 of [24]. So we skip it here.

To derive $\mathcal{M}^\lambda$, we start with a growing mode $[e^{\lambda t} f(x, v), e^{\lambda t} E(x), e^{\lambda t} B(x)]$. Since it was shown in [24] that a growing mode must be purely growing, we can assume $\lambda > 0$. Define the electromagnetic potentials ϕ, ψ and a number $b \in \mathbb{R}$ such that

$$B = \partial_x \psi, \quad E_2 = -\lambda \psi, \quad E_1 = -\partial_x \phi - \lambda b.$$

Then $[f(x, v), \phi, \psi, b]$ must satisfy

$$\lambda f + Df = -\mu_e \hat{v}_1 \partial_x \phi - \lambda b \mu_e \hat{v}_1 - \mu_p \hat{v}_1 \partial_x \psi - (\lambda \mu_e \hat{v}_2 + \lambda \mu_p) \psi \tag{26}$$

and

$$\partial_x E_1 = \rho, \quad \lambda E_1 = -j_1, \quad \lambda E_2 + \partial_x B = -j_2, \quad \lambda B + \partial_x E_2 = 0 \tag{27}$$

with $\rho = -\int f dv$ and $j_i = -\int \hat{v}_i f dv$. Integrating (26) along the particle trajectory, after an integration by parts we have

$$\begin{aligned} f(x, v) = & -\mu_e \phi(x) - \mu_p \psi(x) \\ & + \mu_e \int_{-\infty}^0 \lambda e^{\lambda s} [\phi(X(s)) - \hat{V}_2(s) \psi(X(s)) - b \hat{V}_1(s)] ds. \end{aligned} \tag{28}$$

The first and third equations of (27) are equivalent to $-\partial_x^2 \phi = \rho$ and $(-\partial_x^2 + \lambda^2) \psi = j_2$. After plugging (28) into them, they become

$$\mathcal{A}_1^\lambda \phi = \mathcal{B}^\lambda \psi + bb^\lambda \tag{29}$$

and

$$\mathcal{A}_2^\lambda \psi = -(\mathcal{B}^\lambda)^* \phi + bc^\lambda. \tag{30}$$

The last equation in (27) is automatic.

The second equation in (27) is $\lambda E_1 = -j_1$, from which we will now derive an equation for b. By the continuity equation $\partial_x j_1 + \lambda \rho = 0$, we have $\partial_x^2\phi = -\rho = \frac{1}{\lambda}\partial_x j_1$, which implies that $\partial_x\phi = \frac{1}{\lambda}(j_1 - \frac{1}{P}\int_0^P j_1 dx)$. Thus $\lambda E_1 = -j_1$ is equivalent to $\lambda^2 b = \frac{1}{P}\int_0^P j_1 dx$. Plugging (28) into this result, we obtain

$$\lambda^2 b = \frac{1}{P}\int_0^P \int \hat{v}_1 \mu_e \int_{-\infty}^0 \lambda e^{\lambda s}\{-\phi(X(s)) + b\hat{V}_1(s) + \hat{V}_2(s)\psi(X(s))\} ds dv dx$$
$$= I + II + III.$$

The first term is

$$I = -\frac{1}{P}\int_{-\infty}^0 \lambda e^{\lambda s}\int_0^P \int \mu_e \phi(x)\hat{V}_1(-s) dv dx ds$$
$$= \frac{1}{P}\int_{-\infty}^0 \lambda e^{\lambda s}\int_0^P \int \mu_e \phi(x)\hat{V}_1(s) dv dx ds = \frac{1}{P}(\phi, b^\lambda),$$

where for the first equality we changed variables $(x, v) \to (X(-s), \hat{V}(-s))$ and for the second equality we changed variable $v \to -v$ and used the trajectory property

$$(X(-s; x, -v_1, v_2), -V_1(-s; x, -v_1, v_2), V_2(-s; x, -v_1, v_2))$$
$$= (X(s; x, v_1, v_2), V_1(s; x, v_1, v_2), V_2(s; x, v_1, v_2)).$$

Similarly, $III = -\frac{1}{P}(\psi, c^\lambda)$. By definition, $II = bl^\lambda$. Thus the equation for b is

$$(\lambda^2 - l^\lambda)b = \frac{1}{P}[(\phi, b^\lambda) - (\psi, c^\lambda)]. \tag{31}$$

By (29) we get

$$\phi = \left(\mathcal{A}_1^\lambda\right)^{-1}\mathcal{B}^\lambda\psi + b\left(\mathcal{A}_1^\lambda\right)^{-1}b^\lambda.$$

Plugging this into (30) and (31), we have the pair of equations $\mathcal{L}^\lambda\psi + bd^\lambda = 0$ and $-bk^\lambda + (\psi, d^\lambda) = 0$ by definition of d^λ, k^λ and $\mathcal{L}^\lambda$. That is, the pair (ψ, b) belongs to the kernel of the matrix operator $\mathcal{M}^\lambda$. We note that in the above formulation the equation $\lambda E_1 = -j_1$ is exactly taken care of by the extra constant b.

In a similar way to the proof of Lemma 2.5 of [24], we can show that a non-trivial kernel of $\mathcal{M}^\lambda$ indeed gives a growing mode. Moreover, we also showed in [25] that for any growing mode, $f \in W^{1,1}$ and the linear instability implies non-linear instability in the macroscopic sense.

Lemma 2.6. *If $\mathcal{L}^0 \not\geq 0$, then there exists $\lambda > 0$ such that $\mathcal{M}^\lambda$ has a non-trivial nullspace.*

Proof. Let n^λ be the dimension of the eigenspace of $\mathcal{M}^\lambda$ corresponding to its negative eigenvalues. We first claim that for sufficiently large λ, $n^\lambda \le 1$. Indeed, it is shown in [24] that $\mathcal{L}^\lambda \ge \lambda^2 - C_0$ for some constant C_0 independent of λ. It is also easy to show that $\|d^\lambda\|_{L^2} \le C_1$ for some constant C_1 independent of λ, as in the proof of Lemma 2.4 of [24]. So

$$\begin{aligned}\left\langle \mathcal{M}^\lambda \begin{pmatrix}\psi\\ b\end{pmatrix}, \begin{pmatrix}\psi\\ b\end{pmatrix}\right\rangle &= (\mathcal{L}^\lambda\psi,\psi) + 2b(\psi, d^\lambda) - k^\lambda b^2\\ &\ge (\lambda^2 - C_0)\|\psi\|_2^2 - 2C_1|b|\|\psi\|_2 - |k^\lambda|b^2\\ &\ge -\left(C_1^2 + |k^\lambda|\right)b^2,\end{aligned}$$

provided $\lambda^2 \ge C_0 + 1$. Since $b \in \mathbb{R}$, it follows that $\mathcal{M}^\lambda$ has at most a one-dimensional negative subspace. We now show that if λ is small enough, then $n^\lambda \ge 2$. It is shown in [24] that $\mathcal{L}^\lambda \to \mathcal{L}^0$ strongly when $\lambda \searrow 0$ and

$$\lim_{\lambda\searrow 0}\int_{-\infty}^{0} \lambda e^{\lambda s} h(X(s), V(s))ds = \mathcal{P}h$$

in the norm of $L^2_{|\mu_e|}$ for all $h \in L^2_{|\mu_e|}$. As in the proof of Lemma 3.3 of [24], the projection operator $\mathcal{P}$ maps a function that is odd or even in v_1 to another function with the same symmetry property. So as $\lambda \searrow 0$, $b^\lambda \to \int \mu_e \mathcal{P}(\hat{v}_1)dv = 0$ and similarly $c^\lambda \to 0$ in L^2_P strongly. Thus $d^\lambda \to 0$ and $\mathcal{F}^\lambda \to 0$ in L^2_P strongly as $\lambda \to 0$. So we have

$$\mathcal{M}^\lambda \begin{pmatrix}\psi\\ b\end{pmatrix} \to \mathcal{M}^0 \begin{pmatrix}\psi\\ b\end{pmatrix} = \begin{pmatrix}\mathcal{L}^0 & 0\\ 0 & -k^0\end{pmatrix}\begin{pmatrix}\psi\\ b\end{pmatrix}$$

strongly in $L^2_P \times \mathbb{R}$ as $\lambda \to 0$ for all $\psi \in H^2_P$ and $b \in \mathbb{R}$. Here

$$k^0 = \int_0^P\int |\mu_e|(\mathcal{P}(\hat{v}_1))^2 dvdx > 0.$$

Since $\mathcal{L}^0$ has at least one negative eigenvalue by assumption, $\mathcal{M}^0$ has at least two negative eigenvalues. Thus by [18, IV-3.5], $n^\lambda \ge 2$ if λ is small enough.

For $\lambda > 0$, it was shown in [24] that $\mathcal{L}^\lambda$ is continuous in the operator norm. So $\mathcal{M}^\lambda$ is also continuous in the operator norm for $\lambda > 0$. Thus if $\mathcal{M}^\lambda$ has no kernel for all $\lambda > 0$, then n^λ remains a constant which is inconsistent with the behavior of n^λ near zero and infinity. So we conclude that for some $\lambda > 0$, $\mathcal{M}^\lambda$ must have a non-trivial kernel. This completes the proof of the Lemma. □

Theorem 2.3(i) on the existence of growing modes follows immediately by combining Lemma 2.5 and Lemma 2.6.

For the proof of Theorem 2.3(ii), we need the following two lemmas. We consider real functions below, as all growing modes should be by Theorem 2.1. The following functionals were defined in [24].

$$J(f, E_1, \psi) = \iint \frac{1}{|\mu_e|}(f + \mu_p\psi)^2 dvdx + \int [E_1]^2 dx \tag{32}$$

(33)

$$I(f, E_1, \psi) = J(f, E_1, \psi) - \iint \hat{v}_2\mu_p\psi^2 dvdx + \int \left[(\partial_t\psi)^2 + (\partial_x\psi)^2\right]dx$$

and we denote

$$J(f, E_1, \psi; \tilde{f}, \tilde{E}_1, \tilde{\psi}) = \iint \frac{1}{|\mu_e|}(f + \mu_p\psi)(\tilde{f} + \mu_p\tilde{\psi})dvdx + \int E_1\tilde{E}_1 dx.$$

The next lemma follows immediately by polarization from Lemma 2.7 of [24].

Lemma 2.7. *Consider two real solutions* $(f^i(t), E^i(t), B^i(t) = \partial_x\psi^i(t))$, $i = 1, 2$ *to the linearized system* (18)*, with initial data* $(f^i(0), E^i(0), B^i(0) = \psi^i_x(0)) \in L^1$ *in the constraint set*

$$\mathcal{C} = \left\{\iint f(0)dvdx = 0, \quad \partial_x E_1(0) = -\int f(0)dv, \quad \int B(0)dx = 0\right\},$$

satisfying $J(f(0), E_1(0), \psi(0)) < \infty$. *Then the functional*

$$\begin{aligned} &I\big(f^1, E^1_1, \psi^1; f^2, E^2_1, \psi^2\big)(t) \\ &= J\big(f^1, E^1_1, \psi^1; f^2, E^2_1, \psi^2\big) - \iint \hat{v}_2\mu_p\psi^1\psi^2 dvdx \\ &\quad + \int \left[\partial_t\psi^1\partial_t\psi^2 + \partial_x\psi^1\partial_x\psi^2\right]dx \end{aligned}$$

is independent of t. Furthermore, for all $g \in \ker D$, *the functionals*

$$K_g(f^i, \psi^i) = \iint \left[f^i + (\hat{v}_2\mu_e + \mu_p)\psi^i\right]g dvdx \tag{34}$$

are also independent of t.

Proof of Theorem 2.3 (ii). Assume the linearized system (23), (24) has l independent growing modes and the operator $\mathcal{L}^0$ has a k-dimensional negative eigenspace. By the proof of Lemma 2.6, as λ increases from 0 to $+\infty$, the negative eigenvalues of $\mathcal{M}^\lambda$ must cross the imaginary axis at least $n(\mathcal{M}^0) - 1$ times, with $n(\mathcal{M}^0) = k + 1$ being the number of negative eigenvalues of $\mathcal{M}^0$. Since we get a growing mode at each such crossing, there exist at least k growing modes. Thus $l \geq k$.

It remains to show that $l \le k$. Suppose otherwise, $l > k$. Let $\{\zeta_1, \dots, \zeta_k\} \subset L^2_P$ span the negative eigenspace of $\mathcal{L}^0$. Denote the l linearly independent growing modes by $e^{\lambda_i t}[f^i(x,v), E^i_1(x), \psi^i(x)]$, $i = 1, \dots, l$, where $\psi^i(x)$ is the magnetic potential $\partial_x \psi^i = B^i$. By Theorem 2.1(ii), λ_i are real and positive and we only need to consider real functions below.

First we will prove that $\{\psi^i(x)\}_{i=1}^l$ are linearly independent. Indeed suppose $(c_1, \dots, c_l) \in \mathbb{R}^l$ such that $\psi^c(x) = \sum_{i=1}^l c_i \psi^i(x) = 0$. We denote $f^c = \sum_{i=1}^l c_i f^i$ and $E^c_1 = \sum_{i=1}^l c_i E^i_1$. Applying Lemma 2.7 to any two growing modes $e^{\lambda_i t}[f^i(x,v), E^i_1(x), \psi^i(x)]$ and $e^{\lambda_j t}[f^j(x,v), E^j_1(x), \psi^j(x)]$ with $1 \le i, j \le l$, we have

$$
\begin{aligned}
0 &= I\big(f^i, E^i_1, \psi^i; f^j, E^j_1, \psi^j\big) \\
&= J\big(f^i, E^i_1, \psi^i; f^j, E^j_1, \psi^j\big) - \iint \hat{v}_2 \mu_p \psi^i \psi^j \, dv dx \\
&\quad + \int \big[\lambda_i \lambda_j \psi^i \psi^j + \psi^i_x \psi^j_x\big] dx.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
0 &= J\big(f^c, E^c_1, \psi^c\big) - \iint \hat{v}_2 \mu_p [\psi^c]^2 dv dx \\
&\quad + \int \big[\psi^c_x\big]^2 dx + \int \Big(\sum_{i=1}^l \lambda_i c_i \psi^i\Big)^2 dx.
\end{aligned} \tag{35}
$$

But $\psi^c = 0$ so that

$$
\begin{aligned}
0 &= J\big(f^c, E^c_1, 0\big) + \int \Big(\sum_{i=1}^l \lambda_i c_i \psi^i\Big)^2 dx \\
&\ge J(f^c, E^c, 0) = \iint \frac{1}{|\mu_e|}[f^c]^2 dv dx + \int \big[E^c_1\big]^2 dx.
\end{aligned} \tag{36}
$$

Thus we have $f^c = 0$, $E^c_1 = 0$ and therefore $\sum_{i=1}^l c_i [f^i(x,v), E^i_1(x), \psi^i(x)] = 0$. It follows that $c_1 = \cdots = c_n = 0$ by the linear independence of $[f^i(x,v), E^i_1(x), \psi^i(x)]_{i=1}^l$. This proves our claim that $\{\psi^i(x)\}_{i=1}^l$ is linearly independent.

If $l > k$, there exists a linear combination $\psi^d(x) = \sum_{i=1}^l d_i \psi^i(x)$ with a non-zero vector $(d_1, \dots, d_l) \in \mathbb{R}^l$, such that $\psi^d \perp \zeta_j$ for any $1 \le j \le k$. Using (35) for ψ^d, we have

$$
\begin{aligned}
0 &= J\big(f^d, E^d_1, \psi^d\big) - \iint \hat{v}_2 \mu_p [\psi^d]^2 dv dx \\
&\quad + \int \big[\psi^d_x\big]^2 dx + \int \Big(\sum_{i=1}^l \lambda_i d_i \psi^i\Big)^2 dx.
\end{aligned} \tag{37}
$$

Now by Lemma 2.7, for all $g \in \ker D$ each functional

$$K_g(f^i, \psi^i) = \iint \left[f^i + (\hat{v}_2\mu_e + \mu_p)\psi^i\right] g dv dx$$

vanishes, so that $K_g(f^d, \psi^d) = 0$. Thus by Lemma 2.8 of [24], we have

$$J\left(f^d, E_1^d, \psi^d\right) \geq \iint \left|\mathcal{P}\left(\hat{v}_2\psi^d\right)\right|^2 |\mu_e| dv dx + \left((\mathcal{B}^0)^*\left(\mathcal{A}_1^0\right)^{-1}\mathcal{B}^0\psi^d, \psi^d\right)$$

and (37) implies that

$$\begin{aligned}
0 &\geq \left((\mathcal{B}^0)^*\left(\mathcal{A}_1^0\right)^{-1}\mathcal{B}^0\psi^d, \psi^d\right) \\
&\quad + \iint \left\{|\mu_e|\left|\mathcal{P}\left(\hat{v}_2\psi^d\right)\right|^2 - \hat{v}_2\mu_p[\psi^d]^2\right\} dv dx + \int \left[\psi_x^d\right]^2 dx \\
&\quad + \int \Big(\sum_{i=1}^{l}\lambda_i d_i\psi^i\Big)^2 dx \\
&= \left((\mathcal{B}^0)^*\left(\mathcal{A}_1^0\right)^{-1}\mathcal{B}^0\psi^d, \psi^d\right) + \left(\mathcal{A}_2^0\psi^d, \psi^d\right) + \int \Big(\sum_{i=1}^{l}\lambda_i d_i\psi^i\Big)^2 dx \\
&= (\mathcal{L}^0\psi^d, \psi^d) + \int \Big(\sum_{i=1}^{l}\lambda_i d_i\psi^i\Big)^2 dx.
\end{aligned}$$

Since $(\mathcal{L}^0\psi^d, \psi^d) \geq 0$, we deduce that $\sum_{i=1}^{l}\lambda_i d_i\psi^i = 0$. So $\{\psi^i(x)\}_{i=1}^{l}$ is linearly dependent, which is a contradiction. Therefore $l = k$. This completes the proof of Theorem 2.3. □

3. Formulation of the 3D problem

The 3D RVM for a non-neutral electron plasma with external fields is

$$\begin{gathered}
\partial_t f + \hat{v}\cdot\nabla_x f - (\mathbf{E} + \mathbf{E}^{ext} + \hat{v}\times(\mathbf{B} + \mathbf{B}^{ext}))\cdot\nabla_v f = 0 \\
\partial_t\mathbf{E} - \nabla\times\mathbf{B} = \int \hat{v} f dv = -\mathbf{j} \\
\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0 \\
\nabla\cdot\mathbf{E} = -\int f dv = \rho, \quad \nabla\cdot\mathbf{B} = 0
\end{gathered}$$

where $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$. We consider solutions of finite energy. Thus they vanish in some averaged sense as $|x| \to \infty$.

We use the same notation as in [24]. The cylindrical coordinates in $\mathbb{R}^3$ are (r, θ, z) and the standard cylindrical basis is $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$. The *equilibrium* distribution function is assumed to have the form $f^0 = \mu(e, p)$, with

$$e = \sqrt{1 + |v|^2} - \phi^0(r, z) - \phi^{ext}(r, z),$$
$$p = r\big(v_\theta - A^0_\theta(r, z) - A^{ext}_\theta(r, z)\big)$$

and the equilibrium fields are assumed to have the form

$$\mathbf{E}^0 = -\partial_r\phi^0\mathbf{e}_r - \partial_z\phi^0\mathbf{e}_z, \quad \mathbf{B}^0 = -\partial_z A^0_\theta\mathbf{e}_r + \frac{1}{r}\partial_r\big(rA^0_\theta\big)\mathbf{e}_z,$$

with (A^0_θ, ϕ^0) satisfying the elliptic system (2), (3). We assume f^0 has compact support S in (x, v) space and f^0, E^0, B^0 are everywhere C^1. Such equilibria were constructed in the appendix of [24] for certain ϕ^{ext}, A^{ext}_θ and μ. We assume that

$$\mu_e < 0 \quad \text{on the set } \{\mu > 0\}.$$

For the perturbations $\mathbf{E}, \mathbf{B}$ of the electromagnetic fields, we introduce scalar and vector potentials ϕ and $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

and we impose the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. We will consider only *axisymmetric* perturbations. In cylindrical coordinates we write $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$. We assume that A_r, A_θ, A_z and ϕ are *independent* of θ. Some differentiation rules in cylindrical coordinates are collected in the appendix. Then the corresponding fields are given by

$$\mathbf{E} = (E_r, E_\theta, E_z) = (-\partial_r\phi - \partial_t A_r, -\partial_t A_\theta, -\partial_z\phi - \partial_t A_z),$$
$$\mathbf{B} = (B_r, B_\theta, B_z) = \left(-\partial_z A_\theta,\ \partial_z A_r - \partial_r A_z,\ \frac{1}{r}\partial_r(rA_\theta)\right).$$

Then the linearized Vlasov equation becomes

$$\partial_t f + Df = -\mu_e D\phi - \mu_e\hat{v} \cdot \partial_t\mathbf{A} - r\mu_p\partial_t A_\theta - \mu_p D(rA_\theta), \tag{38}$$

where

$$D = \hat{v} \cdot \nabla_x - (\mathbf{E}^0 + \mathbf{E}^{ext} + \hat{v} \times (\mathbf{B}^0 + \mathbf{B}^{ext})) \cdot \nabla_v$$

(see the appendix). The Maxwell equations become the scalar equation

$$\Delta\phi = -\rho = -\int f dv \tag{39}$$

together with the vector equation

$$\frac{\partial^2}{\partial t^2}\mathbf{A} + \frac{\partial}{\partial t}\nabla\phi - \Delta\mathbf{A} = \mathbf{j} = -\int \hat{v} f dv. \tag{40}$$

We are looking for an axisymmetric growing mode $[e^{\lambda t}f(x,v), e^{\lambda t}\mathbf{E}(x), e^{\lambda t}\mathbf{B}(x)]$, which means that we replace ∂_t by λ everywhere. Here $\mathrm{Re}\lambda > 0$ and $(\mathbf{E},\mathbf{B})$ is independent of θ. By Theorem 1.1 of [24], λ must be real and so $\lambda > 0$. Because of the Coulomb gauge condition, we have

$$0 = \nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_z}{\partial z},$$

so that we can introduce a *super-potential* $\pi(r,z)$ such that

$$A_r = -\partial_z\pi \quad A_z = \frac{1}{r}\partial_r(r\pi) = \partial_r\pi + \frac{1}{r}\pi.$$

(Without this super-potential we would not have been able to deduce the current equation at the end of the construction in Sect. 8.) Replacing ∂_t by λ and substituting $\hat{v}\cdot\mathbf{A} = \hat{v}_\theta A_\theta - \hat{v}_r(-\partial_z\pi) + \hat{v}_z(\partial_r\pi + \frac{1}{r}\pi)$, we rewrite the Vlasov equation (38) as

$$(\lambda + D)f = -\mu_e D\phi - (\lambda + D)(r\mu_p A_\theta) - \mu_e\lambda\hat{v}_\theta A_\theta - \mu_e\lambda\left[-\hat{v}_r\partial_z + \hat{v}_z\left(\partial_r + \frac{1}{r}\right)\right]\pi. \tag{41}$$

We can explicitly invert the operator $(\lambda + D)$ by introducing the particle paths $(X(t;x,v), V(t;x,v))$, which are the characteristics of D. They are defined as the solutions of the ODE

$$\dot{X} = \hat{V}, \quad \dot{V} = -(\mathbf{E}^0 + \mathbf{E}^{ext})(X) - \hat{V}\times(\mathbf{B}^0 + \mathbf{B}^{ext})(X) \tag{42}$$

with the initial conditions $X(0) = x,\ V(0) = v$. Integrating (41) along the path from $t = -\infty$ to $t = 0$, we get

(43)

$$\begin{aligned} f(x,v) = &-\mu_e\phi + \mu_e\int_{-\infty}^0 \lambda e^{\lambda s}\phi(X(s))ds - \mu_p r A_\theta \\ &-\mu_e\int_{-\infty}^0 \lambda e^{\lambda s}\hat{V}_\theta(s)A_\theta(X(s))ds \\ &-\mu_e\int_{-\infty}^0 \lambda e^{\lambda s}\left\{-\hat{V}_r(s)\partial_z\pi(X(s)) + \hat{V}_z(s)\left(\partial_r + \frac{1}{r}\right)\pi(X(s))\right\}ds. \end{aligned}$$

Now it is convenient to introduce several operators depending on a positive parameter λ. These operators will be used throughout the rest of the paper. First, for $k = k(x,v)$ we define the non-local operator

$$(\mathcal{Q}^\lambda k)(x,v) = \int_{-\infty}^0 \lambda e^{\lambda s}k(X(s;x,v), V(s;x,v))ds$$

where

$$Gk = -\hat{v}_r \partial_z k + \hat{v}_z \left(\partial_r + \frac{1}{r}\right) k, \quad G^* k = \hat{v}_r \partial_z k - \hat{v}_z \partial_r k.$$

Then we can rewrite formula (43) as

(44) $$f = -\mu_e \phi + \mu_e \mathcal{Q}^\lambda \phi - \mu_p r A_\theta - \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta A_\theta) - \mu_e \mathcal{Q}^\lambda (G\pi).$$

Moreover, substituting (44) into the Poisson equation $-\Delta\phi = \int f dv$, we obtain

$$\begin{aligned}-\Delta\phi = &-\Big(\int \mu_e dv\Big)\phi + \int \mu_e \mathcal{Q}^\lambda \phi dv - \Big(\int \mu_p dv\Big) r A_\theta \\ &- \int \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta A_\theta) dv - \int \mu_e \mathcal{Q}^\lambda (G\pi) dv.\end{aligned}$$

Furthermore, for $h = h(r, z)$, we define all of the following operators. Each one will appear later in the paper; for purposes of comparison it is convenient to collect their definitions in one place.

$$\mathcal{A}_1^\lambda h = -\Delta h - \Big(\int \mu_e dv\Big) h + \int \mu_e \mathcal{Q}^\lambda h dv$$

$$\mathcal{A}_2^\lambda h = \left(-\Delta + \frac{1}{r^2} + \lambda^2\right) h - r\Big(\int \hat{v}_\theta \mu_p dv\Big) h - \int \hat{v}_\theta \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta h) dv$$

$$\mathcal{B}^\lambda h = -\Big(\int \hat{v}_\theta \mu_e dv\Big) h + \int \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta h) dv$$

$$\mathcal{L}^\lambda = (\mathcal{B}^\lambda)^* \big(\mathcal{A}_1^\lambda\big)^{-1} \mathcal{B}^\lambda + A_2^\lambda$$

$$\mathcal{C}^\lambda h = \int \hat{v}_\theta \mu_e \mathcal{Q}^\lambda (Gh) dv, \quad (\mathcal{C}^\lambda)^* h = \int G^* \big(\mu_e \mathcal{Q}^\lambda (\hat{v}_\theta h)\big) dv$$

$$\mathcal{D}^\lambda h = \int \mu_e \mathcal{Q}^\lambda (Gh) dv, \quad (\mathcal{D}^\lambda)^* h = -\int G^* \big(\mu_e \mathcal{Q}^\lambda (h)\big) dv$$

$$\mathcal{E}^\lambda h = \int G^* \big(\mu_e \mathcal{Q}^\lambda (Gh)\big)$$

$$\mathcal{F}^\lambda = (\mathcal{D}^\lambda)^* \big(\mathcal{A}_1^\lambda\big)^{-1} \mathcal{B}^\lambda - (\mathcal{C}^\lambda)^*.$$

$$\mathcal{G}^\lambda = \mathcal{E}^\lambda + (\mathcal{D}^\lambda)^* \big(\mathcal{A}_!^\lambda\big)^{-1} \mathcal{D}^\lambda$$

$$\mathcal{A}_3^\lambda = \left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right) - \mathcal{E}^\lambda$$

$$\mathcal{A}_4^\lambda = \mathcal{A}_3^\lambda - (\mathcal{D}^\lambda)^* \big(\mathcal{A}_1^\lambda\big)^{-1} \mathcal{D}^\lambda = \left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right) - \mathcal{G}^\lambda.$$

Here these operators are defined formally. In the next section, they will be defined carefully and their key properties will be derived.

Since $r\int \mu_p dv = -\int \hat{v}_\theta \mu_e dv$, the result below (44) can be written as

$$\mathcal{A}_1^\lambda \phi = \mathcal{B}^\lambda A_\theta + \mathcal{D}^\lambda \pi. \tag{45}$$

With ∂_t replaced by λ, the Maxwell equation (40) becomes

$$\lambda^2 \mathbf{A} + \lambda \nabla \phi - \Delta \mathbf{A} = \mathbf{j}. \tag{46}$$

Taking the θ-component of (46) and substituting (44)

$$\begin{aligned}(\lambda^2 - \Delta) A_\theta &= -\int \hat{v}_\theta f dv \\ &= \Big(\int \hat{v}_\theta \mu_e dv\Big)\phi - \int \hat{v}_\theta \mu_e \mathcal{Q}^\lambda \phi dv + \Big(\int \hat{v}_\theta \mu_p dv\Big) r A_\theta \\ &\quad + \int \hat{v}_\theta \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta A_\theta) dv + \int \hat{v}_\theta \mu_e \mathcal{Q}^\lambda (G\pi) dv.\end{aligned}$$

That is,

$$\mathcal{A}_2^\lambda A_\theta = -(\mathcal{B}^\lambda)^* \phi + \mathcal{C}^\lambda \pi. \tag{47}$$

Lemma 3.1.

$$\mathcal{A}_3^\lambda \pi = (\mathcal{D}^\lambda)^* \phi - (\mathcal{C}^\lambda)^* A_\theta. \tag{48}$$

Proof. First we claim that

$$\left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right) \pi = \partial_z j_r - \partial_r j_z. \tag{49}$$

Indeed, let $\mathbf{K} = j_r \mathbf{e}_r + j_z \mathbf{e}_z$ and $\mathbf{I} = (-\Delta)^{-1} \mathbf{K}$ so that $\mathbf{e}_\theta \cdot \mathbf{I} = 0$. By the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$, for a growing mode we have

$$\nabla \cdot \mathbf{K} = \left(\partial_r + \frac{1}{r}\right) j_r + \partial_z j_z = \nabla \cdot \mathbf{j} = -\lambda \rho = \lambda \Delta \phi.$$

Thus the vector identity

$$\nabla \times (\nabla \times \mathbf{K}) = -\Delta \mathbf{K} + \nabla (\nabla \cdot \mathbf{K})$$

takes the form

$$-\nabla \times (\nabla \times \Delta \mathbf{I}) = -\Delta \mathbf{K} + \lambda \nabla \Delta \phi$$

or

$$\nabla \times (\nabla \times \mathbf{I}) = \mathbf{K} - \lambda \nabla \phi.$$

Now the r and z components of the Maxwell equation (46) can be written as

$$(\lambda^2 - \Delta)(A_r \mathbf{e}_r + A_z \mathbf{e}_z) = \mathbf{K} - \lambda \nabla \phi.$$

Furthermore,

$$A_r\mathbf{e}_r + A_z\mathbf{e}_z = -(\partial_z\pi)\mathbf{e}_r - \left(\left(\partial_r + \frac{1}{r}\right)\pi\right)\mathbf{e}_z = \nabla\times(\pi\mathbf{e}_\theta).$$

Combining the last three equations, we have

$$\nabla\times(\lambda^2-\Delta)(\pi\mathbf{e}_\theta) = \nabla\times(\nabla\times\mathbf{I}),$$

which is satisfied if

$$(\lambda^2-\Delta)(\pi\mathbf{e}_\theta) = \nabla\times\mathbf{I} = (\partial_z I_r - \partial_r I_z)\mathbf{e}_\theta.$$

Noting that $\Delta\mathbf{e}_\theta = -\frac{1}{r^2}\mathbf{e}_\theta$, we deduce

$$\left(\lambda^2 - \Delta + \frac{1}{r^2}\right)\pi = \partial_z I_r - \partial_r I_z.$$

Applying $-\Delta + \frac{1}{r^2}$ to this result yields

$$\begin{aligned}\left(-\Delta+\frac{1}{r^2}\right)\left(-\Delta+\frac{1}{r^2}+\lambda^2\right)\pi &= \partial_z\left(-\Delta+\frac{1}{r^2}\right)I_r - \partial_r(-\Delta)I_z\\ &= \partial_z j_r - \partial_r j_z\end{aligned}$$

since $[\partial_r, -\Delta] = \frac{1}{r^2}\partial_r$. This proves the claim.

Upon substituting (44) into $j_r = \int \hat{v}_r f dv$, the first and third terms vanish because they are odd in v_r. The same reasoning is valid for $j_z = \int \hat{v}_z f\, dv$. Therefore

$$\begin{aligned}(50)\quad \partial_z j_r - \partial_r j_z &= -\partial_z\int \hat{v}_r f dv + \partial_r\int \hat{v}_z f dv\\ &= -\partial_z\int \hat{v}_r\mu_e\mathcal{Q}^\lambda\phi dv + \partial_r\int\hat{v}_z\mu_e\mathcal{Q}^\lambda\phi dv\\ &\quad -\partial_z\int\hat{v}_r\mu_e\mathcal{Q}^\lambda(\hat{v}_\theta A_\theta)dv + \partial_r\int\hat{v}_z\mu_e\mathcal{Q}^\lambda(\hat{v}_\theta A_\theta)dv\\ &\quad +\partial_z\int\hat{v}_r\mu_e\mathcal{Q}^\lambda(G\pi)dv - \partial_r\int\hat{v}_z\mu_e\mathcal{Q}^\lambda(G\pi)dv.\end{aligned}$$

The last four terms in (50) equal

$$\begin{aligned}&-\int G^*\big[\mu_e\mathcal{Q}^\lambda(\hat{v}_\theta A_\theta)\big]dv + \int G^*\big[\mu_e\mathcal{Q}^\lambda(G\pi)\big]dv.\\ &= -(\mathcal{C}^\lambda)^* A_\theta + \mathcal{E}^\lambda\pi.\end{aligned}$$

In (50) call the first two terms $T(\phi)$. Then

$$\begin{aligned}(T(\phi), \psi)_{L^2(\mathbb{R}^3)} &= 2\pi \iint T(\phi)\psi r dr dz \\ &= \left\langle G^*\left[\mu_e \mathcal{Q}^\lambda \phi\right], \psi\right\rangle_{L^2(\mathbb{R}^6)} = \left\langle \mathcal{Q}^\lambda \phi, \mu_e G\psi\right\rangle \\ &= \left\langle \phi, \mu_e \mathcal{Q}^\lambda G\psi\right\rangle\end{aligned}$$

by Lemma 4.1(d) below since ϕ is independent of v. The last expression equals

$$\left\langle \phi, \mathcal{Q}^\lambda[\mu_e G\psi]\right\rangle = (\phi, \mathcal{D}^\lambda\psi)_{L^2(\mathbb{R}^3)} = ((\mathcal{D}^\lambda)^*\phi, \psi)_{L^2(\mathbb{R}^3)}.$$

So $T(\phi) = (\mathcal{D}^\lambda)^*\phi$. Thus by (49) and (50),

$$\left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right)\pi = (\mathcal{D}^\lambda)^*\phi - (\mathcal{C}^\lambda)^* A_\theta + \mathcal{E}^\lambda \pi.$$

Hence

$$\mathcal{A}_3^\lambda \pi = (\mathcal{D}^\lambda)^*\phi - (\mathcal{C}^\lambda)^* A_\theta.$$

□

We now have three equations (45), (47) and (48) that link the unknowns ϕ, A_θ and π. Using (45) to eliminate ϕ , we obtain

$$\begin{aligned}\mathcal{A}_2^\lambda A_\theta &= -(\mathcal{B}^\lambda)^*\left(\mathcal{A}_1^\lambda\right)^{-1}\left[\mathcal{B}^\lambda A_\theta + \mathcal{D}^\lambda \pi\right] + \mathcal{C}^\lambda \pi, \\ \mathcal{A}_3^\lambda \pi &= (\mathcal{D}^\lambda)^*\left(\mathcal{A}_1^\lambda\right)^{-1}\left[\mathcal{B}^\lambda A_\theta + \mathcal{D}^\lambda \pi\right] - (\mathcal{C}^\lambda)^* A_\theta.\end{aligned}$$

That is

(51) $$\mathcal{L}^\lambda A_\theta = -(\mathcal{F}^\lambda)^*\pi,$$

and

(52) $$\mathcal{A}_4^\lambda \pi = \mathcal{F}^\lambda A_\theta.$$

These are the basic reduced equations of which we want to find a non-zero solution. Motivated by (51) and (52), we define the matrix operator

(53) $$\mathcal{M}^\lambda = \begin{pmatrix} \mathcal{L}^\lambda & (\mathcal{F}^\lambda)^* \\ \mathcal{F}^\lambda & -\mathcal{A}_4^\lambda \end{pmatrix}$$

of which we want to find a non-trivial nullspace.

4. The operators

Let the space L^2_S consist of the cylindrically symmetric functions (functions of r and z only) in $L^2(\mathbb{R}^3)$. For any positive integer k, let

$$H^{k\dagger} = \left\{\psi \in L^2_S(\mathbb{R}^3) \,\middle|\, e^{i\theta}\psi \in H^k(\mathbb{R}^3)\right\}$$

and $\|\psi\|_{H^{k\dagger}} = \|e^{i\theta}\psi\|^2_{\mathbf{H}^k(\mathbb{R}^3)}$. Furthermore, we define $V^{k\dagger}$ to be the closure of the cylindrically symmetric functions in $C_c^\infty(\mathbb{R}^3)$ with respect to the $\dot{H}^k$ semi-norm

$$\|\psi\|^2_{V^{k\dagger}} = \sum_{|\alpha|=k} \|\partial^\alpha(e^{i\theta}\psi)\|^2_{L^2}.$$

We denote $H^{-k\dagger} = (H^{k\dagger})^*$ and $V^{-k\dagger} = (V^{k\dagger})^*$. It follows easily that $\psi(r,z) \in H^{1\dagger}$ is equivalent to $\psi, \psi_r, \psi_z, \psi/r \in L^2(\mathbb{R}^3)$. Furthermore, $\psi(r,z) \in H^{2\dagger}$ is equivalent to $\psi, \psi_{rr}, \psi_{zz}, (\psi/r)_r \in L^2(\mathbb{R}^3)$, and such a function also satisfies $\psi_r, \psi_z, \psi/r \in L^2(\mathbb{R}^3)$. We also define the space $W^{2\dagger} = V^{2\dagger} \cap V^{1\dagger}$ with the norm

$$\|\psi\|_{W^{2\dagger}} = \|\Delta(e^{i\theta}\psi)\|_{L^2} + \|\nabla(e^{i\theta}\psi)\|_{L^2}$$

and $W^{-2\dagger} = (W^{2\dagger})^*$. We also denote V^k to be the closure of the functions in $C_c^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\psi\|^2_{V^k} = \sum_{|\alpha|=k} \|\partial^\alpha\psi\|^2_{L^2}$$

and $V^{-k} = (V^k)^*$. We note that for any function $\psi(r,z)$

$$-\Delta(\psi e^{i\theta}) = e^{i\theta}\left(-\partial_z^2\psi - \partial_r^2\psi - \frac{1}{r}\partial_r\psi + \frac{1}{r^2}\psi\right). \tag{54}$$

So in spite of the singular factor $1/r^2$, one can apply the usual elliptic regularity theory to the operator $-\partial_z^2 - \partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}$, as pointed out to us by F.H. Lin. The daggered spaces are designed to take account of this singular factor.

We denote by $|\ \ |_2$ the norm in $L^2_S(\mathbb{R}^3)$, by $(\ ,\ )$ the inner product in $L^2_S(\mathbb{R}^3)$, by $\langle\ ,\ \rangle$ the pairing of dual spaces, and by $\langle\ ,\ \rangle_{|\mu_e|}$ the inner product in $L^2_{|\mu_e|}(\mathbb{R}^6)$ where $|\mu_e(x,v)|$ is the weight with $\|\ \ \|_{|\mu_e|}$ the corresponding norm. We defined the operator $\mathcal{Q}^\lambda$ in the previous section.

Lemma 4.1 (Properties of $\mathcal{Q}^\lambda$). *Let* $0 < \lambda < \infty$.

(a) $\mathcal{Q}^\lambda : L^2_{|\mu_e|}(\mathbb{R}^6) \to L^2_{|\mu_e|}(\mathbb{R}^6)$ *with operator norm* $= 1$.
(b) For all $m \in L^2_{|\mu_e|}(\mathbb{R}^6)$, $\|\mathcal{Q}^\lambda m - \mathcal{P}m\|_{|\mu_e|} \to 0$ *as* $\lambda \to 0$, *where* $\mathcal{P}$ *is defined in the introduction.*
(c) If $\sigma > 0$, *then* $\|\mathcal{Q}^\lambda - \mathcal{Q}^\sigma\| = O(|\lambda - \sigma|)$ *as* $\lambda \to \sigma$, *where* $\|\ \ \|$ *denotes the operator norm from* $L^2_{|\mu_e|}$ *to* $L^2_{|\mu_e|}$.
(d) For $v = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$, *denote* $\tilde{v} = -v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta - v_z\mathbf{e}_z$ *and* $\tilde{n}(x,v) = n(x,\tilde{v})$. *Then* $\langle\mathcal{Q}^\lambda m, n\rangle_{|\mu_e|} = \langle m, \mathcal{Q}^\lambda\tilde{n}\rangle_{|\mu_e|}$, *for any* $m, n \in L^2_{|\mu_e|}(\mathbb{R}^6)$.
(e) For all $m \in L^2_{|\mu_e|}(\mathbb{R}^6)$, $\|\mathcal{Q}^\lambda m - m\|_{|\mu_e|} \to 0$ *as* $\lambda \to +\infty$.

Proof. To prove (a),

$$\langle \mathcal{Q}^{\lambda} m, n\rangle_{|\mu_e|} = \int_{-\infty}^{0} \lambda e^{\lambda s} \iint (m\sqrt{|\mu_e|})(X(s), V(s)) \cdot (n\sqrt{|\mu_e|})(x, v) dv dx ds$$
$$\leq \|m\|_{|\mu_e|} \|n\|_{|\mu_e|}.$$

Moreover, $\mathcal{Q}^{\lambda} 1 = 1$.

Assertion (b) was proven in Lemma 2.6 of [24]. As for (c), we estimate

$$\begin{aligned} \|\mathcal{Q}^{\lambda} m - \mathcal{Q}^{\sigma} m\|_{|\mu_e|} &\leq \int_{-\infty}^{0} |\lambda e^{\lambda s} - \sigma e^{\sigma s}| \|m(X(s), V(s))\|_{|\mu_e|} ds \\ &= \int_{-\infty}^{0} |\lambda e^{\lambda s} - \sigma e^{\sigma s}| ds \|m\|_{|\mu_e|} \\ &\leq C |\ln \lambda - \ln \sigma| \|m\|_{|\mu_e|}. \end{aligned}$$

To prove (d), note that the characteristic ODE is invariant under the transformation $s \to -s$, $r \to +r$, $z \to +z$, $v_r \to -v_r$, $v_\theta \to +v_\theta$, $v_z \to -v_e$. Thus

$$n(X(-s; x, v), V(-s; x, v)) = \tilde{n}(X(s; x, v), V(s; x, v)).$$

Now

$$\langle \mathcal{Q}^{\lambda} m, n\rangle_{|\mu_e|} = \int_{-\infty}^{0} \lambda e^{\lambda s} \iint |\mu_e| m(X(s), V(s)) n(x, v) dv dx ds.$$

We change variables $(X(s), V(s)) \to (x, v)$ and $(x, v) \to (X(-s), V(-s))$ with Jacobian $= 1$ to obtain

$$\begin{aligned} \langle \mathcal{Q}^{\lambda} m, n\rangle_{|\mu_e|} &= \int_{-\infty}^{0} \lambda e^{\lambda s} \iint |\mu_e| m(x, v) \tilde{n}(X(-s), V(-s)) dv dx ds \\ &= \langle m, \mathcal{Q}^{\lambda} \tilde{n}\rangle_{|\mu_e|}. \end{aligned}$$

Although Assertion (e) was essentially proven in Lemma 2.6 of [24], we outline the proof here. Letting M denote the spectral measure of the self-adjoint operator $-iD$ in the space $L^2_{|\mu_e|}$, we have

$$\mathcal{Q}^{\lambda} m - m = \int_{\mathbb{R}} \left(\frac{\lambda}{\lambda + i\alpha} - 1 \right) dM(\alpha) m.$$

Thus

$$\|\mathcal{Q}^{\lambda} m - m\|^2_{|\mu_e|} \leq \int_{\mathbb{R}} \left| \frac{\lambda}{\lambda + i\alpha} - 1 \right|^2 d\|M(\alpha) m\|^2_{|\mu_e|} \to 0$$

as $\lambda \to +\infty$. □

Remark 1. Since $\int_{-\infty}^{0} \lambda e^{\lambda s} ds = 1$, the function

$$(\mathcal{Q}^\lambda m)(x, v) = \int_{-\infty}^{0} \lambda e^{\lambda s} m(X(s; x, v), V(s; x, v)) ds \tag{55}$$

is a weighted time average of the observable m along the particle trajectory. By the same proof as in Lemma 4.1(b), we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T m(X(s), V(s)) ds = \mathcal{P}m. \tag{56}$$

But from the standard ergodic theory [1] of Hamiltonian systems, the limit of the time average in (56) equals the phase space average of m in the set traced by the trajectory. Thus $\mathcal{P}m$ has the meaning of the phase space average of m and Lemma 4.1(b) states that the limit of the weighted time average (55) equals the same phase space average. In particular, if the particle motion is ergodic in the set $S_{e,p}$ determined by the two invariants e and p, and if $d\sigma_{e,p}$ denotes the induced measure of $\mathbf{R}^3 \times \mathbf{R}^3$ on $S_{e,p}$, then

$$\mathcal{P}m = \frac{1}{\sigma_{e,p}(S_{e,p})} \int_{S_{e,p}} m(x) d\sigma_{e,p}(x).$$

For non-ergodic particles, we do not have such an explicit expression, but $\mathcal{P}m$ still equals the phase space average of m on the set traced by the particle.

Lemma 4.2. *Let* $0 < \lambda < \infty$.

(a) $\mathcal{B}^\lambda$ *maps* $L^2 \to L^2$ *with operator bound independent of* λ.
(b) $\mathcal{A}_1^\lambda$, $\mathcal{A}_2^\lambda$ *and* $\mathcal{L}^\lambda$ *are self-adjoint on* L^2 *with domains* H^2, $H^{2\dagger}$ *and* $H^{2\dagger}$ *respectively.*
(c) The essential spectrum of $\mathcal{A}_1^\lambda$ *is* $[0, \infty)$, *while that of* $\mathcal{A}_2^\lambda$ *and* $\mathcal{L}^\lambda$ *is* $[\lambda^2, \infty)$.
(d) $\langle \mathcal{A}_1^\lambda h, h\rangle > 0$ *for all* $0 \neq h \in H^2$.
(e) $(\mathcal{A}_1^\lambda)^{-1}$ *maps* V^{-1} *into* V^1 *with operator bound* ≤ 1.
(f) For all $h \in L^2$, $(\mathcal{L}^\lambda - \mathcal{L}^0)h \to 0$ *strongly in* L^2 *as* $\lambda \to 0$.
(g) If $\sigma > 0$, *then as* $\lambda \to \sigma$, *the operator norm from* L^2 *to* L^2 *of* $\mathcal{A}_2^\lambda - \mathcal{A}_2^\sigma$ *tends to zero. The same is true of* $\mathcal{B}^\lambda$, $\mathcal{A}_1^\lambda$, $(\mathcal{A}_1^\lambda)^{-1}$ *and* $\mathcal{L}^\lambda$.

Proof. Assertions (a), (b), (c), (d) and (f) were proven in Lemma 3.1 of [24]. As for (e), let us define $\mathcal{A} = \mathcal{A}_1^\lambda$ for brevity. Let $\phi \in V^1$. Then

$$\begin{aligned} \langle \mathcal{A}\phi, \phi\rangle &= |\nabla\phi|_2^2 + \iint |\mu_e| dv \phi^2 dx - \langle \mathcal{Q}^\lambda \phi, \phi\rangle_{|\mu_e|} \\ &\geq |\nabla\phi|_{L^2}^2 = \|\phi\|_{V^1}^2 \end{aligned}$$

by Lemma 4.1(a). Denoting $h = \mathcal{A}\phi$, we therefore have

$$\|\mathcal{A}^{-1}h\|_{V^1}^2 = |\nabla\phi|_2^2 \le \langle \mathcal{A}\phi, \phi\rangle = \langle h, \mathcal{A}^{-1}h\rangle \le \|h\|_{V^{-1}}\|\mathcal{A}^{-1}h\|_{V^1}.$$

Thus $\|\mathcal{A}^{-1}h\|_{V^1} \le \|h\|_{V^{-1}}$. Finally, Assertion (g) follows directly from Lemma 4.1(c). □

Remark 2. The supports are under control in the following sense. Recall that we assume $f^0(x, v) = \mu(e, p)$ has compact support $\subset S \subset \mathbb{R}_x^3 \times \mathbb{R}_v^3$. We may assume $S = S_x \times S_v$, both balls in $\mathbb{R}^3$. Let $\chi = \chi(r, z)$ be a smooth cut-off function for the spatial support of f^0 in S_x; that is, $\chi = 1$ on the spatial support of f^0 and has compact support inside S_x. Let M_χ be the operator of multiplication by χ. Then

$$\mathcal{B}^\lambda = \mathcal{B}^\lambda M_\chi = M_\chi \mathcal{B}^\lambda = M_\chi \mathcal{B}^\lambda M_\chi$$

and the same is true for all the operators $\mathcal{C}^\lambda$, $\mathcal{D}^\lambda$, $\mathcal{E}^\lambda$, $\mathcal{F}^\lambda$, $(\mathcal{C}^\lambda)^*$, $(\mathcal{D}^\lambda)^*$, $(\mathcal{F}^\lambda)^*$. Indeed,

$$\mu_e(x, v) = \mu_e(X(s; x, v), V(s; x, v))$$

because of the invariance of e and p under the flow. So for example

$$\begin{aligned}(\mathcal{B}^\lambda h)(x) &= -h\int \hat{v}_\theta \mu_e dv + \int \mu_e \mathcal{Q}^\lambda(\hat{v}_\theta h) dv \\ &= -h\int \hat{v}_\theta \mu_e dv + \int \mathcal{Q}^\lambda(\mu_e \hat{v}_\theta h) dv = \big(M_\chi \mathcal{B}^\lambda M_\chi h\big)(x).\end{aligned}$$

Below, for any function space Y, we denote by $Y_c = \{h \in Y \mid supp(h) \subset S_x\}$. Then $V_c^k = \dot{H}_c^k = H_c^k$ and $V_c^{k\dagger} = \dot{H}_c^{k\dagger} = H_c^{k\dagger}$. By mollification, H_c^k is dense in V^k. Furthermore, $(H^{-k})_c \subset V^{-k}$. The multiplication operator M_χ maps V^1 into H^1.

Lemma 4.3. *For any* $\lambda > 0$,

$$\begin{gathered}\mathcal{C}^\lambda,\ \mathcal{D}^\lambda,\ (\mathcal{F}^\lambda)^* : H_{loc}^{1\dagger} \to L_c^2 \\ (\mathcal{C}^\lambda)^*,\ (\mathcal{D}^\lambda)^*,\ \mathcal{F}^\lambda : L_{loc}^2 \to H_c^{-1\dagger} \\ \mathcal{E}^\lambda : H_{loc}^{1\dagger} \to H_c^{-1\dagger}.\end{gathered}$$

All these operator bounds are independent of λ. *Furthermore, all these operators are continuous functions of* λ *in the operator norms. As* $\lambda \to 0+$, *all these operators converge to* 0 *strongly (but not in operator norm).*

Proof. By the preceding remark, the images all have support in the fixed set S_x and the operators act on functions h depending only on χh. Now

$$\langle \mathcal{C}^\lambda h, k\rangle = \iint \hat{v}_\theta \mu_e \mathcal{Q}^\lambda(G\chi h)\chi k\, dvdx = -\big\langle \mathcal{Q}^\lambda(G\chi h), \hat{v}_\theta \chi k\big\rangle_{|\mu_e|}$$

so that

$$|\langle \mathcal{C}^\lambda h, k\rangle| \leq C\|\chi h\|_{H^{1\dagger}}\|\chi k\|_{L^2}$$

with C independent of λ. The same proof works for all of the operators (in their appropriate spaces), except $\mathcal{F}^\lambda$ and $(\mathcal{F}^\lambda)^*$. For $\mathcal{F}^\lambda = (\mathcal{D}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda - (\mathcal{C}^\lambda)^*$, it follows from Lemma 4.2(a) that the operator $(\mathcal{D}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda = (\mathcal{D}^\lambda)^* M_\chi (\mathcal{A}_1^\lambda)^{-1}\mathcal{B}^\lambda$ maps

$$L^2_{loc} \to L^2_c \subset H_c^{-1} \subset V^{-1} \to V^1 \to H_c^1 \subset L^2 \to H_c^{-1\dagger}.$$

Similarly for $(\mathcal{F}^\lambda)^*$.

The continuity follows directly from Lemma 4.1(c). Now let us consider the behavior as $\lambda \to 0$. For any function $\phi \in H_c^{1\dagger}$, by Lemma 4.1(b) we have

$$\mathcal{C}^\lambda\phi \to \int \hat{v}_\theta \mu_e \mathcal{P}(G\phi)dv$$

strongly in L^2 as $\lambda \to 0+$. Clearly $G\phi$ is odd in (v_r, v_z). By Lemma 3.3 in [24], it follows that $\mathcal{P}(Gh)$ is also odd in (v_r, v_z). But $\hat{v}_\theta\mu_e$ is even, so that the integral $\int \hat{v}_\theta\mu_e\mathcal{P}(Gh)dv$ vanishes. Therefore $\mathcal{C}^\lambda \to 0$ strongly as $\lambda \to 0+$. The proof is the same for the other operators. □

We study the mapping properties of the operator $\mathcal{A}_4^\lambda$ in the following lemma.

Lemma 4.4. *There exists $\lambda_1 > 0$ such that for any $0 < \lambda < \lambda_1$, the operator $\mathcal{A}_4^\lambda$ maps $W^{2\dagger}$ in a one-to-one manner onto $W^{-2\dagger}$. Therefore it has a bounded inverse from $W^{-2\dagger}$ onto $W^{2\dagger}$. Furthermore, $(\mathcal{A}_4^\lambda)^{-1}$, if restricted to $V^{-2\dagger}$, maps $V^{-2\dagger}$ into $V^{2\dagger}$ with operator bound independent of λ.*

Proof. It is convenient to introduce yet another operator $\mathcal{A}_5^\lambda$ so that

$$\mathcal{A}_4^\lambda = \mathcal{U}^\lambda\mathcal{A}_5^\lambda\mathcal{U}^\lambda$$

where $\mathcal{U}^\lambda = (-\Delta + \frac{1}{r^2} + \lambda^2)^{\frac{1}{2}}$. Then

$$\mathcal{A}_5^\lambda = -\Delta + \frac{1}{r^2} - (\mathcal{U}^\lambda)^{-1}\mathcal{G}^\lambda(\mathcal{U}^\lambda)^{-1},$$

where $\mathcal{G}^\lambda = \mathcal{E}^\lambda + (\mathcal{D}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{D}^\lambda$. We remark that the operator $\mathcal{E}^\lambda \leq 0$; however, this fact is not useful because the other operator $(\mathcal{D}^\lambda)^*(\mathcal{A}_1^\lambda)^{-1}\mathcal{D}^\lambda \geq 0$ so that the two signs are in conflict.

By (54), for $\phi \in L^2_S$ we have

$$e^{i\theta}(\mathcal{U}^\lambda)^{-1}\phi = (-\Delta + \lambda^2)^{-\frac{1}{2}}(e^{i\theta}\phi)$$

so that $(\mathcal{U}^\lambda)^{-1} : L^2_S \to H^{1\dagger}$ and $H^{-1\dagger} \to L^2_S$. We consider the two terms in $\mathcal{G}^\lambda$ separately. The operator $(\mathcal{U}^\lambda)^{-1}\mathcal{E}^\lambda(\mathcal{U}^\lambda)^{-1} = (\mathcal{U}^\lambda)^{-1} M_\chi \mathcal{E}^\lambda M_\chi (\mathcal{U}^\lambda)^{-1}$ maps

$$H^{2\dagger} \to H^{3\dagger} \to H^{1\dagger} \to H_c^{-1\dagger} \to L^2.$$

Because the multiplication operator $M_\chi : H^{3\dagger} \to H^{1\dagger}$ is compact, the operator $(\mathcal{U}^\lambda)^{-1}\mathcal{E}^\lambda(\mathcal{U}^\lambda)^{-1}$ is relatively compact with respect to $-\Delta + \frac{1}{r^2}$. Similarly, the operator

$$(\mathcal{U}^\lambda)^{-1}(\mathcal{D}^\lambda)^*\left(\mathcal{A}_1^\lambda\right)^{-1}\mathcal{D}^\lambda(\mathcal{U}^\lambda)^{-1} = (\mathcal{U}^\lambda)^{-1}(\mathcal{D}^\lambda)^* M_\chi \left(\mathcal{A}_1^\lambda\right)^{-1}\mathcal{D}^\lambda M_\chi (\mathcal{U}^\lambda)^{-1}$$

maps

$$\begin{aligned} H^{2\dagger} &\to H^{3\dagger} \to H_c^{1\dagger} \to L_c^2 \subset H_c^{-1} \subset V^{-1} \to V^1 \\ &\to H_c^1 \subset L_c^2 \to H_c^{-1\dagger} \to L^2 \end{aligned}$$

and it is relatively compact with respect to $-\Delta + \frac{1}{r^2}$. Therefore by the Kato–Rellich and Weyl theorems, $\mathcal{A}_5^\lambda$ is self-adjoint on L^2_S with domain $H^{2\dagger}$ and its essential spectrum equals $[0, +\infty)$.

We split $\mathcal{A}_5^\lambda$ into two parts as

$$\mathcal{A}_5^\lambda = \frac{1}{2}\left(-\Delta + \frac{1}{r^2}\right) + \mathcal{A}_6^\lambda, \quad \mathcal{A}_6^\lambda = \frac{1}{2}\left(-\Delta + \frac{1}{r^2}\right) - (\mathcal{U}^\lambda)^{-1}\mathcal{G}^\lambda(\mathcal{U}^\lambda)^{-1}$$

and claim that

$$\mathcal{A}_6^\lambda \geq 0$$

for sufficiently small λ. To prove the claim, first note that $\mathcal{A}_6^\lambda$ too is self-adjoint on L^2_S with domain $H^{2\dagger}$ and its essential spectrum equals $[0, \infty)$. So we merely need to show that the point spectrum of $\mathcal{A}_6^\lambda$ is also contained in $[0, \infty)$ for sufficiently small λ. We prove this by contradiction. If it were not true, then there would be sequences $\lambda_n \searrow 0$, $\kappa_n > 0$ and $0 \neq u_n \in H^{2\dagger}$ such that $\mathcal{A}_6^{\lambda_n} u_n = -\kappa_n^2 u_n$. Let $h_n = e^{i\theta}\mathcal{U}^{\lambda_n} u_n$. Then $0 \neq h_n \in H^3$ and

$$\frac{1}{2}(-\Delta)\left(-\Delta + \lambda_n^2\right)h_n = e^{i\theta}\mathcal{G}^{\lambda_n}e^{-i\theta}h_n - \kappa_n^2\left(-\Delta + \lambda_n^2\right)h_n.$$

Because of the support properties of $\mathcal{G}^\lambda$, we can insert the cut-off function χ freely both before and after the exponentials. So if $\chi h_n = 0$, then $\frac{1}{2}(-\Delta + \kappa_n^2)(-\Delta + \lambda_n^2)h_n = 0$, whence $h_n = 0$. Therefore $\chi h_n \neq 0$. We normalize $\|\chi h_n\|_{V^1} = 1$.

By Lemma 4.3, $e^{i\theta}\mathcal{G}^{\lambda_n}e^{-i\theta}$ is bounded from H^1 to H^{-1} uniformly in λ. Hence

$$\left(-\frac{1}{2}\Delta + \kappa_n^2\right)\left(-\Delta + \lambda_n^2\right)h_n = \chi e^{i\theta}\mathcal{G}^{\lambda_n}e^{-i\theta}\chi h_n$$

is a bounded sequence in H^{-1}. Multiplying this equation by h_n, we get $\|h_n\|_{V^2}^2 \le C\|\chi h_n\|_{H^1} \le C'\|h_n\|_{V^2}$. Thus h_n is bounded in V^2.

Taking a subsequence, we therefore have $h_n \rightharpoonup h$ weakly in V^2. Since χ has compact support, it follows that $\chi h_n \to \chi h$ strongly in V^1 and that $\|\chi h\|_{V^1} = 1$. Now for any $\ell \in H^1$, we have

$$\left|\left\langle e^{i\theta}\mathcal{G}^{\lambda_n}e^{-i\theta}h_n, \ell\right\rangle\right| = \left|\left\langle \chi h_n, e^{i\theta}\mathcal{G}^{\lambda_n}e^{-i\theta}\chi\ell\right\rangle\right| \le \|\mathcal{G}^{\lambda_n}e^{-i\theta}\chi\ell\|_{H^{-1\dagger}}$$

since χh_n is bounded in V^1. By Lemma 4.3, the right side tends to zero as $n \to \infty$. Thus $e^{i\theta}\mathcal{G}^{\lambda_n}e^{-i\theta}h_n \rightharpoonup 0$ weakly in H^{-1}.

Letting $n \to \infty$, $\lambda_n \to 0$, $\kappa_n \to \kappa_0$, the limit satisfies $(-\frac{1}{2}\Delta + \kappa_0^2)(-\Delta)h = 0$, where $h \in V^2$. Since $\Delta h \in L^2$, we deduce $\Delta h = 0$. We do not know that h or ∇h belong to L^2, but we can use Hardy's inequality (valid for functions in V^2) to estimate

$$\begin{aligned}|\nabla h(x_0)| &\le \frac{C}{R^3}\int_{\{|x-x_0|<R\}} |\nabla h|dx \\ &\le \frac{C'}{R^3}\left(\int \frac{|\nabla h|^2}{|x-x_0|^2}dx\right)^{\frac{1}{2}}(R^5)^{\frac{1}{2}} = O(R^{-\frac{1}{2}})\end{aligned}$$

for every point x_0. Therefore h is a constant. Since $h \in V^2$, $h \equiv 0$. This contradicts $\|\chi h\|_{V^1} = 1$, which proves the claim.

The claim we have just proven means that $\langle \mathcal{A}_6^\lambda u, u\rangle \ge 0$ for all u in the domain $H^{2\dagger}$ of the operator. Thus

$$\left\langle \mathcal{A}_5^\lambda u, u\right\rangle \ge \frac{1}{2}\int \left(|\nabla u|^2 + \frac{1}{r^2}u^2\right)dx.$$

The right side is the squared norm of u in $V^{1\dagger}$. The left side defines a bilinear form $a(u,u)$ that extends continuously to $V^{1\dagger} \times V^{1\dagger}$. So by the Lax–Milgram lemma, the operator $\mathcal{A}_5^\lambda : V^{1\dagger} \to V^{-1\dagger}$ is one-to-one onto.

But $\mathcal{A}_4^\lambda = \mathcal{U}^\lambda\mathcal{A}_5^\lambda\mathcal{U}^\lambda$. Since for fixed $\lambda > 0$, the operator $\mathcal{U}^\lambda$ is an isomorphism: $W^{2\dagger} \to V^{1\dagger}$ and also $V^{-1\dagger} \to W^{-2\dagger}$, we deduce that $\mathcal{A}_4^\lambda$ maps $W^{2\dagger}$ to $W^{-2\dagger}$ in a one-to-one onto fashion. It is also clear that $\|h\|_{V^{2\dagger}} \le C\|\mathcal{U}^\lambda h\|_{V^{1\dagger}}$ so that $(U^\lambda)^{-1} : V^{1\dagger} \to V^{2\dagger}$ with a bound independent of λ and $(U^\lambda)^{-1} : V^{-2\dagger} \to V^{-1\dagger}$ with a bound independent of λ. Therefore

$$\left(\mathcal{A}_4^\lambda\right)^{-1} : V^{-2\dagger} \to V^{-1\dagger} \to V^{+1\dagger} \to V^{+2\dagger}$$

with a bound independent of λ. □

Lemma 4.5. *If S is a ball in $\mathbb{R}^3$, there exist constants $C > 0$ and $\lambda_2 \in (0, \lambda_1)$ such that*

$$\left\langle \mathcal{A}_4^\lambda u, u\right\rangle \ge C\|u\|_{V^{1\dagger}}^2$$

for all $u \in V^{2\dagger}$ with support in S and all $\lambda \in (0, \lambda_2]$.

Proof. We argue by contradiction in a similar way to the preceding proof. If the lemma were false, then there would be sequences $\lambda_n \to 0$ and $u_n \in V^{2\dagger}$ with supports in S such that $\|u_n\|_{V^{1\dagger}} = 1$ but $\langle \mathcal{A}_4^{\lambda_n} u_n, u_n\rangle \to 0$. By definition of $\mathcal{A}_4^\lambda$,

$$\left\langle \left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda_n^2\right) u_n - \mathcal{G}^{\lambda_n} u_n, u_n \right\rangle \to 0.$$

Letting $h_n = e^{i\theta} u_n$, we have

$$\left\langle (-\Delta)\left(-\Delta + \lambda_n^2\right) h_n, h_n \right\rangle - \left\langle e^{i\theta} \mathcal{G}^{\lambda_n} e^{-i\theta} h_n, h_n \right\rangle \to 0.$$

Thus

$$\|\Delta h_n\|_{L^2}^2 + \lambda_n^2 \|\nabla h_n\|_{L^2}^2 \le \|\mathcal{G}^{\lambda_n}\|_{H^{1\dagger} \mapsto H^{-1\dagger}} \|h_n\|_{H^1}^2 + 1.$$

Because the right side is bounded, we therefore have a bound for Δh_n so that h_n is bounded in V^2. Taking a subsequence, we have $h_n \rightharpoonup h_0$ weakly in V^2 and consequently $u_n \rightharpoonup e^{-i\theta} h_0$ weakly in $V^{2\dagger}$. Because of the uniformly bounded support, we can replace $V^{2\dagger}$ by $H^{2\dagger}$ and use the compact embedding to deduce that $u_n \to e^{-i\theta} h_0$ strongly in $H^{1\dagger}$. Therefore $1 = \|e^{-i\theta} h_0\|_{V^{1\dagger}} = \|h_0\|_{V^1}$. By the strong convergence of u_n in $H^{1\dagger}$, and the strong convergence of $\mathcal{G}^{\lambda_n}$ as $\lambda_n \to 0$ from Lemma 4.3, we have $\langle \mathcal{G}^{\lambda_n} u_n, u_n\rangle \to 0$. Therefore

$$\left\langle (-\Delta)\left(-\Delta + \lambda_n^2\right) h_n, h_n \right\rangle \to 0.$$

So h_n tends to zero strongly in V^2 and so also in V^1 (due to the bounded support), which contradicts $\|h_0\|_{V^1} = 1$. □

It follows immediately from either of the two preceding lemmas that $M_\chi (\mathcal{A}_4^\lambda)^{-1} M_\chi$ maps $H^{-1\dagger}$ into $H^{1\dagger}$ with a bound independent of λ.

5. Behavior for small λ

Lemma 5.1. *There exists $\lambda_3 > 0$ such that for any $\lambda \in (0, \lambda_3]$ the operator*

$$\mathcal{N}^\lambda = \mathcal{L}^\lambda + (\mathcal{F}^\lambda)^* \left(\mathcal{A}_4^\lambda\right)^{-1} \mathcal{F}^\lambda$$

is self-adjoint on L_S^2 with domain $H^{2\dagger}$ and has essential spectrum $[\lambda^2, \infty)$. Moreover, if $\mathcal{L}^0$ has a negative eigenvalue, then $\mathcal{N}^\lambda$ also has a negative eigenvalue.

Proof. The bound λ_2 is given in Lemma 4.5. By the proof of Lemma 3.1 in [24], the operator $\mathcal{L}^\lambda$ is relatively compact with respect to $-\Delta + 1/r^2 + \lambda^2$.

By Lemmas 4.3 and 4.5, the operator $(\mathcal{F}^\lambda)^*(\mathcal{A}_4^\lambda)^{-1}\mathcal{F}^\lambda = M_\chi \cdot (\mathcal{F}^\lambda)^* \cdot \{M_\chi(\mathcal{A}_4^\lambda)^{-1}M_\chi\} \cdot \mathcal{F}^\lambda \cdot M_\chi$ maps

$$H^{2\dagger} \to L_c^2 \to H^{-1\dagger} \to H^{1\dagger} \to L^2 \to L_c^2,$$

which implies that it is relatively compact with respect to $-\Delta + 1/r^2 + \lambda^2$. So the self-adjoint and the essential spectrum properties follow from the Kato–Rellich and Weyl theorems.

Assume now that $\mathcal{L}^0$ has a negative eigenvalue $k^0 < 0$ and let $\zeta^0 \in H^{2\dagger}$ be a normalized eigenvector. Write

$$\begin{aligned}\langle \mathcal{N}^\lambda \zeta^0, \zeta^0\rangle - k^0 &= \langle \mathcal{N}^\lambda \zeta^0, \zeta^0\rangle - \langle \mathcal{L}^0 \zeta^0, \zeta^0\rangle \\ &= \langle (\mathcal{L}^\lambda - \mathcal{L}^0)\zeta^0, \zeta^0\rangle + \big\langle (\mathcal{F}^\lambda)^* \big(\mathcal{A}_4^\lambda\big)^{-1} \mathcal{F}^\lambda \zeta^0, \zeta^0\big\rangle.\end{aligned}$$

By Lemma 4.2(f), the first term on the right is less than $\|(\mathcal{L}^\lambda - \mathcal{L}^0)\zeta^0\|_{L^2} \to 0$, as $\lambda \searrow 0$. By Lemma 4.5, the second term is bounded by

$$\begin{aligned}\Big|\Big\langle \big(\mathcal{A}_4^\lambda\big)^{-1} M_\chi \mathcal{F}^\lambda \zeta^0, M_\chi \mathcal{F}^\lambda \zeta^0\Big\rangle\Big| &\le \big\|M_\chi \big(\mathcal{A}_4^\lambda\big)^{-1} M_\chi\big\|_{H^{-1\dagger} \mapsto H^{1\dagger}} \|\mathcal{F}^\lambda \zeta^0\|_{H^{-1\dagger}}^2 \\ &\le C\|\mathcal{F}^\lambda \zeta^0\|_{H^{-1\dagger}}^2 \to 0 \quad \text{as } \lambda \searrow 0\end{aligned}$$

because C is independent of λ, and using Lemma 4.3. Thus $\langle \mathcal{N}^\lambda \zeta^0, \zeta^0\rangle \to k^0 < 0$ as $\lambda \searrow 0$. So if λ_3 is small enough and $0 < \lambda \le \lambda_3$, then $\mathcal{N}^\lambda$ has a negative eigenvalue. □

Now we perform a finite-dimensional truncation of the matrix operator (53). Let $\{\sigma_1, \sigma_2, \dots\}$ be a sequence of functions in $H_c^{2\dagger}$, for which the finite linear combinations are dense in $V^{2\dagger}$. Orthogonalize them so that they form an orthonormal set in L_S^2. As before, $\langle\ ,\ \rangle$ denotes the usual L^2 pairing and we will denote the standard inner product in $\mathbb{R}^n$ by a dot. Let n be a positive integer. Define the projection operator $P_n : V^{-2\dagger} \to \mathbb{R}^n$ and its L^2-adjoint $P_n^* : \mathbb{R}^n \to V^{2\dagger}$ by

$$P_n h = \{\langle h, \sigma_j\rangle\}_{j=1}^n, \qquad P_n^* b = \sum_{j=1}^n b^j \sigma_j,$$

where $h \in V^{-2\dagger}$ and $b = (b^1, \dots, b^n) \in \mathbb{R}^n$. Then $P_n P_n^* b = b$ for any $b \in \mathbb{R}^n$, and $P_n^* P_n h = \sum_{j=1}^n \langle h, \sigma_j\rangle \sigma_j$ for any $h \in V^{-2\dagger}$. Define the "approximate matrix operator"

$$\mathcal{M}_n^\lambda = \begin{pmatrix} \mathcal{L}^\lambda & (\mathcal{F}^\lambda)^* P_n^* \\ P_n \mathcal{F}^\lambda & -P_n \mathcal{A}_4^\lambda P_n^* \end{pmatrix}$$

which takes $V^{2\dagger} \times \mathbb{R}^n$ into $L_S^2 \times \mathbb{R}^n$.

Lemma 5.2. *Let $0 < \lambda \le \lambda_3$. For any $\eta \in L_{loc}^2$, let us define $d_n = (P_n \mathcal{A}_4^\lambda P_n^*)^{-1} P_n \mathcal{F}^\lambda \eta$. Then*

$$\sup_n \|P_n^* d_n\|_{V^{2\dagger}} < \infty.$$

Proof. Because λ is fixed, for brevity we denote $\mathcal{A} = \mathcal{A}_4^\lambda$ and $\mathcal{F} = \mathcal{F}^\lambda$. Note that $\alpha = P_n \mathcal{A} P_n^*$ is the $n \times n$ symmetric positive-definite matrix with entries $\alpha_{jk} = \langle \mathcal{A}_4 \sigma_k, \sigma_j \rangle$. Let $c(n) = \|\chi P_n^* d_n\|_{H^{1\dagger}}$. We will show that $c(n)$ is bounded. Suppose on the contrary that $c(n) \to \infty$. Let $u_n = P_n^* d_n / c(n)$ so that $\|\chi u_n\|_{H^{1\dagger}} = 1$. Then $P_n \mathcal{A} u_n = P_n \mathcal{A} P_n^* d_n / c(n) = P_n \mathcal{F} \eta / c(n)$ so that

$$\begin{aligned}\langle \mathcal{A} u_n, u_n \rangle &= \left\langle \mathcal{A} u_n, \frac{1}{c(n)} P_n^* d_n \right\rangle = \frac{1}{c(n)} P_n \mathcal{A} u_n \cdot d_n \\ &= \frac{1}{c^2(n)} P_n \mathcal{F} \eta \cdot d_n = \frac{1}{c(n)} \langle \mathcal{F} \eta, u_n \rangle.\end{aligned}$$

Thus

$$\left\langle \left(-\Delta + \frac{1}{r^2} \right) \left(-\Delta + \frac{1}{r^2} + \lambda^2 \right) u_n, u_n \right\rangle = \left\langle \mathcal{G}^\lambda u_n, u_n \right\rangle + \frac{1}{c(n)} \langle \mathcal{F} \eta, \chi u_n \rangle$$

so that, as in the proof of Lemma 4.4,

$$\|u_n\|_{W^{2\dagger}}^2 \le C \|\chi u_n\|_{H^{1\dagger}}^2 + \frac{1}{c(n)} \|\chi u_n\|_{H^{1\dagger}} \le C + 1.$$

We take a subsequence so that $u_n \rightharpoonup u_0$ weakly in $W^{2\dagger}$. Then $\chi u_n \to \chi u_0$ strongly in $H^{1\dagger}$, so that $\|\chi u_0\|_{H^{1\dagger}} = 1$. Fix an integer $m \ge 1$ and let $n \ge m$. Then $P_n^* \delta_m = \sigma_m$, where $(\delta_m)_j = 1$ for $j = m$ and is otherwise 0. Then

$$\langle \mathcal{A} u_n, \sigma_m \rangle = P \mathcal{A} u_n \cdot \delta_m = \frac{1}{c(n)} P_n \mathcal{F} \eta \cdot \delta_m = \frac{1}{c(n)} \langle \mathcal{F} \eta, \sigma_m \rangle \to 0$$

as $n \to \infty$ since $\langle \mathcal{F} \eta, \sigma_m \rangle$ is independent of n. Thus $\langle \mathcal{A} u_0, \sigma_m \rangle = 0$ for all m, so that $\mathcal{A} u_0 = 0$. So $u_0 = 0$, which contradicts $\|\chi u_0\|_{H^{1\dagger}} = 1$. Thus $c(n)$ is indeed bounded.

Now substituting $u_n = P_n^* d_n / c(n)$ into the inequality above, we get

$$\left\| \frac{1}{c(n)} P_n^* d_n \right\|_{W^{2\dagger}}^2 = \|u_n\|_{W^{2\dagger}}^2 \le C \|\chi u_n\|_{H^{1\dagger}}^2 + \frac{C}{c(n)} \|\chi u_n\|_{H^{1\dagger}}.$$

Multiplying by $c^2(n)$, we find

$$\left\| P_n^* d_n \right\|_{W^{2\dagger}}^2 \le C \left\| \chi P_n^* d_n \right\|_{H^{1\dagger}}^2 + C \left\| \chi P_n^* d_n \right\|_{H^{1\dagger}} = C c^2(n) + C c(n) \le C'.$$

Therefore $P_n^* d_n$ is bounded in $W^{2\dagger}$, hence in $V^{2\dagger}$. □

Lemma 5.3. *Fix $0 < \lambda \le \lambda_3$. There exists a positive integer $N = N(\lambda)$ such that for $n \ge N$, the matrix operator*

$$\mathcal{M}_n^\lambda = \begin{pmatrix} \mathcal{L}^\lambda & (\mathcal{F}^\lambda)^* P_n^* \\ P_n \mathcal{F}^\lambda & -P_n \mathcal{A}_4^\lambda P_n^* \end{pmatrix}$$

is self-adjoint on $L_S^2 \times \mathbb{R}^n$ with domain $H^{2\dagger} \times \mathbb{R}^n$, has essential spectrum $[\lambda^2, \infty)$ and has at least $n + 1$ negative eigenvalues.

Proof. We recall that $\mathcal{L}^\lambda$ is self-adjoint with essential spectrum $[\lambda^2, \infty)$. However, the symmetric operator

$$\begin{pmatrix} 0 & (\mathcal{F}^\lambda)^* P_n^* \\ P_n \mathcal{F}^\lambda & -P_n \mathcal{A}_4^\lambda P_n^* \end{pmatrix}$$

has finite-dimensional range and so it is compact. The theorems of Kato–Rellich and Weyl apply here directly to prove the first two assertions of the lemma. It remains to consider the negative spectrum.

The last assertion is equivalent to saying that there is an $(n+1)$-dimensional subspace $S \subset H^{2\dagger} \times \mathbb{R}^n$ such that $\langle \mathcal{M}_n^\lambda z, z\rangle < 0$ for all $z \in S \setminus \{0\}$. For simplicity, we temporarily drop the superscript λ as it is fixed in this proof. As above, let α be the $n \times n$ symmetric positive matrix with entries $\alpha_{jk} = \langle \mathcal{A}_4 \sigma_k, \sigma_j\rangle$. Let

$$\mathcal{J}_n = \begin{pmatrix} I & 0 \\ \alpha^{-1} P_n \mathcal{F} & I \end{pmatrix}.$$

Then

$$\mathcal{J}_n^* \mathcal{M}_n \mathcal{J}_n = \begin{pmatrix} \mathcal{L} + (\mathcal{F})^* P_n^* \alpha^{-1} P_n \mathcal{F} & 0 \\ 0 & -\alpha \end{pmatrix}$$

has the same number of negative eigenvalues as $\mathcal{M}_n$. But $-\alpha$ has exactly n negative eigenvalues, so it suffices to prove that

$$\mathcal{N}_n = \mathcal{L} + (\mathcal{F})^* P_n^* \alpha^{-1} P_n \mathcal{F}$$

has a negative eigenvalue when n is large.

By Lemma 5.1, the untruncated operator $\mathcal{N} = \mathcal{L} + (\mathcal{F})^*(\mathcal{A}_4)^{-1}\mathcal{F}$ has a negative eigenvalue. Let $\eta = \eta^\lambda$ be an eigenvector of $\mathcal{N} = \mathcal{N}^\lambda$ as in Lemma 5.2 with eigenvalue $\mu < 0$ and $\|\eta\|_{L^2} = 1$. Let $\xi = \mathcal{A}_4^{-1}\mathcal{F}\eta$. Since $\eta \in L^2$, we have $\mathcal{F}\eta \in H^{-1\dagger}$ and $\xi \in V^{2\dagger}$. Recall that $d_n = \alpha^{-1} P_n \mathcal{F}\eta$. By these definitions, we have

$$\begin{aligned} &\langle \mathcal{N}_n \eta, \eta\rangle - \langle \mathcal{N}\eta, \eta\rangle \\ &= \big\langle \big(\mathcal{F}^* P_n^* \alpha^{-1} P_n \mathcal{F} - \mathcal{F}^* \mathcal{A}_4^{-1}\mathcal{F}\big)\eta, \eta\big\rangle = \big\langle \big(P_n^* \alpha^{-1} P_n \mathcal{F} - \mathcal{A}_4^{-1}\mathcal{F}\big)\eta, \mathcal{F}\eta\big\rangle \\ &= \big\langle P_n^* d_n - \xi, \mathcal{A}_4 \xi\big\rangle = \big\langle \mathcal{A}_4(P_n^* d_n - \xi), \xi\big\rangle. \end{aligned}$$

Choose a sequence ξ_n such that $\|\xi_n - \xi\|_{V^{2\dagger}} \to 0$ and such that each ξ_n is a linear combination of $\{\sigma_1, \ldots, \sigma_n\}$. Then ξ_n belongs to the range of P_n^*. Because $P_n \mathcal{A}_4(P_n^* d_n - \xi) = \alpha d_n - P_n \mathcal{F}\eta = 0$, it follows that

$$\begin{aligned} |\langle \mathcal{N}_n \eta, \eta\rangle - \langle \mathcal{N}\eta, \eta\rangle| &= \big|\big\langle \mathcal{A}_4(P_n^* d_n - \xi), \xi - \xi_n\big\rangle\big| \\ &\le \big\| \mathcal{A}_4\big(P_n^* d_n - \xi\big)\big\|_{V^{-2\dagger}} \|\xi - \xi_n\|_{V^{2\dagger}} \\ &\le C\big\| P_n^* d_n - \xi\big\|_{V^{2\dagger}} \|\xi - \xi_n\|_{V^{2\dagger}} \\ &\le C' \|\xi - \xi_n\|_{V^{2\dagger}} \to 0 \end{aligned}$$

as $n \to \infty$. Since $\langle \mathcal{N}\eta, \eta\rangle < 0$, it follows that $\langle \mathcal{N}_n \eta, \eta\rangle < 0$ for sufficiently large n, so that $\mathcal{N}_n$ must have a negative eigenvalue. □

6. Approximate growing mode

Now we consider the behavior for large λ.

Lemma 6.1. *There exists $\lambda_4 > 0$ such that if $\lambda \geq \lambda_4$, then for each n the operator $\mathcal{M}_n^\lambda$ has at most n negative eigenvalues.*

Proof. For $h \in H^{2\dagger}$,

$$(\mathcal{L}^\lambda h, h) \geq \left(\mathcal{A}_2^\lambda h, h\right) = \left(\left(-\Delta + \frac{1}{r^2} + \lambda^2\right) h, h\right) - \iint r\hat{v}_\theta \mu_p dv|h|^2 dx \\ - \iint \hat{v}_\theta \mu_e \mathcal{Q}^\lambda(\hat{v}_\theta h) dv h dx.$$

The last term is bounded by

$$\left|\left\langle \mathcal{Q}^\lambda(\hat{v}_\theta h), \hat{v}_\theta h\right\rangle_{|\mu_e|}\right| \leq \|\hat{v}_\theta h\|_{|\mu_e|}^2 \leq \left(\sup_x \int |\mu_e| dv\right)|h|_2^2$$

by Lemma 4.1 (a). Therefore $(\mathcal{L}^\lambda h, h) \geq (\lambda^2 - C_0)|h|_2^2$, where $C_0 = \sup_x(\int(|r\mu_p| + |\mu_e|)dv)$. Now for any $(h, b) \in H^{2\dagger} \times \mathbb{R}^n$,

$$\begin{aligned}\left\langle \mathcal{M}_n^\lambda \begin{pmatrix} h \\ b \end{pmatrix}, \begin{pmatrix} h \\ b \end{pmatrix} \right\rangle &= (\mathcal{L}^\lambda h, h) + 2\left\langle \mathcal{F}^\lambda h, P_n^* b\right\rangle - \left\langle \mathcal{A}_4^\lambda P_n^* b, P_n^* b\right\rangle \\ &\geq \left(\lambda^2 - C_0\right)|h|_2^2 - \|\mathcal{F}^\lambda h\|_{H^{-1\dagger}} \left\|\chi P_n^* b\right\|_{H^{1\dagger}} \\ &\quad - C(\lambda)\left\|P_n^* b\right\|_{V^{2\dagger}}^2 \\ &\geq \left(\lambda^2 - C_0\right)|h|_2^2 - 2C_1|h|_2 \left\|P_n^* b\right\|_{V^{2\dagger}} - C(\lambda)\left\|P_n^* b\right\|_{V^{2\dagger}}^2 \\ &\geq -\left(C_1^2 + C(\lambda)^2\right)\left\|P_n^* b\right\|_{V^{2\dagger}}^2\end{aligned}$$

provided $\lambda \geq \lambda_4 = \sqrt{C_0 + 1}$. Since $b \in \mathbb{R}^n$, it follows that $\mathcal{M}_n^\lambda$ has at most n negative eigenvalues. □

Now we are ready to exhibit an approximate growing mode.

Lemma 6.2. *For each positive integer $n \geq N(\lambda_3)$, there exists $\lambda_n \in [\lambda_3, \lambda_4]$ such that $\mathcal{M}_n^{\lambda_n}$ has a non-trivial kernel. Here λ_3 and λ_4 are in Lemmas 5.3 and 6.1.*

Proof. We emphasize that λ_3 and λ_4 do not depend on n. We use continuity with respect to λ. First, $\mathcal{M}_n^\lambda$ is a continuous family of operators of λ in the sense that if $\sigma > 0$, then there exists $C, \delta > 0$ such that

$$\left\|\mathcal{M}_n^\lambda - \mathcal{M}_n^\sigma\right\| \leq C|\lambda - \sigma|$$

for $|\lambda - \sigma| < \delta$, $\lambda, \sigma \in (0, \infty)$, where $\|\|$ denotes the operator norm from $L_S^2 \times \mathbb{R}^n$ to $L_S^2 \times \mathbb{R}^n$. This continuity property follows immediately from Lemma 4.2.

By Lemma 5.3, $\mathcal{M}_n^{\lambda_3}$ has at least $(n+1)$ negative eigenvalues. By Lemma 6.1, $\mathcal{M}_n^{\lambda_4}$ has at most n negative eigenvalues. By [18, IV-3.5], the eigenvalues of $\mathcal{M}_n^{\lambda}$ within the interval $[\lambda_3, \lambda_4]$ are continuous functions of λ. In particular, the dimension of the corresponding eigenspace is a constant. Hence at least one eigenvalue must cross from negative to positive. So there exists some $\lambda_n \in [\lambda_3, \lambda_4]$ such that $\mathcal{M}_n^{\lambda_n}$ has a non-trivial kernel. □

7. Limit as $n \to +\infty$

Lemma 7.1. *There exist* λ_0, h_0, k_0 *such that* $0 < \lambda_0 < \infty$, $h_0 \in H^{2\dagger}$, $k_0 \in H^{2\dagger}$ *and*

$$\mathcal{L}^{\lambda_0} h_0 + (\mathcal{F}^{\lambda_0})^* k_0 = 0, \tag{57}$$

$$\mathcal{F}^{\lambda_0} h_0 - \mathcal{A}_4^{\lambda_0} k_0 = 0 \tag{58}$$

with $(h_0, k_0) \neq (0, 0)$.

Proof. By Lemma 6.2, for each $n \geq N(\lambda_3)$ there exists $\lambda_n \in [\lambda_3, \lambda_4]$ and a non-zero solution $(h_n, b_n) \in H^{2\dagger} \times \mathbb{R}^n$ such that

$$\mathcal{L}^{\lambda_n} h_n + (\mathcal{F}^{\lambda_n})^* P_n^* b_n = 0, \tag{59}$$

$$P_n \mathcal{F}^{\lambda_n} h_n - P_n \mathcal{A}_4^{\lambda_n} P_n^* b_n = 0. \tag{60}$$

We normalize h_n, b_n such that

$$\|h_n\|_{L^2} + \left\|\chi P_n^* b_n\right\|_{H^{1\dagger}} = 1$$

by Lemma 4.5. We claim that h_n is bounded in $H^{2\dagger}$. Indeed, $\|\chi P_n^* b_n\|_{H^{1\dagger}} \leq 1$, so that $(\mathcal{F}^{\lambda_n})^* P_n^* b_n$ is bounded in L^2, and $\mathcal{L}^{\lambda_n} h_n$ is bounded in L^2. Since $\|h_n\|_{L^2} \leq 1$, $(\mathcal{B}^{\lambda_n})^* (\mathcal{A}_1^{\lambda_n})^{-1} \mathcal{B}^{\lambda_n} h_n$ is also bounded in L^2, and so are $\mathcal{A}_2^{\lambda_n} h_n$ and $(-\Delta + \frac{1}{r^2} + \lambda_n^2) h_n$. Therefore h_n is bounded in $H^{2\dagger}$. By (60) we have

$$\left\langle \mathcal{A}_4^{\lambda_n} P_n^* b_n, P_n^* b_n \right\rangle = \left\langle \mathcal{F}^{\lambda_n} h_n, P_n^* b_n \right\rangle.$$

The right side of this equation is bounded. So $\langle (-\Delta + \frac{1}{r^2})(-\Delta + \frac{1}{r^2} + \lambda_n^2) P_n^* b_n, P_n^* b_n \rangle$ is also bounded. Therefore $P_n^* b_n$ is bounded in $V^{1\dagger} \cap V^{2\dagger}$. Now we take subsequences such that $\lambda_n \to \lambda_0 \in [\lambda_3, \lambda_4]$, $h_n \to h_0$ weakly in $H^{2\dagger}$, $P_n^* b_n \to k_0$ weakly in $V^{2\dagger}$. We look at each term for (59), (60) separately. First, for any $l \in H^{2\dagger}$,

$$\begin{aligned} &\left|\left(\mathcal{L}^{\lambda_n} h_n - \mathcal{L}^{\lambda_0} h_0, l\right)\right| \\ &\leq \left|\left(\mathcal{L}^{\lambda_0}(h_n - h_0), l\right)\right| + \left|\left((\mathcal{L}^{\lambda_n} - \mathcal{L}^{\lambda_0}) h_n, l\right)\right| \\ &\leq \left|\left((h_n - h_0), \mathcal{L}^{\lambda_0} l\right)\right| + \|\mathcal{L}^{\lambda_n} - \mathcal{L}^{\lambda_0}\|_{L^2 \to L^2} \|h_n\|_{L^2} \|l\|_{L^2} \to 0 \end{aligned}$$

as $n \to \infty$, by Lemma 4.2. Thus $\mathcal{L}^{\lambda_n} h_n \to \mathcal{L}^{\lambda_0} h_0$ weakly in $H^{-2\dagger}$.

Secondly, for any $l \in L_S^2$,

$$\begin{aligned}
&\left|\left\langle (\mathcal{F}^{\lambda_n})^* P_n^* b_n - (\mathcal{F}^{\lambda_0})^* k_0, l \right\rangle\right| \\
&\le \left|\left\langle ((\mathcal{F}^{\lambda_n})^* - (\mathcal{F}^{\lambda_0})^*) k_0, l \right\rangle\right| + \left|\left\langle (\mathcal{F}^{\lambda_0})^* \left(P_n^* b_n - k_0\right), l \right\rangle\right| \\
&\le \|\mathcal{F}^{\lambda_n} - \mathcal{F}^{\lambda_0}\|_{L^2 \to H_C^{-1\dagger}} \|\chi k_0\|_{H^{1\dagger}} \|l\|_{L^2} + \left|\left\langle \left(P_n^* b_n - k_0\right), \mathcal{F}^{\lambda_0} l \right\rangle\right| \to 0
\end{aligned}$$

as $n \to \infty$ by Lemma 4.3. Thus $(\mathcal{F}^{\lambda_n})^* P_n^* b_n \to (\mathcal{F}^{\lambda_0})^* k_0$ weakly in L_S^2.

Thirdly, for any $g \in H^{2\dagger}$, let $g_n = \sum_{j=1}^n c_n^j \sigma_j \to g$ strongly in $H^{2\dagger}$ as $n \to \infty$. Let $\gamma_n = \{c_n^j\}_{j=1}^n \in \mathbb{R}^n$. Then $P_n^* \gamma_n = g_n$ and $\langle P_n \mathcal{F}^{\lambda_n} h_n, \gamma_n \rangle = \langle \mathcal{F}^{\lambda_n} h_n, g_n \rangle$. Hence again using Lemma 4.3,

$$\begin{aligned}
&\left|\left\langle P_n \mathcal{F}^{\lambda_n} h_n, \gamma_n \right\rangle - \left\langle \mathcal{F}^{\lambda_0} h_0, g \right\rangle\right| \\
&\le \left|\left\langle (\mathcal{F}^{\lambda_n} - \mathcal{F}^{\lambda_0}) h_n, g_n \right\rangle\right| + \left|\left\langle \mathcal{F}^{\lambda_0} h_n, g_n - g \right\rangle\right| + \left|\left\langle \mathcal{F}^{\lambda_0} (h_n - h_0), g \right\rangle\right| \\
&\le \|\mathcal{F}^{\lambda_n} - \mathcal{F}^{\lambda_0}\|_{L^2 \to H^{-1\dagger}} \|h_n\|_{L^2} \|\chi g_n\|_{H^{1\dagger}} \\
&\quad + \left\|\mathcal{F}^{\lambda_0} h_n\right\|_{H^{-1\dagger}} \|\chi(g_n - g)\|_{H^{1\dagger}} + \left|\left\langle (h_n - h_0), (\mathcal{F}^{\lambda_0})^* g \right\rangle\right| \\
&\le C_1 \|\mathcal{F}^{\lambda_n} - \mathcal{F}^{\lambda_0}\|_{L^2 \to H^{-1\dagger}} \|h_n\|_{L^2} \|g\|_{H^{2\dagger}} + C_2 \|h_n\|_{L^2} \|g_n - g\|_{H^{2\dagger}} \\
&\quad + \left|\left\langle (h_n - h_0), (\mathcal{F}^{\lambda_0})^* g \right\rangle\right| \\
&\to 0
\end{aligned}$$

as $n \to 0$. Thus $\langle P_n \mathcal{F}^{\lambda_n} h_n, \gamma_n \rangle \to \langle \mathcal{F}^{\lambda_0} h_0, g \rangle$ for all $g \in H^{2\dagger}$.

Fourthly, using the same $g \in H^{2\dagger}$ as above,

$$\begin{aligned}
&\left|\left\langle P_n \mathcal{A}_4^{\lambda_n} P_n^* b_n, \gamma_n \right\rangle - \left\langle \mathcal{A}_4^{\lambda_0} k_0, g \right\rangle\right| \\
&= \left|\left\langle \mathcal{A}_4^{\lambda_n} P_n^* b_n, g_n \right\rangle - \left\langle \mathcal{A}_4^{\lambda_0} k_0, g \right\rangle\right| \\
&\le \left|\left\langle \left(\mathcal{A}_4^{\lambda_n} - \mathcal{A}_4^{\lambda_0}\right) P_n^* b_n, g \right\rangle\right| + \left|\left\langle \mathcal{A}_4^{\lambda_0} P_n^* b_n, g_n - g \right\rangle\right| \\
&\quad + \left|\left\langle \mathcal{A}_4^{\lambda_0} (P_n^* b_n - k_0), g \right\rangle\right| \\
&= I + II + III.
\end{aligned}$$

The first term on the right is estimated as

$$\begin{aligned}
I &\le \left(\lambda_n^2 - \lambda_0^2\right) \left\| P_n^* b_n \right\|_{V^{1\dagger}} \|g\|_{V^{1\dagger}} \\
&\quad + \|\mathcal{G}^{\lambda_n} - \mathcal{G}^{\lambda_0}\|_{H^{1\dagger} \to H^{-1\dagger}} \left\| \chi P_n^* b_n \right\|_{H^{1\dagger}} \|\chi g\|_{H^{1\dagger}} \\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$\mathcal{G}^\lambda = \mathcal{E}^\lambda + (\mathcal{D}^\lambda)^* \left(\mathcal{A}_1^\lambda\right)^{-1} \mathcal{D}^\lambda : H^{1\dagger} \to H^{-1\dagger}.$$

By Lemma 4.4, $\|\mathcal{A}_4^{\lambda_0}\|_{V^{2\dagger}\to V^{-2\dagger}} \le C$, so

$$II \le C \left\| P_n^* b_n \right\|_{V^{2\dagger}} \|g_n - g\|_{V^{2\dagger}} \to 0, \quad \text{as } n \to \infty.$$

As for the third term, $\mathcal{A}_4^{\lambda_0} g \in H^{-2\dagger}$ so that

$$III = \left| \left\langle \left(P_n^* b_n - k_0 \right), \mathcal{A}_4^{\lambda_0} g \right\rangle \right| \to 0, \quad \text{as } n \to \infty.$$

So $\langle P_n \mathcal{A}_4^{\lambda_n} P_n^* b_n, \gamma_n \rangle \to \langle \mathcal{A}_4^{\lambda_0} k_0, g \rangle$ for all $g \in H^{2\dagger}$. Thus all four terms in (59), (60) converge and the limits satisfy (57) and (58).

It remains to show that $(h_0, k_0) \ne (0, 0)$. Let us write (59) explicitly, using the definition of $\mathcal{L}^{\lambda_n}$, as

$$\left(-\Delta + \lambda_n^2\right)\left(e^{i\theta} h_n\right) = e^{i\theta} \left(-\Delta + \frac{1}{r^2} + \lambda_n^2 \right) h_n = f_n,$$

where

$$f_n = -e^{i\theta} (\mathcal{F}^{\lambda_n})^* P_n^* b_n + e^{i\theta} \int r \hat{v}_\theta \mu_p dv \, h_n$$
$$+ e^{i\theta} \int \hat{v}_\theta \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta h) dv - e^{i\theta} (\mathcal{B}^{\lambda_n})^* \left(\mathcal{A}_1^{\lambda_n}\right)^{-1} \mathcal{B}^{\lambda_n} h_n.$$

By Lemmas 4.2 and 4.3, f_n is bounded in $L^2(\mathbb{R}^3)$ and has support in the fixed bounded set $S_x \subset \mathbb{R}^3$. Therefore the inversion of the operator $(-\Delta + \lambda_n^2)$ with $\lambda_n \ge \lambda_3 > 0$ implies that h_n decays exponentially as $|x| \to \infty$, uniformly in n. Thus $\{h_n\}$ is compact in L^2, so that $h_n \to h_0$ strongly in L^2. Since $\|P_n^* b_n\|_{V^{2\dagger}}$ is uniformly bounded, $\chi P_n^* b_n \to \chi k_0$ strongly in $H^{1\dagger}$. Therefore, we have $\|h_0\|_{L^2} + \|\chi k_0\|_{H^{1\dagger}} = 1$ and so $(h_0, k_0) \ne (0, 0)$. □

8. Growing mode

Changing notation, $A_\theta = h_0,\ \pi = k_0,$ and replacing λ_0 by λ, we have from (57) and (58) the pair of equations

$$\mathcal{L}^\lambda A_\theta = -(\mathcal{F}^\lambda)^* \pi, \quad \mathcal{A}_4^\lambda \pi = \mathcal{F}^\lambda A_\theta \tag{61}$$

where $(A_\theta, \pi) \ne (0, 0)$, $A_\theta \in H^{2\dagger}$, $\pi \in H^{2\dagger}$, $\lambda \in (0, +\infty)$. We must define f, ϕ and $\mathbf{A}$ so that (38), (39) and (40) are satisfied by $e^{\lambda t}(f, \phi, \mathbf{A})$. Indeed, motivated by Sect. 3, we define

$$A_r = -\partial_z \pi, \quad A_z = \frac{1}{r}\partial_r (r\pi), \quad \mathbf{A} = (A_r, A_\theta, A_z),$$
$$\phi = \left(\mathcal{A}_1^\lambda\right)^{-1} \left(\mathcal{B}^\lambda A_\theta + \mathcal{D}^\lambda \pi\right), \tag{62}$$
$$\mathbf{E} = -\nabla\phi - \lambda \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

and

$$f(x, v) = -\mu_e \phi + \mu_e \mathcal{Q}^\lambda \phi - \mu_p r A_\theta - \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta A_\theta) - \mu_e \mathcal{Q}^\lambda (G\pi). \tag{63}$$

It follows directly that $\nabla \cdot \mathbf{A} = 0$, $\mathbf{A} \in H^1$, $\phi \in V^1$, $\mathbf{E} \in L^2$, $\mathbf{B} \in L^2$, $A_\theta \in L^\infty$ and by Lemmas 4.2, 4.3 and 4.1, $f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$.

Lemma 8.1. *The Poisson equation $-\Delta\phi = \rho$ is satisfied. Moreover, $\phi \in H^2(\mathbb{R}^3)$ and $f \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.*

Proof. By (62), we have $\mathcal{A}_1^\lambda \phi = \mathcal{B}^\lambda A_\theta + \mathcal{D}^\lambda \pi$, which is written explicitly as

$$\begin{aligned}-\Delta\phi = \Big(\int \mu_e dv\Big)\phi - \int \mu_e \mathcal{Q}^\lambda \phi dv - \Big(\int \hat{v}_\theta \mu_e dv\Big) A_\theta \\ + \int \mu_e \mathcal{Q}^\lambda (\hat{v}_\theta A_\theta) dv + \int \mu_e \mathcal{Q}^\lambda (G\pi) dv.\end{aligned}$$

On the other hand, by (63) and $\int (r\mu_p + \hat{v}_\theta \mu_e) dv = 0$, we get exactly the same expression for $\rho = -\int f dv$. Now integrating (63) in v and x, we find that the first and second terms cancel, the third and fourth terms cancel, and the fifth term $\iint \mu_e G\pi dv dx$ vanishes by the oddness of the integrand in (v_r, v_z). Thus $\int \rho dx = -\iint f dx dv = 0$. Furthermore, ρ has compact support. So by the proof of Lemma 3.2 of [24], $\phi \in L^2$. Since $\rho \in L^2$, by elliptic regularity we have $\phi \in H^2 \subset L^\infty$.

Moreover,

$$\begin{aligned}\left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right)\pi &= \mathcal{A}_4^\lambda \pi + \mathcal{G}^\lambda \pi \\ &= \mathcal{F}^\lambda A_\theta + \mathcal{E}^\lambda \pi + (\mathcal{D}^\lambda)^* \big(\mathcal{A}_1^\lambda\big)^{-1} \mathcal{D}^\lambda \pi \\ &\in H^{-1\dagger}\end{aligned}$$

so that $\pi \in V^{3\dagger}$ and $G\pi \in L^\infty$. Therefore from (63), $f \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. □

Lemma 8.2. *The function f defined by* (63) *satisfies* (38).

Proof. We have

$$f = -\mu_e \phi + \mu_e \mathcal{Q}^\lambda \phi - \mu_p r A_\theta - \mu_e \mathcal{Q}^\lambda (\hat{v} \cdot \mathbf{A}).$$

To show that f is a weak solution of (38), we take any $g \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$, and compute

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{R}^3} (Dg) f dx dv \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} (Dg)(-\mu_e\phi) dx dv + \iint_{\mathbb{R}^3\times\mathbb{R}^3} (Dg) \mu_e \mathcal{Q}^\lambda \phi dx dv \\ &\quad + \iint_{\mathbb{R}^3\times\mathbb{R}^3} (Dg)(-\mu_p r A_\theta) dx dv - \iint_{\mathbb{R}^3\times\mathbb{R}^3} (Dg) \mu_e \mathcal{Q}^\lambda (\hat{v} \cdot \mathbf{A}) dx dv \\ &= I + II + III + IV.\end{aligned}$$

Since D is skew-adjoint, the first term is

$$I = \iint_{\mathbb{R}^3\times\mathbb{R}^3} gD(\mu_e\phi)dxdv = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mu_e gD\phi dxdv.$$

Similarly,

$$III = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mu_p gD(rA_\theta)dxdv.$$

By definition of $\mathcal{Q}$,

$$\begin{aligned}
II &= \int_{-\infty}^{0} \lambda e^{\lambda s} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mu_e Dg(x,v)\phi(X(s;x,v))dxdvds \\
&= \int_{-\infty}^{0} \lambda e^{\lambda s} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mu_e (Dg)(X(-s),V(-s))\phi(x)dxdvds \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mu_e \int_{-\infty}^{0} \lambda e^{\lambda s}\left(-\frac{d}{ds}g(X(-s),V(-s))\right) ds\phi(x)dxdv \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mu_e\left\{-\lambda g(x,v) + \int_{-\infty}^{0} \lambda^2 e^{\lambda s} g(X(-s),V(-s))ds\right\}\phi(x)dxdv \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left\{-\mu_e\lambda\phi(x) + \mu_e\int_{-\infty}^{0} \lambda^2 e^{\lambda s}\phi(X(s),V(s))ds\right\}g(x,v)dxdv \\
&= \lambda \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left\{-\mu_e\phi + \mu_e\mathcal{Q}^\lambda\phi\right\}gdxdv.
\end{aligned}$$

The preceding calculations are valid since ϕ belongs to H^2 and thus is continuous. Similarly,

$$IV = -\lambda \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left\{-\mu_e\hat{v}\cdot A + \mu_e\mathcal{Q}^\lambda(\hat{v}\cdot\mathbf{A})\right\}gdxdv.$$

So we have

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} (Dg)fdxdv \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \lambda\left\{-\mu_e\phi + \mu_e\mathcal{Q}^\lambda\phi + \mu_e\mathcal{Q}^\lambda(\hat{v}\cdot\mathbf{A})\right\}gdxdv \\
&\quad + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \{\mu_e D\phi + \mu_p D(rA_\theta) + \lambda\mu_e\hat{v}\cdot\mathbf{A}\}gdxdv \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \{\lambda(f+\mu_p rA_\theta) + \mu_e D\phi + \mu_p D(rA_\theta) + \lambda\mu_e\hat{v}\cdot\mathbf{A}\}gdxdv.
\end{aligned}$$

So f weakly satisfies the equation

$$(\lambda + D)f = -\mu_e D\phi - \mu_p D(rA_\theta) - \lambda\mu_p rA_\theta - \lambda\mu_e\hat{v}\cdot\mathbf{A}$$

which is exactly (38). □

Lemma 8.3. *Denoting* $\rho = -\int f dv$ *and* $\mathbf{j} = -\int \hat{v} f\, dv$, *we have the continuity equation* $\lambda\rho + \nabla \cdot \mathbf{j} = 0$.

Proof. By the last lemma, f satisfies (38) weakly, which can be written as

$$\lambda f + \nabla_x \cdot (\hat{v} f) - \nabla_v \cdot \{(\mathbf{E}^0 + \mathbf{E}^{ext} + \hat{v} \times (\mathbf{B}^0 + \mathbf{B}^{ext}))f\} = -\nabla_v \cdot \{(\mathbf{E} + \hat{v} \times \mathbf{B}) f^0\}. \tag{64}$$

The last equality follows in the same way that (38) was derived. Let $\varsigma(v) \in C_c^1(\mathbb{R}^3)$ to be a cut-off function for the v-support of $\mu(e, p)$. Taking any $h(x) \in C_c^1(\mathbb{R}^3)$ and using $\varsigma(v)h(x)$ as a test function for (64), all the terms coming from v-divergences vanish and we have

$$\int \lambda\rho(x)h(x)dx - \int \mathbf{j} \cdot \nabla h dx = 0.$$

So $\lambda\rho + \nabla \cdot \mathbf{j} = 0$ weakly. □

Lemma 8.4. *The Maxwell equation* (40) *is satisfied.*

Proof. By (63), we have

$$\mathbf{j} = -\int \hat{v} f dv = \Big(\int \hat{v}\mu_e dv\Big)\phi - \int \hat{v}\mu_e \mathcal{Q}^\lambda \phi dv + \Big(\int \hat{v}\mu_p dv\Big) r A_\theta + \int \hat{v}\mu_e \mathcal{Q}^\lambda (\hat{v}_\theta A_\theta) dv + \int \hat{v}\mu_e \mathcal{Q}^\lambda (G\pi) dv.$$

Its θ-component can be written as

$$j_\theta = -(\mathcal{B}^\lambda)^*\phi - \mathcal{A}_2^\lambda A_\theta + \left(-\Delta + \frac{1}{r^2} + \lambda^2\right) A_\theta + \mathcal{C}^\lambda \pi.$$

By the definition of ϕ,

$$-(\mathcal{B}^\lambda)^*\phi = -(\mathcal{B}^\lambda)^*\big(\mathcal{A}_1^\lambda\big)^{-1}\mathcal{B}^\lambda A_\theta - (\mathcal{B}^\lambda)^*\big(\mathcal{A}_1^\lambda\big)^{-1}\mathcal{D}^\lambda \pi.$$

By the definition of $\mathcal{L}^\lambda$,

$$-\mathcal{A}_2^\lambda A_\theta = -\mathcal{L}^\lambda A_\theta + (\mathcal{B}^\lambda)^*\big(\mathcal{A}_1^\lambda\big)^{-1}\mathcal{B}^\lambda A_\theta.$$

By (61),

$$-\mathcal{L}^\lambda A_\theta = (\mathcal{F}^\lambda)^*\pi = (\mathcal{B}^\lambda)^*\big(\mathcal{A}_1^\lambda\big)^{-1}\mathcal{D}^\lambda \pi - \mathcal{C}^\lambda \pi.$$

Adding the last three equations, we obtain

$$-(\mathcal{B}^\lambda)^*\phi - \mathcal{A}_2^\lambda A_\theta + \mathcal{C}^\lambda \pi = 0,$$

so that

$$j_\theta = \left(-\Delta + \frac{1}{r^2} + \lambda^2\right) A_\theta$$

and

$$j_\theta \mathbf{e}_\theta = (\lambda^2 - \Delta)(A_\theta \mathbf{e}_\theta).$$

Because $\nabla\phi$ has no θ-component, this result is the θ-component of the Maxwell equation (40).

It remains to derive the r and z components of (40). By (61), (62) and (63), it follows exactly as in the proof of Lemma 3.1 that

$$\begin{aligned}\left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right)\pi &= (\mathcal{D}^\lambda)^*\phi - (\mathcal{C}^\lambda)^* A_\theta + \mathcal{E}^\lambda \pi \\ &= \partial_z j_r - \partial_r j_z.\end{aligned}$$

As in that proof, we introduce $\mathbf{K} = j_r\mathbf{e}_r + j_z\mathbf{e}_z$ and $\mathbf{I} = (-\Delta)^{-1}\mathbf{K}$. Then

$$\left(-\Delta + \frac{1}{r^2}\right)\left(-\Delta + \frac{1}{r^2} + \lambda^2\right)\pi = \left(-\Delta + \frac{1}{r^2}\right)(\partial_z I_r - \partial_r I_z)$$

so that

$$\left(-\Delta + \frac{1}{r^2} + \lambda^2\right)\pi = \partial_z I_r - \partial_r I_z.$$

This result can be rewritten as

$$(\lambda^2 - \Delta)(\pi \mathbf{e}_\theta) = \nabla \times \mathbf{I}.$$

Taking the curl of both sides,

$$(\lambda^2 - \Delta)(A_r\mathbf{e}_r + A_z\mathbf{e}_z) = -\Delta\mathbf{I} + \nabla(\nabla\cdot\mathbf{I}).$$

But

$$\nabla\cdot\mathbf{I} = \nabla\cdot(-\Delta)^{-1}\mathbf{K} = (-\Delta)^{-1}\nabla\cdot\mathbf{j} = \lambda\Delta^{-1}\rho = -\lambda\phi,$$

so that

$$(\lambda^2 - \Delta)(A_r\mathbf{e}_r + A_z\mathbf{e}_z) = \mathbf{K} - \lambda\nabla\phi.$$

In components, this means

$$(\lambda^2 - \Delta)A_r = j_r - \lambda\partial_r\phi, \quad (\lambda^2 - \Delta)A_z = j_z - \lambda\partial_z\phi,$$

which are precisely the r and z components of (40). □

This completes the proof of Theorem 1.2(i). To prove Theorem 1.2(ii) on the number of growing modes, we first note that for each n-truncated problem, it follows from the continuity argument that the number of approximate growing modes is bounded below by the dimension of the negative eigenspace of $\mathcal{L}^0$. Since we have the uniform control of the converging process as $n \to \infty$, the lower bound for the number of exact growing modes follows. The proof of the upper bound is the same as in the $1\frac{1}{2}$D case and we omit it. □

Remark 3. In this 3D case we do not have much regularity of f and the growing mode is only shown to satisfy the linearized equation weakly. This is mainly due to the complicated behavior of the 3D particle trajectories. To see this difficulty more clearly, we formally differentiate f given by (63) and look at a typical term

$$\int_{-\infty}^{0} \iint \mu_e \lambda e^{\lambda s} \nabla_x \phi(X(s;x,v)) \frac{\partial X(s;x,v)}{\partial v} dxdvds.$$

If the stretching factor $\frac{\partial X(s;x,v)}{\partial v}$ grows like $e^{a|s|}$ with $a > \lambda$, the integral diverges and we lose the differentiability of f. In the $1\frac{1}{2}$D case it is possible to prove (see [25]) some regularity of f by estimating an averaged Liapunov exponent for the quantity $\iint |\frac{\partial X(s;x,v)}{\partial v}| dxdv$. This idea was first introduced in the 1D Vlasov–Poisson in [23] and it works for integrable trajectories. However, the 3D trajectory in general is non-integrable so that the idea fails. For this reason we have had to study the operators $(\mathcal{C}^\lambda)^*$, $(\mathcal{D}^\lambda)^*$, $\mathcal{E}^\lambda$, $\mathcal{F}^\lambda$ and $\mathcal{A}_4^\lambda$ with ranges in negative Sobolev spaces. We note as well that the non-integrability of trajectories is the main reason for the difficulty of passing from linear to non-linear instability.

9. Non-monotone equilibria

In case μ_e changes sign, it does not seem possible to extend the methods of [24] to get linear stability. However, we can still get sufficient conditions for linear *instability* by extending the matrix formulation of this paper. If μ_e changes sign, we will reformulate the growing mode problem as a 3×3 matrix operator $\mathcal{M}^\lambda$ depending on a positive parameter $\lambda > 0$ and then look for the change of the signature of $\mathcal{M}^\lambda$ as λ goes from 0 to $+\infty$.

In the discussion below, we illustrate this idea only for a simple case, namely a purely magnetic equilibrium of $1\frac{1}{2}D$ RVM system with two species. Assume now that

$$\mu^+(e,p) = \mu^-(e,-p). \tag{65}$$

Then an purely magnetic equilibrium is obtained with electric potential $\phi^0 \equiv 0$ and magnetic potential ψ^0 satisfying the ODE

$$\partial_x^2 \psi^0 = 2 \int \hat{v}_2 \mu^- \big(\langle v \rangle, v_2 - \psi^0(x)\big) dv.$$

We use the same notation as in [24] and [25]. Define

$$\mathcal{A}_1^0 h = -\partial_x^2 h - \Big(\int 2\mu_e^- dv\Big)h + \int 2\mu_e^- \mathcal{P}^- h dv,$$

$$\mathcal{A}_2^0 h = -\partial_x^2 h - \Big(2\int \hat{v}_2 \mu_p^- dv\Big)h - \int 2\mu_e^- \hat{v}_2 \mathcal{P}^-(\hat{v}_2 h) dv,$$

$$k^0 = \int_0^P \int \mu_e^- (\mathcal{P}^-(\hat{v}_1))^2 dv dx$$

where $\mathcal{P}^-$ is the projection operator of $L^2_{|\mu_e^-|}$ onto $\ker D^-$ and $D^- = \hat{v}_1 \partial_x - \hat{v}_2 B^0 \partial_{v_1} + \hat{v}_1 B^0 \partial_{v_2}$. Denote by $n(\mathcal{A}_1^0)$ and $n(\mathcal{A}_2^0)$ the number of negative eigenvalues of $\mathcal{A}_1^0$ and $\mathcal{A}_2^0$.

Theorem 9.1. *Consider a periodic purely magnetic equilibrium as above. Assume* $\ker \mathcal{A}_1^0 = \{0\}$. *Then the equilibrium is spectrally unstable if either*

(i) $l^0 < 0$ *and* $n(\mathcal{A}_1^0) \neq n(\mathcal{A}_2^0)$ *or*
(ii) $l^0 > 0$ *and* $n(\mathcal{A}_1^0) + 1 \neq n(\mathcal{A}_2^0)$.

Proof (sketched). As we are merely sketching the extension of our results to this case, let us take just one species and use the notation in Sect. 2. Finding a growing mode $e^{\lambda t}(f, E_1, E_2, B)$ with $\lambda > 0$ is equivalent to solving (29)–(31) for (ϕ, ψ, b) where (ϕ, ψ) is the electromagnetic potential and $b \in \mathbb{R}^1$. We define the rank-one operators $\mathcal{C}^\lambda,\ \mathcal{D}^\lambda : \mathbb{R}^1 \to L^2_P$ by $\mathcal{C}^\lambda(b) = bb^\lambda$ and $\mathcal{D}^\lambda(b) = bc^\lambda$. Then (ϕ, ψ, b) satisfies the matrix equation

$$\begin{pmatrix} -\mathcal{A}_1^\lambda & \mathcal{B}^\lambda & C^\lambda \\ (\mathcal{B}^\lambda)^* & \mathcal{A}_2^\lambda & -\mathcal{D}^\lambda \\ (C^\lambda)^* & -(\mathcal{D}^\lambda)^* & -P(\lambda^2 - l^\lambda) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ b \end{pmatrix} = \mathcal{M}^\lambda \begin{pmatrix} \phi \\ \psi \\ b \end{pmatrix} = 0.$$

This 3×3 matrix $\mathcal{M}^\lambda$ is different from the 2×2 one of the previous sections. Notice that $\mathcal{M}^\lambda$ is formally self-adjoint.

Let us look at the asymptotic behavior of $\mathcal{M}^\lambda$. As $\lambda \to +\infty$, we can show that the off-diagonal terms $\mathcal{B}^\lambda, \mathcal{C}^\lambda, \mathcal{D}^\lambda \to 0$ and $\mathcal{A}_1^\lambda \to -\frac{d}{dx^2} > 0$, by noticing that

$$\lim_{\lambda\to\infty} \int_{-\infty}^0 \lambda e^{\lambda s} h(X(s)) ds \to h(x)$$

strongly in L^2_P, which is the analogue of Lemma 4.1(e). We also have $\mathcal{A}_2^\lambda > 0$ for large λ. As $\lambda \searrow 0$, we have $\mathcal{C}^\lambda,\ \mathcal{D}^\lambda \to 0$ as shown in the proof of Lemma 2.6. Moreover, it was shown in Lemmas 4.2 and 3.1 of [24] that for a purely magnetic equilibrium, $\mathcal{B}^\lambda \to 0$ strongly as $\lambda \searrow 0$. So $\mathcal{M}^\lambda$ tends to a diagonal operator as λ tends to 0 and the same as λ tends to ∞. Now $\mathcal{A}_1^\lambda,\ \mathcal{A}_2^\lambda$ and l^λ tend to $\mathcal{A}_1^0,\ \mathcal{A}_2^0$ and l^0 as $\lambda \searrow 0$.

We want to show that $\mathcal{M}^\lambda$ has a different signature for small and large λ. For then a continuity argument should ensure the existence of a non-trivial kernel for some $\mathcal{M}^\lambda$. However since $\mathcal{M}^\lambda$ is not bounded either from below or from above, in order to make the argument rigorous we must truncate as in the $3D$ case. We truncate the ϕ-component (but not the other components) to an n-dimensional subspace which does not spoil the negative space of $\mathcal{A}_1^0$; that is, we project onto the lowest n modes of $\mathcal{A}_1^0$. We denote the resulting truncated matrix operator by $\mathcal{M}_n^\lambda$. Then for large λ, say $\lambda \geq \Lambda$, $\mathcal{M}_n^\lambda$ has $n + 0 + 1$ negative eigenvalues. In case $l^0 < 0$, $\mathcal{M}_n^0$ has $(n - n(\mathcal{A}_1^0)) + n(\mathcal{A}_2^0) + 1$ negative eigenvalues. In case $l^0 > 0$, $\mathcal{M}_n^0$ has $(n - n(\mathcal{A}_1^0)) + n(\mathcal{A}_2^0) + 0$ negative eigenvalues. Therefore $\mathcal{M}_n^0$ and $\mathcal{M}_n^\Lambda$ have a different number of negative eigenvalues in both cases (i) and (ii). By continuity, $\mathcal{M}_n^\lambda$ has a non-trivial kernel for some $\lambda > 0$. Then we let n go to $+\infty$ to obtain a non-trivial kernel for $\mathcal{M}^\lambda$. As the details are somewhat similar to the 3D cylindrical case, we omit them. □

For purely magnetic equilibria, in case $\mu_e < 0$, we have $\mathcal{A}_1^0 > 0$ and $l^0 < 0$. In this case, it was shown in [24] that $n(\mathcal{A}_2^0) \neq 0$ is the sharp condition for linear instability. So Theorem 9.1 is a generalization of that instability result to the case of a general purely magnetic equilibrium with non-monotone μ. Moreover, it was shown in [25] that these linear instability results imply non-linear instability in the macroscopic sense.

For the $3D$ case with μ_e of general sign, one can also use the same idea. The equations (45), (47) and (48) for (ϕ, A_θ, π) can be rewritten as

$$\begin{pmatrix} -\mathcal{A}_1^\lambda & \mathcal{B}^\lambda & -\mathcal{D}^\lambda \\ (\mathcal{B}^\lambda)^* & \mathcal{A}_2^\lambda & C^\lambda \\ -(\mathcal{D}^\lambda)^* & (\mathcal{C}^\lambda)^* & \mathcal{A}_3^\lambda \end{pmatrix} \begin{pmatrix} \phi \\ A_\theta \\ \pi \end{pmatrix} = \mathcal{M}^\lambda \begin{pmatrix} \phi \\ A_\theta \\ \pi \end{pmatrix} = 0.$$

Again $\mathcal{M}^\lambda$ is formally self-adjoint. By studying the difference of the signatures of $\mathcal{M}^\lambda$ at 0 and at ∞, one can obtain sufficient conditions for linear instability of general equilibria, which will generalize the instability criterion of the monotone case. However we do not pursue the details here.

10. Appendix

In this appendix, we list some common formulae in the cylindrical coordinates. Assume $\psi = \psi(r, \theta, z)$ is a scalar function and $\mathbf{A} = (A_r, A_\theta, A_z)$ is a vector function.

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\psi}{\partial z}\mathbf{e}_z,$$

$$\Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2},$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z},$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_z$$

$$\Delta \mathbf{A} = \left(\Delta A_r - \frac{1}{r^2}A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\mathbf{e}_r + \left(\Delta A_\theta - \frac{1}{r^2}A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_\theta + \Delta A_z \mathbf{e}_z.$$

We now present the derivation of (38) in detail. The linearized Vlasov equation can be written as

$$\partial_t f + Df = (\mathbf{E} + \hat{v} \times \mathbf{B}) \cdot \nabla_v f^0.$$

Since $f^0 = \mu(e, p)$, we have

$$\nabla_v f^0 = \mu_e \hat{v} + \mu_p r \mathbf{e}_\theta.$$

So

$$\begin{aligned}\mathbf{E} \cdot \nabla_v f^0 &= (-\nabla_x \phi - \partial_t \mathbf{A}) \cdot (\mu_e \hat{v} + \mu_p r \mathbf{e}_\theta) \\ &= -\mu_e \hat{v} \cdot \nabla_x \phi - \mu_e \hat{v} \cdot \partial_t \mathbf{A} - \mu_p r \partial_t A_\theta\end{aligned}$$

Moreover,

$$\begin{aligned}\hat{v} \times \mathbf{B} \cdot \nabla_v f^0 &= \{\hat{v} \times (\nabla_x \times \mathbf{A})\} \cdot \{\mu_e \hat{v} + \mu_p r \mathbf{e}_\theta\} \\ &= r\mu_p \{\hat{v} \times (\nabla_x \times \mathbf{A})\} \cdot \mathbf{e}_\theta \\ &= -\mu_p(\hat{v}_r \partial_r (rA_\theta) + \hat{v}_z \partial_z (rA_\theta)) = -\mu_p D(rA_\theta).\end{aligned}$$

The last line is a consequence of the identity

$$\begin{aligned}&\hat{v} \times (\nabla_x \times \mathbf{A}) \cdot \mathbf{e}_\theta \\ &= \Bigg\{(\hat{v}_r \mathbf{e}_r + \hat{v}_\theta \mathbf{e}_\theta + \hat{v}_z \mathbf{e}_z) \\ &\qquad \times \left(-\frac{\partial A_\theta}{\partial z}\mathbf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\frac{\partial (rA_\theta)}{\partial r}\mathbf{e}_z\right)\Bigg\} \cdot \mathbf{e}_\theta \\ &= -\hat{v}_r \frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \hat{v}_z \frac{\partial A_\theta}{\partial z} = \frac{1}{r} D(rA_\theta).\end{aligned}$$

Combining the above computations, we obtain (38).[1]

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