Small BGK waves and nonlinear Landau damping

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Abstract

Consider 1D Vlasov-poisson system with a fixed ion background and periodic condition on the space variable. First, we show that for general homogeneous equilibria, within any small neighborhood in the Sobolev space $W^{s,p}$ $\left(p>1, s<1+\frac{1}{p}\right)$ of the steady distribution function, there exist nontrivial travelling wave solutions (BGK waves) with arbitrary minimal period and traveling speed. This implies that nonlinear Landau damping is not true in $W^{s,p}$ $\left(s<1+\frac{1}{p}\right)$ space for any homogeneous equilibria and any spatial period. Indeed, in $W^{s,p}$ $\left(s<1+\frac{1}{p}\right)$ neighborhood of any homogeneous state, the long time dynamics is very rich, including travelling BGK waves, unstable homogeneous states and their possible invariant manifolds. Second, it is shown that for homogeneous equilibria satisfying Penrose's linear stability condition, there exist no nontrivial travelling BGK waves and unstable homogeneous states in some $W^{s,p}$ $\left(p>1, s>1+\frac{1}{p}\right)$ neighborhood. Furthermore, when $p=2$,we prove that there exist no nontrivial invariant structures in the H^s $\left(s>\frac{3}{2}\right)$ neighborhood of stable homogeneous states. These results suggest the long time dynamics in the $W^{s,p}$ $\left(s>1+\frac{1}{p}\right)$ and particularly, in the H^s $\left(s>\frac{3}{2}\right)$ neighborhoods of a stable homogeneous state might be relatively simple. We also demonstrate that linear damping holds for initial perturbations in very rough spaces, for linearly stable homogeneous state. This suggests that the contrasting dynamics in $W^{s,p}$ spaces with the critical power $s=1+\frac{1}{p}$ is a trully nonlinear phenomena which can not be traced back to the linear level.

1 Introduction

Consider a one-dimensional collisionless electron plasma with a fixed homogeneous neutralizing ion background. The fixed ion background is a good physical approximation since the motion of ions is much slower than electrons. But we consider fixed ion mainly to simplify notations and the main results in this paper are also true for electrostatic plasmas with two or more species. The time evolution of such electron plasmas can be modeled by the Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = 0, \tag{1a}$$

$$\frac{\partial E}{\partial x} = -\int_{-\infty}^{+\infty} f dv + 1, \tag{1b}$$

where $f(x,v,t)$ is the electron distribution function, $E(x,t)$ the electric field, and 1 is the ion density. The one-dimensional assumption is proper for a high temperature and dilute plasma immersed in a constant magnetic field oriented in the x-direction. For example, recent discovery by satellites of electrostatic structures near geomagnetic fields can be justified by using such Vlasov-Poisson models ([27], [39]). We assume: 1) $f(x,v,t) \geq 0$ and $E(x,t)$ are $T-$periodic in x. 2) Neutral condition: $\int_0^T \int_{\mathbf{R}} f(x,v,0)\,dxdv = T$. 3) $\int_0^T E(x,t)\,dx = 0$, so $E(x,t) = -\partial_x \phi(x,t)$, where the electric potential $\phi(x,t)$ is $T-$periodic in x. Since $\int\int f(x,v,t)\ dxdv$ is an invariant, the neutral condition 2) is preserved for all time. The condition 3) ensures that $E(t)$ is determined uniquely by $f(t)$ from (1b) and the system (1) can be considered to be an evolution equation of f only. It is shown in [21] that with condition 3), the system (1) is equivalent to the following one-dimensional Vlasov-Maxwell system

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = 0,$$

$$\frac{\partial E}{\partial x} = -\int_{-\infty}^{+\infty} f dv + 1,$$

$$\frac{\partial E}{\partial t} = \int_{\mathbf{R}} vf(x,v,t)\,dv - U,$$

where U is the bulk velocity of the ion background. The system (1) is non-dissipative and time-reversible. It has infinitely many equilibria, including the homogeneous states $(f_0(v), 0)$ where $f_0(v)$ is any nonnegative function satisfying $\int_{\mathbf{R}} f_0(v)\,dv = 1$.

In 1946, Landau [29], looking for analytical solutions of the linearized Vlasov-Poisson system around Maxwellian $\left(e^{-\frac{1}{2}v^2}, 0\right)$, pointed out that the electric field is subject to time decay even in the absence of collisions. The effect of this Landau damping, as it is subsequently called, plays a fundamental role in the study of plasma physics. However, Landau's treatment is in the linear regime; that is, only for infinitesimally small initial perturbations. Despite many numerical, theoretical and experimental efforts, no rigorous justification of the Landau damping has been given in a nonlinear dynamical sense. In the past decade, there has been renewed interest [26] [37] [28] [18] [19] [50] [13] [25] [38] [47] as well as controversy about the Landau damping. In [13] [25], it was shown that there exist certain analytical perturbations for which electric fields decay exponentially in the nonlinear level. More recently, in [38] nonlinear Landau damping was shown for general analytical perturbations of stable equilibria with linear exponential decay. For non-analytic perturbations, the linear decay rate of electric fields is known to be only algebraic (i.e. [48]) and the nonlinear damping is more difficult to justify if it is true. Moreover, in the nonlinear regime, it has been known ([42]) that the damping can be prevented by particles trapped in the potential well of the wave. Such particle trapping effect is ignored in Landau's linearized analysis as well as other physically equivalent linear theories ([12], [49]), which assume that the small amplitude of waves have a negligible effect on the evolution of distribution functions. As early as in 1949, Bohm and Gross ([8]) already recognized the importance of particle trapping effects and the possibility of nonlinear travelling waves of small but constant amplitude. In 1957, Bernstein, Greene and Kruskal ([6]) formalized the ideas of Bohm and Gross and found a general class of exact nonlinear steady imhomogeneous solutions of the Vlasov-Poisson system. Since then, such steady solutions have been known as BGK modes, BGK waves or BGK equilibria. The nontrivial steady waves of this type are made possible by the existence of particles trapped forever within the electrostatic potential wells of the wave. The existence of such undamped waves in any small neighborhood of an equilibrium will certainly imply that nonlinear damping is not true.

Furthermore, numerical simulations [37] [16] [36] [14] [9] indicate that for certain small initial data near a stable homogeneous state including Maxwellian, there is no decay of electric fields and the asymptotic state is a BGK wave or superposition of BGK waves which were formally constructed in [11]. Moreover, BGK waves also appear as the asymptotic states for the saturation of an unstable homogeneous state ([3]). These suggest that small BGK waves play important role in understanding the long time behaviors of Vlasov-Poisson system, near homogeneous equilibria. In this paper, we provide a sharp characterization of the Sobolev spaces in which small BGK waves exist in any small neighborhood of a homogeneous equilibrium. Denote the fractional order Sobolev spaces by $W^{s,p}(\mathbf{R})$ or $W^{s,p}_{x,v}((0,T)\times\mathbf{R})$ with $p\geq 1, s\geq 0$. These spaces are the interpolation spaces (see [1], [46]) of L^p space and Sobolev space $W^{m,p}$ (m positive integer).

Theorem 1 *Assume the homogeneous distribution function $f_0(v)\in W^{s,p}(\mathbf{R})$ $\left(p>1, s\in[0,1+\frac{1}{p})\right)$ satisfies*

$$f_0(v)\geq 0,\ \int f_0(v)\,dv=1,\ \int v^2 f_0(v)<+\infty.$$

Fix $T>0$ and $c\in\mathbf{R}$. Then for any $\varepsilon>0$, there exist travelling BGK wave solutions of the form $(f_\varepsilon(x-ct,v), E_\varepsilon(x-ct))$ to (1), such that $(f_\varepsilon(x,v), E_\varepsilon(x))$ has minimal period T in x, $f_\varepsilon(x,v)\geq 0$, $E_\varepsilon(x)$ is not identically zero, and

$$\|f_\varepsilon-f_0\|_{L^1_{x,v}}+\int_0^T\int_{\mathbf{R}} v^2\left|f_\varepsilon(x,v)-f_0(v)\right|dxdv+\|f_\varepsilon-f_0\|_{W^{s,p}_{x,v}}<\varepsilon. \tag{2}$$

The first two terms in (2) imply that the BGK wave is close to the homogeneous state $(f_0, 0)$ in the norms of total mass and energy. When $p > 1, s = 1$, the fractional Sobolev space is equivalent to the usual Sobolev space $W^{1,p}$. The conclusions in Theorem 1 are also true for the Sobolev space $W^{1,1}_{x,v}$ by the same proof. Above theorem immediately implies that nonlinear Landau damping is *not true* for perturbations in any $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$ space, for any homogeneous equilibrium in $W^{s,p}$ and any spatial period.

As a corollary of the proof, we show that there exist unstable homogeneous states in $W^{s,p}(\mathbf{R})$ $\left(s < 1 + \frac{1}{p}\right)$ neighborhood of any homogeneous equilibrium.

Corollary 1 *Under the assumption of Theorem 1, for any fixed $T > 0$, $\exists$ $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$, there exists a homogeneous state $(f_\varepsilon(v), 0)$ which is linearly unstable under perturbations of $x-$period T,*

$$f_\varepsilon(v) \geq 0, \ \int_{\mathbf{R}} f_\varepsilon(v) \ dv = 1,$$

and

$$\|f_\varepsilon(v) - f_0(v)\|_{L^1(\mathbf{R})} + \int_{\mathbf{R}} v^2 |f_\varepsilon(v) - \ f_0(v)| \, dv + \|f_\varepsilon(v) - f_0(v)\|_{W^{s,p}(\mathbf{R})} < \varepsilon.$$

By above Corollary and Remark 1 following the proof of Theorem 1, in $W^{s,p}_{x,v}$ $\left(s < 1 + \frac{1}{p}\right)$ neighborhood of any homogeneous state there exist lots of unstable homogeneous states and unstable nontrivial BGK waves. In a work in progress, we are constructing stable and unstable manifolds near an unstable equilibrium of Vlasov-Poisson system by extending our work ([33]) on invariant manifolds of Euler equations. Such (possible) invariant manifolds might reveal more complicated global invariant structures such as heteroclinic or homoclinic orbits. Moreover, in some physical reference ([11]), small BGK waves are formally shown to follow a nonlinear superposition principle to form time-periodic or quasi-periodic orbits. We note that Maxwellian or any homogeneous equilibria $f_0(v) = \mu\left(\frac{1}{2}v^2\right)$ with μ monotonically decreasing, were shown by Newcomb in 1950s (see Appendix I, pp. 20-21 of [7]) to be nonlinearly stable in the norm $\|f\|_{L^2}$. So our result suggests, in particular, that in any invariant small L^2 neighborhood of Maxwellian, the long time dynamical behaviors are very rich.

The following Theorem shows that there exist no nontrivial BGK waves near a stable homogeneous state in $W^{s,p}_{x,v}$ space when $p > 1, \ s > 1 + \frac{1}{p}$.

Theorem 2 *Assume $f_0(v) \in W^{s,p}(\mathbf{R})$ $\left(p > 1, s > 1 + \frac{1}{p}\right)$. Let $S = \{v_i\}_{i=1}^{l}$ be the set of all extrema points of f_0. Let $0 < T_0 \leq +\infty$ be defined by*

$$\left(\frac{2\pi}{T_0}\right)^2 = \max\left\{0, \max_{v_i \in S} \int \frac{f_0'(v)}{v - v_i} dv\right\}. \tag{3}$$

Then for any $T < T_0$, $\exists$ $\varepsilon_0(T) > 0$, such that there exist no nontrivial travelling wave solutions $(f(x - ct, v), E(x - ct))$ to (1) for any $c \in \mathbf{R}$, satisfying *that* $(f(x,v), E(x))$ *has period* T *in* x*,* $E(x)$ *not identically* 0*,*

$$\int_0^T \int_{\mathbf{R}} v^2 f(x,v)\, dvdx < \infty, \text{ (assumption of finite energy)}$$

and $\|f - f_0\|_{W^{s,p}_{x,v}} < \varepsilon_0$.

By Penrose's stability criterion ([41] or Lemma 7) the homogeneous equilibrium $(f_0(v), 0)$ is linearly stable to perturbations of $x-$period $T < T_0$. Moreover, in Proposition 3, the linear damping of electrical field is shown for such stable states in a rough function space. Theorems 1 and 2 imply that for any $p > 1$, $s = 1 + \frac{1}{p}$ is the critical index for existence or non-existence of small BGK waves in $W^{s,p}$ neighborhood of a stable homogeneous state. In Lemma 5, we show that the stability condition $0 < T < T_0$ is in some sense also necessary for the above non-existence result in $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$.

The following corollary shows that all homogeneous equilibria in a sufficiently small $W^{s,p}(\mathbf{R})$ $\left(s > 1 + \frac{1}{p}\right)$ neighborhood of a stable homogeneous state remain linearly stable. With Corollary 1, it implies that $s = 1 + \frac{1}{p}$ is also the critical index for persistence of linear stability of homogeneous states under perturbations in $W^{s,p}(\mathbf{R})$ space.

Corollary 2 *Assume* $f_0(v) \in W^{s,p}(\mathbf{R})$ $\left(p > 1, s > 1 + \frac{1}{p}\right)$. *Let* $S = \{v_i\}_{i=1}^{l}$ *be the set of all extrema points of* f_0 *and* T_0 *be defined in (3). Then for any* $T < T_0$*,* $\exists\, \varepsilon_0(T) > 0$ *such that any homogeneous state* $(f(v), 0)$ *satisfying*

$$\|f(v) - f_0(v)\|_{W^{s,p}(\mathbf{R})} < \varepsilon_0$$

is linearly stable under perturbations of $x-$*period* T.

Theorem 2 and the above Corollary suggest that the dynamical structures in small $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$ neighborhood of a stable homogeneous equilibrium might be relatively simple, since the only nearby steady structures, including travelling waves, are stable homogeneous states. The physical implication of Theorem 2 is that when the initial perturbation is small in $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$, the potential well of the wave is unable to trap particles forever to form BGK waves. So the particles will get out of the potential well sooner or later and perform free flights, then the linear damping effect might manifest itself at the nonlinear level.

Furthermore, when $p = 2$, we get a much stronger result that any invariant structure near a stable homogeneous state in H^s space $\left(s > \frac{3}{2}\right)$ must be trivial, that is, the electric field is identically zero.

Theorem 3 *Assume the homogeneous profile* $f_0(v) \in H^s(\mathbf{R})$ $\left(s > \frac{3}{2}\right)$. *For any* $T < T_0$ *(defined by (3)), there exists* $\varepsilon_0 > 0$*, such that if* $(f(t), E(t))$ *is a solution to the nonlinear VP equation (1a)-(1b) and*

$$\|f(t) - f_0\|_{L^2_x H^s_v} < \varepsilon_0, \text{ for all } t \in \mathbf{R}, \tag{4}$$

then $E(t) \equiv 0$ *for all* $t \in \mathbf{R}$.

The space $L_x^2 H_v^s$ is contained in the Sobolev space $H_{x,v}^s$. The above theorem excludes any nontrivial invariant structure, such as almost periodic solutions and heteroclinic (homoclinic) orbits, in the H^s $\left(s > \frac{3}{2}\right)$ neighborhood of a stable homogeneous state. In Theorem 4, we also show that nonlinear decay of electric field is true for any positive or negative invariant structure (see Section 5 for definition) in the H^s $\left(s > \frac{3}{2}\right)$ neighborhood of stable homogeneous states. These results reveal that in contrary to the H^s $\left(s < \frac{3}{2}\right)$ case, there are no obstacles in the H^s $\left(s > \frac{3}{2}\right)$ neighborhood of stable homogeneous states to prevent nonlinear Landau damping.

We note that Theorems 1, 2 and 3 about the contrasting nonlinear dynamics in $W^{s,p}$ spaces with $s < 1 + \frac{1}{p}$ or $s > 1 + \frac{1}{p}$ (particularly when $p = 2$), have no any analogue at the linear level. Indeed, under Penrose's stability condition, it is shown in Section 4 that the linear decay of electrical fields holds true for very rough initial data, particularly, no derivatives of $f(t = 0)$ is required for linear damping. We refer to Propositions 4 and 3, as well as Remark 5 in Section 4 for more details. This shows once again the importance of particle trapping effects on nonlinear dynamics, which are completely ignored at the linear level.

Finally, we briefly describe main ideas in the proof of Theorems 1, 2 and 3. For simplicity, we look at steady BGK waves. The first attempt would be to construct BGK waves near $(f_0(v), 0)$ directly by the bifurcation theory. However, this requires a bifurcation condition: for bifurcation period $T > 0$,

$$\left(\frac{2\pi}{T}\right)^2 = \int_{\mathbf{R}} \frac{f_0'(v)}{v} dv. \tag{5}$$

For general homogeneous equilibria and period T, the bifurcation condition (5) is not satisfied. For example, for Maxwellian, this condition fails for any $T > 0$. Our strategy is to modify $f_0(v)$ to get a nearby homogeneous state satisfying (5) and then do bifurcation near this modified state. In the modification step, we introduce two parameters, one is to to obtain (5) and the other one is to ensure that the modification results in a small $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$ norm change. For the proof of non-existence of travelling waves in $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$, our idea is to get an second order equation for the electrical field $E(x)$ from steady Vlasov-Poisson equations and show the integral form of this equation is not compatible when $T < T_0$ and the perturbation is small in $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$. Interestingly, T_0 (defined by (3)) is exactly the critical period for linear stability by Penrose's criterion, which is also used in the proof of Corollaries 1 and 2. To prove Theorem 3, we use the integral form of the linear decay estimate (Proposition 4) and the H^s $\left(s > \frac{3}{2}\right)$ invariant assumption to obtain similar nonlinear decay estimates in the integral form. From such integral estimates, we can show the homogeneous nature of the invariant structures and the decay of electric field for semi-invariant solutions.

Here we are in a position to offer a conceptual explanation why $s = 1 + \frac{1}{p}$ appears as the critical Sobolev exponent for the existence of small BKG waves and possibly also in the nonlinear Landau damping. By Penrose' stability criterion, the critical spatial period T_0 for linear stability of $(f_0(v), 0)$ is determined in (3) by integrals $\int \frac{f_0'}{v-c_i} dv$, where c_i are critical points of f_0. These integrals are controlled by $\|f_0\|_{W^{s,p}}$ if $s > 1 + \frac{1}{p}$, but not if $s < 1 + \frac{1}{p}$. In the latter case, a small homogeneous perturbation to f_0 in $W^{s,p}$ space may dramatically change its stability for any fixed spatial period T. Due to this change of stability, bifurcations occur and produce small BKG waves and possibly other complicated structures. In the opposite case when $s > 1 + \frac{1}{p}$, small homogeneous perturbations does not change the stability of $(f_0(v), 0)$, therefore the bifurcation of nontrivial waves cannot occur and the nonlinear Landau damping may be expected.

The result of this paper has also been extended to a related problem of inviscid decay of Couette flow $\vec{v}_0 = (y, 0)$ of 2D Euler equations. The linear decay of vertical velocity near Couette flow was already known by Orr ([40]) in 1907. This inviscid decay problem is important to understand the formation of coherent structures in 2D turbulence. In [34], we are able to obtain similar results near the Couette flow.

This paper is organized as follows. In Section 2, we prove the existence result in $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$. In Section 3, non-existence of BGK waves in $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$ is shown. In Section 4, we study the linear damping problem in Sobolev spaces. In Section 5, we use the linear decay estimate in Section 4 to show that all invariant structures in H^s $\left(s > \frac{3}{2}\right)$ are trivial. The appendix is to reformulate Penrose's linear stability criterion used in this paper. Throughout this paper, we use C to denote a generic constant in the estimates and the dependence of C is indicated only when it matters in the proof.

2 Existence of BGK waves in $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$

In this Section, we construct small BGK waves near any homogeneous state in the space $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$. Our strategy is to first construct BGK waves near proper smooth homogeneous states. Then we show that any homogeneous state can be approximated by such smooth states in $W^{s,p}$.

Lemma 1 *Assume $u(x) \in C^\infty(\mathbf{R})$, supp $u \subset [-b, b]$, and $u(x)$ is even, then there exists $g \in C^\infty(\mathbf{R})$, supp $g \subset \left[-\sqrt{b}, \sqrt{b}\right]$, such that $u(x) = g\left(x^2\right)$.*

Proof. The proof is essentially given in [24, P. 394]. We repeat it here for completeness. When k is odd, since $u^{(k)}(x)$ is odd we have $u^{(k)}(0) = 0$. By Theorem 1.2.6 in [24], we can choose $g_0 \in C_0^\infty\left(-\sqrt{b}, \sqrt{b}\right)$ with the Taylor expansion $\sum u^{(2k)}(0)\, x^k / (2k)!$. Then all derivatives of $u_1(x) = u(x) -$

$g_0\left(x^2\right)$ vanish at 0. Define

$$g\left(x\right)=\left\{\begin{array}{cl} g_0\left(x\right)+u_1\left(\sqrt{x}\right) & \text{if } x>0\\ g_0\left(x\right) & \text{if } x\leq 0\end{array}\right. .$$

Then $g\left(x\right)$ satisfies all the required properties. In particular, $g\left(x\right)$ is C^{∞} at $x=0$ because all derivatives of $u_1\left(x\right)$ vanish there. ■

Proposition 1 *Assume*

$$f_0\left(v\right)\in C^{\infty}\left(\mathbf{R}\right)\cap W^{2,p}\left(\mathbf{R}\right)\ \ \left(p>1\right),$$

f_0 *is even near* $v=0$, *and*

$$f_0>0,\ \int_{\mathbf{R}}f_0\left(v\right)dv=1,\ \int_{\mathbf{R}}v^2f_0\left(v\right)dv<\infty.$$

Then for any fixed $s<1+\frac{1}{p}$, $T>0$, *and any* $\varepsilon>0$, *there exist steady BGK solutions of the form* $\left(f_{\varepsilon}\left(x,v\right),E_{\varepsilon}\left(x\right)\right)$ *to (1), such that* $\left(f_{\varepsilon}\left(x,v\right),E_{\varepsilon}\left(x\right)\right)$ *has period* T *in* x, $f_{\varepsilon}\left(x,v\right)>0$, $E_{\varepsilon}\left(x\right)$ *is not identically zero, and*

$$\left\|f_{\varepsilon}-f_0\right\|_{L^1_{x,v}}+\int_0^T\int_{\mathbf{R}}v^2\left|f_0-f_{\varepsilon}\right|\ dxdv+\left\|f_{\varepsilon}-f_0\right\|_{W^{s,p}_{x,v}}<\varepsilon. \tag{6}$$

Proof. Assume $f_0\left(v\right)$ is even in $\left[-2a,2a\right]$ $\left(a>0\right)$. Let $\sigma\left(x\right)=\sigma\left(\left|x\right|\right)$ to be the cut-off function such that $\sigma\left(x\right)\in C_0^{\infty}\left(\mathbf{R}\right)$,

$$0\leq\sigma\left(x\right)\leq 1;\ \sigma\left(x\right)=1 \text{ when } \left|x\right|\leq 1;\ \sigma\left(x\right)=0 \text{ when } \left|x\right|\geq 2. \tag{7}$$

By Lemma 1, there exists $g_0\left(x\right)\in C^{\infty}\left(\mathbf{R}\right)$, supp $g_0\subset\left[-\sqrt{2a},\sqrt{2a}\right]$, such that

$$f_0\left(v\right)\sigma\left(\frac{v}{a}\right)=g_0\left(v^2\right).$$

Define $g_+\left(x\right),\ g_-\left(x\right)\in C^{\infty}\left(\mathbf{R}\right)$ by

$$g_{\pm}\left(x\right)=\left\{\begin{array}{cc} f_0\left(\pm\sqrt{x}\right)\left(1-\sigma\left(\frac{\sqrt{x}}{a}\right)\right)+g_0\left(x\right) & \text{if } x>\sqrt{a}\\ g_0\left(x\right) & \text{if } -\sqrt{2a}<x\leq\sqrt{a}\\ 0 & \text{if } x\leq-\sqrt{2a}\end{array}\right. .$$

Then

$$f_0\left(v\right)=\left\{\begin{array}{cl} g_+\left(v^2\right) & \text{if } v>0\\ g_-\left(v^2\right) & \text{if } v\leq 0\end{array}\right. .$$

Since $f_0'\left(0\right)=0$, $f_0\in W^{2,p}\left(\mathbf{R}\right)\cap C^{\infty}\left(\mathbf{R}\right)$, we have $\left|\int_{\mathbf{R}}\frac{f_0'(v)}{v}dv\right|<\infty$. Indeed,

$$\begin{aligned}\left|\int_{\mathbf{R}}\frac{f_0'\left(v\right)}{v}dv\right| &\leq\int_{\left|v\right|\leq 1}\left|\frac{f_0'\left(v\right)}{v}\right|dv+\int_{\left|v\right|\geq 1}\left|\frac{f_0'\left(v\right)}{v}\right|dv\\ &\leq 2\max_{\left|v\right|\leq 1}\left|f_0''\left(v\right)\right|+\left(\int_{\left|v\right|\geq 1}\frac{1}{\left|v\right|^{p'}}dv\right)^{\frac{1}{p'}}\left\|f_0'\right\|_{L^p}\\ &=2\max_{\left|v\right|\leq 1}\left|f_0''\left(v\right)\right|+\left(\frac{1}{p'-1}\right)^{\frac{1}{p'}}\left\|f_0'\right\|_{L^p}<\infty.\end{aligned}$$

We consider three cases.

Case 1: $\int_{\mathbf{R}} \frac{f_0'(v)}{v} dv < \left(\frac{2\pi}{T}\right)^2$. Choose a function $F(v) \in C^\infty(\mathbf{R})$, such that $F \in W^{2,p}(\mathbf{R})$, $F(v)$ is even,

$$F(v) > 0, \int_{\mathbf{R}} F(v)\, dv < \infty, \quad \int_{\mathbf{R}} v^2 F(v)\, dv < \infty, \quad \int_{\mathbf{R}} \frac{F'(v)}{v} dv > 0. \tag{8}$$

An example of such functions is given by

$$F(v) = \exp\left(-\frac{(v-v_0)^2}{2}\right) + \exp\left(-\frac{(v+v_0)^2}{2}\right),$$

where v_0 is a large positive constant. Indeed,

$$\int_{\mathbf{R}} \frac{F'(v)}{v} dv = \int \frac{F(v) - F(0)}{v^2} dv > 0, \text{ when } v_0 \text{ is large enough,}$$

and other properties in (8) are easy to check. Since $F(v)$ is even, by Lemma 1, there exists $G(x) \in C^\infty(\mathbf{R})$ such that $F(v) = G(v^2)$. Let $\gamma, \delta > 0$ be two small parameters to be fixed, define

$$f_{\gamma,\delta}(v) = \frac{1}{1 + C_0\gamma^2}\left[f_0(v) + \frac{\gamma}{\delta} F\left(\frac{v}{\gamma\delta}\right)\right], \tag{9}$$

where $C_0 = \int F(v)\, dv > 0$. Note that $f_{\gamma,\delta} \in C^\infty(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$, $\int_{\mathbf{R}} f_{\gamma,\delta}(v)\, dv = 1$, and

$$\int_{\mathbf{R}} \frac{f_{\gamma,\delta}'(v)}{v} dv = \frac{1}{1 + C_0\gamma^2}\left[\int_{\mathbf{R}} \frac{f_0'(v)}{v} dv + \frac{1}{\delta^2}\int_{\mathbf{R}} \frac{F'(v)}{v} dv\right].$$

Since $\int_{\mathbf{R}} \frac{f_0'(v)}{v} dv < \left(\frac{2\pi}{T}\right)^2$, there exists $0 < \delta_1 < \delta_2$ such that

$$0 < \int_{\mathbf{R}} \frac{f_0'(v)}{v} dv + \frac{1}{\delta_2^2}\int_{\mathbf{R}} \frac{F'(v)}{v} dv < \left(\frac{2\pi}{T}\right)^2 < \int_{\mathbf{R}} \frac{f_0'(v)}{v} dv + \frac{1}{\delta_1^2}\int_{\mathbf{R}} \frac{F'(v)}{v} dv.$$

Thus there exists $\gamma_0 > 0$ small enough, such that

$$0 < \int_{\mathbf{R}} \frac{f_{\gamma,\delta_2}'(v)}{v} dv < \left(\frac{2\pi}{T}\right)^2 < \int_{\mathbf{R}} \frac{f_{\gamma,\delta_1}'(v)}{v} dv, \text{ when } 0 < \gamma < \gamma_0. \tag{10}$$

We look for steady BGK waves near the homogeneous states $(f_{\gamma,\delta}(v), 0)$. Consider a steady BGK solution $\left(f^0(x,v), E^0(x) = -\beta_x(x)\right)$ to (1). Denote $e = \frac{1}{2}v^2 - \beta(x)$ to be the particle energy. From the steady Vlasov equation, $f^0(x,v)$ is constant along each particle trajectory. So for trapped particles with $-\max\beta < e < -\min\beta$, f^0 depends only on e, and for free particles with $e > -\min\beta$, f^0 depends on e and the sign of the initial velocity v. We look for BGK waves near $(f_{\gamma,\delta}, 0)$ of the form

$$f_{\gamma,\delta}^{\beta}(x,v) = \begin{cases} \frac{1}{1+C_0\gamma^2}\left[g_+(2e) + \frac{\gamma}{\delta} G\left(\frac{2e}{(\gamma\delta)^2}\right)\right] & \text{if } v > 0 \\ \frac{1}{1+C_0\gamma^2}\left[g_-(2e) + \frac{\gamma}{\delta} G\left(\frac{2e}{(\gamma\delta)^2}\right)\right] & \text{if } v \le 0 \end{cases}. \tag{11}$$

For $\|\beta\|_{L^\infty}$ sufficiently small, $f^{\beta}_{\gamma,\delta}(x,v) > 0$ and it satisfies the steady Vlasov equation, since in particular for trapped particles $f^{\beta}_{\gamma,\delta}(x,v) = f^{\beta}_{\gamma,\delta}(x,-v)$. To satisfy Poisson's equation, we solve the ODE

$$\begin{aligned}
\beta_{xx} &= \int_{\mathbf{R}} f^{\beta}_{\gamma,\delta}(x,v)\ dv - 1 \\
&= \frac{1}{1+C_0\gamma^2}\left[\int_{v>0} g_+(2e)\,dv + \int_{v\leq 0} g_-(2e)\,dv + \int_{\mathbf{R}} \frac{\gamma}{\delta} G\left(\frac{2e}{(\gamma\delta)^2}\right) dv\right] - 1 \\
&:= h_{\gamma,\delta}(\beta).
\end{aligned}$$

Then $h_{\gamma,\delta} \in C^\infty(\mathbf{R})$. Since $f^{\beta=0}_{\gamma,\delta}(x,v) = f_{\gamma,\delta}(v)$, so

$$h_{\gamma,\delta}(0) = \int_{\mathbf{R}} f_{\gamma,\delta}(v)\ dv - 1 = 0$$

and

$$\begin{aligned}
h'_{\gamma,\delta}(0) &= \frac{-2}{1+C_0\gamma^2}\left\{\int_{v>0} g'_+\left(v^2\right) dv + \int_{v\leq 0} g'_-\left(v^2\right) dv + \int_{\mathbf{R}} \frac{\gamma}{\delta}\frac{1}{(\gamma\delta)^2} G'\left(\frac{v^2}{(\gamma\delta)^2}\right) dv\right\} \\
&= -\int_{\mathbf{R}} \frac{f'_{\gamma,\delta}(v)}{v} dv.
\end{aligned}$$

Thus when $0 < \gamma < \gamma_0,\ \delta_1 < \delta < \delta_2$, we have $h'_{\gamma,\delta}(0) < 0$, which implies that $\beta = 0$ is a center of the second order ODE

$$\beta_{xx} = h_{\gamma,\delta}(\beta). \tag{12}$$

So by the standard bifurcation theory of periodic solutions near a center, for any fixed $\gamma \in (0,\gamma_0)$, there exists $r_0 > 0$ (independent of $\delta \in (\delta_1,\delta_2)$), such that for each $0 < r < r_0$, there exists a $T(\gamma,\delta;r)-$periodic solution $\beta_{\gamma,\delta;r}$ to the ODE (12) with $\|\beta_{\gamma,\delta;r}\|_{H^2(0,T(\gamma,\delta;r))} = r$. Moreover,

$$\left(\frac{2\pi}{T(\gamma,\delta;r)}\right)^2 \to \int_{\mathbf{R}} \frac{f'_{\gamma,\delta}(v)}{v} dv, \text{ when } r \to 0.$$

By (10), when r is small enough,

$$T(\gamma,\delta_1;r) < T < T(\gamma,\delta_2;r).$$

Since $T(\gamma,\delta;r)$ is continuous in δ, for each $\gamma, r > 0$ small enough, there exists $\delta_T(\gamma,r) \in (\delta_1,\delta_2)$, such that $T(\gamma,\delta_T;r) = T$. Define $f^T_{\gamma,r}(x,v) = f^{\beta}_{\gamma,\delta_T}(x,v)$ from (11) by setting $\beta = \beta_{\gamma,\delta_T;r}$ and let $E_{\gamma,r}(x) = -\beta'_{\gamma,\delta_T;r}(x)$. Then $\left(f^T_{\gamma,r}(x,v), E_{\gamma,r}(x)\right)$ is a nontrivial BGK solution to (1) with $x-$period T. For any fixed $\gamma > 0$, let

$$\delta(\gamma) = \lim_{r\to 0} \delta_T(\gamma,r) \in [\delta_1,\delta_2].$$

By the dominant convergence theorem, it is easy to show that

$$\left\|f_{\gamma,r}^{T}(x,v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{L_{x,v}^{1}}+\int_{0}^{T}\int_{\mathbf{R}}v^{2}\left|f_{\gamma,r}^{T}(x,v)-f_{\gamma,\delta(\gamma)}(v)\right|\,dxdv$$
$$+\left\|f_{\gamma,r}^{T}(x,v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{W_{x,v}^{2,p}}\to 0,$$

when $r=|\beta_{\gamma,\delta_T;r}|_{H^2(0,T)}\to 0$. Since $s<1+\frac{1}{p}<2$, for any $\gamma>0$ small, there exists $r=r(\gamma,\varepsilon)>0$ such that

$$\left\|f_{\gamma,r}^{T}(x,v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{L_{x,v}^{1}}+\int_{0}^{T}\int_{\mathbf{R}}v^{2}\left|f_{\gamma,r}^{T}(x,v)-f_{\gamma,\delta(\gamma)}(v)\right|\,dxdv$$
$$+\left\|f_{\gamma,r}^{T}(x,v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{W_{x,v}^{s,p}}<\frac{\varepsilon}{2},$$

Next, we show that the modified homogeneous state $f_{\gamma,\delta(\gamma)}(v)$ is arbitrarily close to $f_0(v)$ in the sense that

$$\left\|f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{L^1}+T\int_{\mathbf{R}}v^2\left|f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right|\,dv+\left\|f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{W_{x,v}^{s,p}}\to 0,$$

when $\gamma\to 0$. Note that the deviation is

$$f_0(v)-f_{\gamma,\delta(\gamma)}(v)=\frac{1}{1+C_0\gamma^2}\left[-C_0\gamma^2 f_0(v)-\frac{\gamma}{\delta}F\left(\frac{v}{\gamma\delta}\right)\right].$$

Since $\delta(\gamma)\in[\delta_1,\delta_2]$, when $\gamma\to 0$,

$$\int_{\mathbf{R}}\frac{\gamma}{\delta(\gamma)}F\left(\frac{v}{\gamma\delta(\gamma)}\right)dv=\gamma^2\int_{\mathbf{R}}F(v)\,dv\to 0,$$

$$\begin{aligned}
\int_{\mathbf{R}}v^2\frac{\gamma}{\delta(\gamma)}F\left(\frac{v}{\gamma\delta(\gamma)}\right)dv &= \gamma^4\delta(\gamma)^2\int_{\mathbf{R}}v^2F(v)\,dv\to 0,\\
\left\|\frac{\gamma}{\delta(\gamma)}F\left(\frac{v}{\gamma\delta(\gamma)}\right)\right\|_{L^p} &= \gamma^{1+\frac{1}{p}}\delta(\gamma)^{\frac{1}{p}-1}\left\|F(v)\right\|_{L^p}\to 0,\\
\left\|\frac{d}{dv}\left(\frac{\gamma}{\delta(\gamma)}F\left(\frac{v}{\gamma\delta(\gamma)}\right)\right)\right\|_{L^p} &= \gamma^{\frac{1}{p}}\delta(\gamma)^{\frac{1}{p}-2}\left\|F'(v)\right\|_{L^p}\to 0,
\end{aligned}$$

and thus

$$\left\|f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{L^1}+T\int_{\mathbf{R}}v^2\left|f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right|\,dv+\left\|f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right\|_{W_{x,v}^{1,p}}\to 0.$$

It remains to check

$$\left\||D|^{s-1}\frac{d}{dv}\left(f_0(v)-f_{\gamma,\delta(\gamma)}(v)\right)\right\|_{L^p}\to 0,\ \text{when}\ \gamma\to 0,$$

where $|D|^{\delta}$ $(\delta > 0)$ is the fractional differentiation operator with the Fourier symbol $|\xi|^{\delta}$. By using the scaling equality

$$\left(|D|^{\delta}\chi_d\right)(v) = \frac{1}{d^{\delta}}\left(|D|^{\delta}\chi\right)\left(\frac{v}{d}\right),$$

where $\chi_d(v) = \chi(v/d)$, we have

$$\left\| |D|^{s}\frac{d}{dv}\left(\frac{\gamma}{\delta(\gamma)}F\left(\frac{v}{\gamma\delta(\gamma)}\right)\right)\right\|_{L^p} = \gamma^{1-s+\frac{1}{p}}\delta(\gamma)^{-1-s+\frac{1}{p}}\left\|\left(|D|^{s}F'\right)(v)\right\|_{L^p} \to 0,$$

when $\gamma \to 0$, since $s < 1 + \frac{1}{p}$. So we can choose $\gamma = \gamma(\varepsilon) > 0$ small such that

$$\left\|f_0 - f_{\gamma,\delta(\gamma)}\right\|_{L^1(\mathbf{R})} + T\int_{\mathbf{R}} v^2\left|f_0(v) - f_{\gamma,\delta(\gamma)}(v)\right| dv + \left\|f_0 - f_{\gamma,\delta(\gamma)}\right\|_{W^{s,p}(\mathbf{R})} < \frac{\varepsilon}{2}.$$

Then

$$(f_\varepsilon, E_\varepsilon) = \left(f^{T}_{\gamma(\varepsilon),r(\gamma(\varepsilon),\varepsilon)}(x,v), E_{\gamma(\varepsilon),r(\gamma(\varepsilon),\varepsilon)}(x)\right)$$

is a steady BGK wave solution satisfying (6).

Case 2: $\int_{\mathbf{R}}\frac{f_0'(v)}{v}dv > \left(\frac{2\pi}{T}\right)^2$. Choose $F(v) = \exp\left(-\frac{v^2}{2}\right)$, then $\int_{\mathbf{R}}\frac{F'(v)}{v}dv < 0$. Define $f_{\gamma,\delta}(v)$ as in Case 1 (see (9)). Then there exists $0 < \delta_1 < \delta_2$ such that

$$0 < \int_{\mathbf{R}}\frac{f_0'(v)}{v}dv + \frac{1}{\delta_1^2}\int_{\mathbf{R}}\frac{F'(v)}{v}dv < \left(\frac{2\pi}{T}\right)^2 < \int_{\mathbf{R}}\frac{f_0'(v)}{v}dv + \frac{1}{\delta_2^2}\int_{\mathbf{R}}\frac{F'(v)}{v}dv.$$

The rest of the proof is the same as in Case 1.

Case 3: $\int_{\mathbf{R}}\frac{f_0'(v)}{v}dv = \left(\frac{2\pi}{T}\right)^2$. For $\delta > 0$, define

$$f_\delta(v) = \frac{1}{\delta}f_0\left(\frac{v}{\delta}\right).$$

Then $f_\delta \in C^{\infty}(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$, $f_\delta(v) > 0$, $\int_{\mathbf{R}} f_\delta(v)\,dv = 1$, and

$$\int_{\mathbf{R}}\frac{f_\delta'(v)}{v}dv = \frac{1}{\delta^2}\int_{\mathbf{R}}\frac{f_0'(v)}{v}dv.$$

For any $\varepsilon > 0$ small, there exist $0 < \delta_1(\varepsilon) < 1 < \delta_2(\varepsilon)$ such that

$$0 < \frac{1}{\delta_2^2}\int_{\mathbf{R}}\frac{f_0'(v)}{v}dv < \left(\frac{2\pi}{T}\right)^2 < \frac{1}{\delta_1^2}\int_{\mathbf{R}}\frac{f_0'(v)}{v}dv$$

and when $\delta \in (\delta_1(\varepsilon), \delta_2(\varepsilon))$,

$$\left\|f_0(v) - f_\delta(v)\right\|_{L^1(\mathbf{R})} + T\int_{\mathbf{R}} v^2\left|f_0(v) - f_\delta(v)\right| dv + \left\|f_0(v) - f_\delta(v)\right\|_{W^{2,p}(\mathbf{R})} < \frac{\varepsilon}{2}.$$

For $\delta \in (\delta_1(\varepsilon), \delta_2(\varepsilon))$, we consider bifurcation of steady BGK waves near $(f_\delta(v), 0)$, which are of the form

$$f_\delta^{\beta}(x,v) = \begin{cases} \frac{1}{\delta}g_+\left(\frac{2e}{\delta^2}\right) & \text{if } v > 0 \\ \frac{1}{\delta}g_-\left(\frac{2e}{\delta^2}\right) & \text{if } v \le 0 \end{cases}, \quad e = \frac{1}{2}v^2 - \beta(x), \; E = -\beta_x. \tag{13}$$

The existence of BGK waves is then reduced to solve the ODE

$$\beta_{xx} = \int_{\mathbf{R}} f_\delta^\beta (x,v) \; dv - 1 := h_\delta (\beta) \tag{14}$$

As in Case 1, for any $\delta \in (\delta_1 (\varepsilon), \delta_2 (\varepsilon))$, $\exists\, r_0 (\varepsilon) > 0$ (independent of δ) such that for each $0 < r < r_0$, there exists a $T(\delta; r)-$periodic solution $\beta_{\delta;r}$ to the ODE (14) with $\|\beta_{\delta;r}\|_{H^2(0,T(\delta;r))} = r$. Moreover,

$$\left(\frac{2\pi}{T(\delta;r)}\right)^2 \to \int_{\mathbf{R}} \frac{f_\delta'(v)}{v} dv, \text{ when } r \to 0.$$

So when r is small enough, $T(\delta_1; r) < T < T(\delta_2; r)$ and there exists $\delta_T (r, \varepsilon) \in (\delta_1 (\varepsilon), \delta_2 (\varepsilon))$ such that $T(\delta_T; r) = T$. Define $f_{r,\varepsilon}^T (x,v) = f_{\delta_T(r,\varepsilon)}^\beta (x,v)$ with $\beta = \beta_{\delta_T(r,\varepsilon);r}$ in (13) and $E_{r,\varepsilon}(x) = -\beta'_{\delta_T(r,\varepsilon);r}(x)$. Then $\left(f_{r,\varepsilon}^T (x,v), E_{r,\varepsilon}(x)\right)$ is a nontrivial BGK solution to (1) with $x-$period T. Let

$$\delta(\varepsilon) = \lim_{r\to 0} \delta_T (r,\varepsilon) \in [\delta_1 (\varepsilon), \delta_2 (\varepsilon)]$$

As in Case 1, by dominance convergence theorem, we can choose $r = r(\varepsilon) > 0$ small enough, such that

$$\left\| f_{r(\varepsilon),\varepsilon}^T (x,v) - f_{\delta(\varepsilon)}(v) \right\|_{L^1_{x,v}} + \int_0^T \int_{\mathbf{R}} v^2 \left| f_{r(\varepsilon),\varepsilon}^T (x,v) - f_{\delta(\varepsilon)}(v) \right| \; dxdv$$
$$+ \left\| f_{r(\varepsilon),\varepsilon}^T (x,v) - f_{\delta(\varepsilon)}(v) \right\|_{W^{2,p}_{x,v}} < \frac{\varepsilon}{2}.$$

Then

$$(f_\varepsilon, E_\varepsilon) = \left(f_{r(\varepsilon),\varepsilon}^T (x,v), E_{r(\varepsilon),\varepsilon}(x) \right)$$

is a steady BGK wave solution satisfying

$$\|f_\varepsilon - f_0\|_{L^1_{x,v}} + \int_0^T \int_{\mathbf{R}} v^2 |f_0 - f_\varepsilon| \; dxdv + \|f_\varepsilon - f_0\|_{W^{2,p}_{x,v}} < \varepsilon,$$

which certainly implies (6). This finishes the proof of the Proposition. ■

To finish the proof of Theorem 1, we need the following approximation result.

Lemma 2 *Fixed $p > 1$, $0 \le s < 1 + \frac{1}{p}$ and $c \in \mathbf{R}$. Assume $f_0 \in W^{s,p}(\mathbf{R})$, $f_0 > 0$, $\int_{\mathbf{R}} f_0 (v)\, dv = 1$, and $\int_{\mathbf{R}} v^2 f_0 (v)\, dv < \infty$. Then for any $\varepsilon > 0$, there exists $f_\varepsilon (v) \in C^\infty (\mathbf{R}) \cap W^{2,p}(\mathbf{R})$, such that f_ε is even near $v = c$, and*

$$\|f_\varepsilon - f_0\|_{L^1(\mathbf{R})} + \int_{\mathbf{R}} v^2 |f_\varepsilon - f_0| \; dxdv + \|f_\varepsilon - f_0\|_{W^{s,p}(\mathbf{R})} \le \varepsilon.$$

Proof. Let $\eta(x)$ be the standard mollifier function, that is,

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{x^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1 \end{cases},$$

and $\eta_{\delta_1}(x) = \frac{1}{\delta_1}\eta\left(\frac{x}{\delta_1}\right)$. Define $f_{\delta_1}(v) := \eta_{\delta_1}(v) * f_0(v)$. Then by the properties of mollifiers, we have

$$f_{\delta_1} \in C^\infty(\mathbf{R}),\ f_{\delta_1}(v) > 0,\ \int_{\mathbf{R}} f_{\delta_1}(v)\, dv = 1,$$

and when δ_1 is small enough

$$\|f_{\delta_1} - f_0\|_{L^1(\mathbf{R})} + \int_{\mathbf{R}} v^2 |f_{\delta_1} - f_0|\ dxdv + \|f_{\delta_1} - f_0\|_{W^{s,p}(\mathbf{R})} \leq \frac{\varepsilon}{2}. \tag{15}$$

We can assume $f_{\delta_1}(v) \in W^{2,p}$. Since otherwise, we can modify $f_{\delta_1}(v)$ near infinity by cut-off to get $\tilde{f}_{\delta_1}(v)$ such that $\tilde{f}_{\delta_1}(v) \in W^{2,p}$ and

$$\left\|f_{\delta_1} - \tilde{f}_{\delta_1}\right\|_{L^1(\mathbf{R})} + \int_{\mathbf{R}} v^2 \left|f_{\delta_1} - \tilde{f}_{\delta_1}\right|\ dxdv + \left\|f_{\delta_1} - \tilde{f}_{\delta_1}\right\|_{W^{s,p}(\mathbf{R})} \leq \frac{\varepsilon}{2}.$$

Solely to simplify notations, we set $c = 0$ below. Let $\sigma(x) = \sigma(|x|)$ to be the cut-off function defined by (7). Let $\delta_2 > 0$ be a small number, and define

$$\begin{aligned} f_{\delta_1,\delta_2}(v) &= f_{\delta_1}(v)\left(1 - \sigma\left(\frac{v}{\delta_2}\right)\right) + \left(\frac{f_{\delta_1}(v) + f_{\delta_1}(-v)}{2}\right)\sigma\left(\frac{v}{\delta_2}\right) \\ &= f_{\delta_1}(v) - \left(\frac{f_{\delta_1}(v) - f_{\delta_1}(-v)}{2}\right)\sigma\left(\frac{v}{\delta_2}\right) \end{aligned}$$

Then obviously,

$$f_{\delta_1,\delta_2} \in C^\infty(\mathbf{R}),\ f_{\delta_1,\delta_2}(v) > 0,\ \int_{\mathbf{R}} f_{\delta_1,\delta_2}(v)\, dv = \int_{\mathbf{R}} f_{\delta_1}(v)\, dv = 1,$$

and $f_{\delta_1,\delta_2}(v)$ is even on the interval $[-\delta_2, \delta_2]$. Below, we prove that: when δ_2 is small enough

$$\|f_{\delta_1} - f_0\|_{L^1(\mathbf{R})} + \int_{\mathbf{R}} v^2 |f_{\delta_1} - f_0|\ dv + \|f_{\delta_1} - f_0\|_{W^{s,p}(\mathbf{R})} \leq \frac{\varepsilon}{2}. \tag{16}$$

Since

$$\|f_{\delta_1} - f_{\delta_1,\delta_2}\|_{L^1} \leq \int_{|v|\leq 2\delta_2} f_{\delta_1}(v)\, dv$$

$$\int_{\mathbf{R}} v^2 |f_{\delta_1} - f_{\delta_1,\delta_2}|\ dv \leq (2\delta_2)^2 \int_{|v|\leq 2\delta_2} f_{\delta_1}(v)\, dv,$$

$$\|f_{\delta_1} - f_{\delta_1,\delta_2}\|_{L^p} \leq \|f_{\delta_1}\|_{L^p[-2\delta_2, 2\delta_2]},$$

and

$$\partial_v\left(f_{\delta_1}-f_{\delta_1,\delta_2}\right)=\left(\frac{f'_{\delta_1}(v)+f'_{\delta_1}(-v)}{2}\right)\sigma\left(\frac{v}{\delta_2}\right)+\sigma'\left(\frac{v}{\delta_2}\right)\frac{f_{\delta_1}(v)-f_{\delta_1}(-v)}{2\delta_2},$$

$$\begin{aligned}\left\|\partial_v\left(f_{\delta_1}-f_{\delta_1,\delta_2}\right)\right\|_{L^p} &\leq \left\|f'_{\delta_1}\right\|_{L^p[-2\delta_2,2\delta_2]}+\max|\sigma'|\left\|\frac{f_{\delta_1}(v)-f_{\delta_1}(-v)}{2\delta_2}\right\|_{L^p\{\delta_2\leq|v|\leq 2\delta_2\}}\\ &\leq \left\|f'_{\delta_1}\right\|_{L^p[-2\delta_2,2\delta_2]}+\frac{1}{2\delta_2}\max|\sigma'|\left\|\int_{-2\delta_2}^{2\delta_2}\left|f'_{\delta_1}\right|dv\right\|_{L^p\{\delta_2\leq|v|\leq 2\delta_2\}}\\ &\leq \left\|f'_{\delta_1}\right\|_{L^p[-2\delta_2,2\delta_2]}+\frac{1}{2\delta_2}\max|\sigma'|\left(4\delta_2\right)^{\frac{1}{p'}}\left\|f'_{\delta_1}\right\|_{L^p[-2\delta_2,2\delta_2]}\left(2\delta_2\right)^{\frac{1}{p}}\\ &\leq \left(1+2\max|\sigma'|\right)\left\|f'_{\delta_1}\right\|_{L^p[-2\delta_2,2\delta_2]},\end{aligned}$$

so when $\delta_2\to 0$,

$$\left\|f_{\delta_1}-f_{\delta_1,\delta_2}\right\|_{L^1}+\int_{\mathbf{R}}v^2\left|f_{\delta_1}-f_{\delta_1,\delta_2}\right|\,dv+\left\|f_{\delta_1}-f_{\delta_1,\delta_2}\right\|_{W^{1,p}}\to 0.$$

Next, we show

$$\left\|\partial_v\left(f_{\delta_1}-f_{\delta_1,\delta_2}\right)\right\|_{W^{s-1,p}(\mathbf{R})}\to 0,\text{ when }\delta_2\to 0.$$

This follows from Lemma 3 below, since $s-1<\frac{1}{p}$ and

$$\begin{aligned}&\left\|\frac{f_{\delta_1}(v)-f_{\delta_1}(-v)}{2v}\right\|_{W^{s-1,p}(\mathbf{R})}\\ &=\left\|\int_0^1 f'_{\delta_1}\left((2\tau-1)v\right)d\tau\right\|_{W^{s-1,p}(\mathbf{R})}\leq\int_0^1\left\|f'_{\delta_1}\left((2\tau-1)v\right)\right\|_{W^{s-1,p}(\mathbf{R})}d\tau\\ &\leq C\int_0^1\left(|2\tau-1|^{-\frac{1}{p}}\left\|f'_{\delta_1}\right\|_{L^p}+|2\tau-1|^{-\left(\frac{1}{p}-s+1\right)}\left\||D|^{s-1}f'_{\delta_1}\right\|_{L^p}\right)d\tau\leq C\left\|f_{\delta_1}\right\|_{W^{s,p}(\mathbf{R})}.\end{aligned}$$

So when $\delta_2\to 0$,

$$\left\|f_{\delta_1}-f_{\delta_1,\delta_2}\right\|_{L^1_v}+\int_{\mathbf{R}}v^2\left|f_{\delta_1}-f_{\delta_1,\delta_2}\right|\,dv+\left\|f_{\delta_1}-f_{\delta_1,\delta_2}\right\|_{W^{s,p}}\to 0.$$

Thus by choosing δ_2 small enough, (16) is satisfied. By setting $f_\varepsilon=f_{\delta_1,\delta_2}$, the conclusion of the lemma follows from (15) and (16). ■

Lemma 3 *Given* $f\in W^{\frac{1}{p},p}(\mathbf{R})\cap L^\infty,\ g\in W^{s,p}(\mathbf{R})\left(p>1,0\leq s<\frac{1}{p}\right)$, *then for any* $\delta>0,\ f\left(\frac{x}{\delta}\right)g(x)\in W^{s,p}(\mathbf{R})$ *and*

$$\left\|f\left(\frac{x}{\delta}\right)g\right\|_{W^{s,p}}\to 0,\ \textit{when }\delta\to 0. \tag{17}$$

Proof. First, we cite a result of Strichartz ([44]): Given $h_1 \in W^{\frac{1}{p},p}(\mathbf{R}) \cap L^\infty$, $h_2 \in W^{s,p}(\mathbf{R})$, then $h_1h_2 \in W^{s,p}(\mathbf{R})$ and

$$\|h_1h_2\|_{W^{s,p}} \leq C\left(\|h_1\|_{W^{\frac{1}{p},p}}, \|h_1\|_{L^\infty}\right)\|h_2\|_{W^{s,p}}.$$

Above result immediately implies that $f\left(\frac{x}{\delta}\right)g(x) \in W^{s,p}(\mathbf{R})$. To show (17), for any $\varepsilon > 0$, we pick $g_1 \in C_0^\infty(\mathbf{R})$ such that $\|g - g_1\|_{W^{s,p}} < \varepsilon$. Since

$$\left\|f\left(\frac{x}{\delta}\right)\right\|_{L^p} + \left\||D|^{\frac{1}{p}}\left(f\left(\frac{x}{\delta}\right)\right)\right\|_{L^p} = \delta^{\frac{1}{p}}\|f(x)\|_{L^p} + \left\||D|^{\frac{1}{p}}f\right\|_{L^p},$$

so when $\delta \leq 1$,

$$\left\|f\left(\frac{x}{\delta}\right)\right\|_{W^{\frac{1}{p},p}} \leq C\|f\|_{W^{\frac{1}{p},p}}, \text{ for some } C \text{ independent of } \delta.$$

Thus

$$\begin{aligned}
&\left\|f\left(\frac{x}{\delta}\right)g\right\|_{W^{s,p}} \\
\leq\ &\left\|f\left(\frac{x}{\delta}\right)(g-g_1)\right\|_{W^{s,p}} + \left\|f\left(\frac{x}{\delta}\right)g_1\right\|_{W^{s,p}} \\
\leq\ &C\left(\|f\|_{W^{\frac{1}{p},p}}, \|f\|_{L^\infty}\right)\|g-g_1\|_{W^{s,p}} + \left\|f\left(\frac{x}{\delta}\right)\right\|_{W^{s,p}} C\left(\|g_1\|_{W^{\frac{1}{p},p}}, \|g_1\|_{L^\infty}\right) \\
\leq\ &C\left(\|f\|_{W^{\frac{1}{p},p}}, \|f\|_{L^\infty}\right)\varepsilon + \left(\delta^{\frac{1}{p}}\|f(x)\|_{L^p} + \delta^{\frac{1}{p}-s}\left\||D|^{\frac{1}{p}}f\right\|_{L^p}\right)C\left(\|g_1\|_{W^{\frac{1}{p},p}}, \|g_1\|_{L^\infty}\right).
\end{aligned}$$

Letting $\delta \to 0$, we get

$$\lim_{\delta\to 0}\left\|f\left(\frac{x}{\delta}\right)g\right\|_{W^{s,p}} \leq C\left(\|f\|_{W^{\frac{1}{p},p}}, \|f\|_{L^\infty}\right)\varepsilon.$$

Since ε is arbitrarily small, (17) is proved. ■

Proof of Theorem 1. Fixed the period $T > 0$ and the travel speed $c \in \mathbf{R}$.Then by Lemma 2, for any ε small enough, there exists $f_1(v) \in C^\infty(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$, such that $f_1(v)$ is even near $v = c$ and

$$\|f_1 - f_0\|_{L^1(\mathbf{R})} + T\int_{\mathbf{R}} v^2|f_1 - f_0|\ dv + \|f_1 - f_0\|_{W^{s,p}(\mathbf{R})} \leq \varepsilon/2.$$

Our goal is to construct travelling BGK wave solutions of the form

$$(f_\varepsilon(x-ct,v), E_\varepsilon(x-ct))$$

near $(f_1(v), 0)$, such that

$$\|f_\varepsilon(x,v) - f_1(v)\|_{L^1_{x,v}} + \int_{\mathbf{R}} v^2|f_\varepsilon(x,v) - f_1(v)|\ dxdv + \|f_\varepsilon(x,v) - f_1(v)\|_{W^{s,p}_{x,v}} < \frac{\varepsilon}{2}.$$

It is equivalent to find steady BGK solutions $(f_\varepsilon(x,v+c), E_\varepsilon(x))$ near $(f_1(v+c), 0)$. By Proposition 1, there exists steady BGK solution $(f_2(x,v), E_2(x))$ near $(f_1(v+c), 0)$ such that $E_2(x)$ not identically 0,

$$\|f_2(x,v) - f_1(v+c)\|_{L^1_{x,v}} + \int_{\mathbf{R}} v^2|f_2(x,v) - f_1(v+c)|\ dxdv$$

$$+\left\|f_2(x,v)-f_1(v+c)\right\|_{W^{s,p}_{x,v}}<\frac{\varepsilon}{2\left(5+4c^2\right)}.$$

Setting

$$f_\varepsilon(x,v)=f_2(x,v-c)\,,\ \ E_\varepsilon(x)=E_2(x)\,,$$

then $\left(f_\varepsilon(x-ct,v),E_\varepsilon(x-ct)\right)$ is a travelling BGK solution and

$$\left\|f_\varepsilon-f_1(v)\right\|_{L^1_{x,v}}+\int_{\mathbf{R}}(v-c)^2\left|f_\varepsilon-f_1(v)\right|\ dxdv$$

$$+\left\|f_\varepsilon-f_1(v)\right\|_{W^{s,p}_{x,v}}<\frac{\varepsilon}{2\left(5+4c^2\right)}.$$

Since $|v-c|\geq|v|/2$ when $|v|\geq2|c|$,so

$$\begin{aligned}
&\int_{\mathbf{R}}v^2\left|f_\varepsilon(x,v)-f_1(v)\right|\ dxdv\\
&\leq\int_{|v|\geq2|c|}v^2\left|f_\varepsilon(x,v)-f_1(v)\right|\ dxdv+\int_{|v|\leq2|c|}v^2\left|f_\varepsilon(x,v)-f_1(v)\right|\ dxdv\\
&\leq4\int(v-c)^2\left|f_\varepsilon(x,v)-f_1(v)\right|\ dxdv+4c^2\left\|f_\varepsilon-f_1\right\|_{L^1_{x,v}}\\
&<\frac{\left(4+4c^2\right)\varepsilon}{2\left(5+4c^2\right)},
\end{aligned}$$

and thus

$$\begin{aligned}
&\left\|f_\varepsilon-f_1(v)\right\|_{L^1_{x,v}}+\int_{\mathbf{R}}v^2\left|f_\varepsilon(x,v)-f_1(v)\right|\ dxdv+\left\|f_\varepsilon(x,v)-f_1(v)\right\|_{W^{s,p}_{x,v}}\\
&<\frac{\varepsilon}{2\left(5+4c^2\right)}+\frac{\left(4+4c^2\right)\varepsilon}{2\left(5+4c^2\right)}=\frac{\varepsilon}{2},
\end{aligned}$$

So

$$\left\|f_\varepsilon-f_0(v)\right\|_{L^1_{x,v}}+\int_{\mathbf{R}}v^2\left|f_\varepsilon(x,v)-f_0(v)\right|\ dxdv+\left\|f_\varepsilon(x,v)-f_0(v)\right\|_{W^{s,p}_{x,v}}<\varepsilon.$$

and the proof of Theorem 1 is finished. ■

Remark 1 *For steady BGK waves* $(f(x,v),E(x))$ *of the form* $E(x)=-\beta_x$ *and*

$$f(x,v)=\left\{\begin{array}{ll}\mu^+(e) & \text{if } v\geq0\\ \mu^-(e) & \text{if } v<0\end{array}\right\},\quad e=\frac{1}{2}v^2-\beta(x)\,,\tag{18}$$

with $\mu^+,\mu^-\in C^1(\mathbf{R})$ *, such as constructed in the proof of Theorem 1,* $E(x)$ *has only two zeros in one minimal period. This is because the electric potential* β *satisfying the 2nd order autonomous ODE*

$$\beta_{xx}=\int_{v\geq0}\mu^+\left(\frac{1}{2}v^2-\beta\right)\ dv+\int_{v<0}\mu^-\left(\frac{1}{2}v^2-\beta\right)\ dv-1=h(\beta)\tag{19}$$

with $h \in C^1(\mathbf{R})$. Any periodic solution of minimal period to the ODE (19) has only one minimum and maximum, and therefore $E = -\beta_x$ vanishes at only two points. By Theorem 1, for $T > 0$, near any homogeneous equilibria we can construct small BGK waves such that multiple of its minimal period equal T. By [31] and [32], any of such multi-BGK waves are linearly and nonlinearly unstable under perturbations of period T. So far, the existence of stable BGK wave of minimal period remains open, although some numerical evidences suggest the existence of such stable BGK wave. For example, in [5] starting near a unstable multi-BGK wave, numerical simulations shows that the long time asymptotics is to tend to a seemingly stable BGK wave of minimal period.

Remark 2 *In ([22] [23]), Dorning and Holloway (see also [10], [17]) studied the bifurcation of small travelling BGK waves with speed v_p near homogeneous equilibria $(f_0(v), 0)$ under the bifurcation condition*

$$\nu(v_p) = P\int \frac{f_0'(v)}{v - v_p} dv > 0, \tag{20}$$

where P denotes the principal value integral. It is equivalent to find steady BGK waves near $(f_0(v + v_p), 0)$. The approach in ([22] [23]) is as follows. Define

$$\begin{aligned} f^{e,v_p}(v) &= \frac{1}{2}\left(f_0(v + v_p) + f_0(-v + v_p)\right), \\ f^{o,v_p}(v) &= \frac{1}{2}\left(f_0(v + v_p) - f_0(-v + v_p)\right) \ . \end{aligned}$$

Then

$$\int \frac{\frac{d}{dv} f^{e,v_p}(v)}{v} dv = P\int \frac{f_0'(v)}{v - v_p} dv = \nu(v_p) > 0.$$

So by the bifurcation theory, there exist small BGK waves $(f^e(x,v), -\beta_x)$ near $(f^{e,v_p}(v), 0)$ with periods close to $\frac{2\pi}{\sqrt{\nu(v_p)}}$, and $f^e(x,v)$ is even in v. Next, the odd part $f^{o,v_p}(x,v)$ is defined by

$$f^{o,v_p}(x,v) = \left(1 - \sigma\left(\frac{e}{-2\min\beta}\right)\right)\left\{\begin{array}{cc} G^o(e) & \text{if } v \geq 0 \\ -G^o(e) & \text{if } v < 0 \end{array}\right. ,$$

where $\sigma(x)$ is the cut-off function as defined in (7) and $G^o(e) = f^{o,v_p}\left(\sqrt{2e}\right)$ when $e > 0$. Define

$$f(x,v) = f^{e,v_p}(x,v) + f^{o,v_p}(x,v),$$

then $(f(x,v), -\beta_x)$ is a steady BGK wave, since for trapped particles with $e < -\min\beta$, $f^{o,v_p}(x,v) = 0$ and $f(x,v)$ only depends on e. It can be shown that $(f(x,v), -\beta_x)$ is close to $(f_0(v + v_p), 0)$ in $L^p_{x,v}$ norm. The periods of the BGK waves constructed above are only near $\frac{2\pi}{\sqrt{\nu(v_p)}}$. In [22] [23], [17], it was suggested that BGK waves with exact period $\frac{2\pi}{\sqrt{\nu(v_p)}}$ and $\varepsilon-$close to $(f_0(v + v_p), 0)$ in $L^p_{x,v}$

norm can be constructed by performing above bifurcation from $((1+\mu(\varepsilon)) f_0(v+v_p), 0)$ *for proper small parameter* μ. *It should be pointed out that this strategy actually does not work to get exact period* $\frac{2\pi}{\sqrt{\nu(v_p)}}$. *Since to ensure that* $((1+\mu(\varepsilon)) f^{e,v_p}(v), 0)$ *is a bifurcation point, it is required that*

$$\int_{\mathbf{R}} (1+\mu(\varepsilon)) f^{e,v_p}(v)\, dv = 1,$$

and thus $\mu(\varepsilon) = 0$, *i,e,* μ *is not adjustable at all.*

Second, by Lemma 4 below,

$$|\nu(v_p)| = \left| \int \frac{\frac{d}{dv} f^{e,v_p}(v)}{v} dv \right| \leq \left\| f^{e,v_p}(v) \right\|_{W^{2,p}} = \left\| f_0(v) \right\|_{W^{2,p}}.$$

So by the method in [22] [23], one can not get small BGK waves with spatial periods less than $2\pi/\sqrt{\|f_0(v)\|_{W^{2,p}}}$. *By comparison, we construct BGK waves with any minimal period near any homogeneous equilibrium* $(f_0(v), 0)$ *in any* $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$ *neighborhood.*

It is also claimed in [22] [23] that for v_p *such that* $\nu(v_p) < 0$, *there exist no travelling BGK waves with travel speed* v_p, *arbitrarily near* $(f_0(v), 0)$. *For Maxwellian* $f_0(v) = e^{-\frac{1}{2}v^2}$, *the critical speed is about* $v_p = 1.35$ *since* $\nu(v_p) < 0$ *when* $v_p < 1.35$. *However, by our Theorem 1, BGK waves with arbitrary travel speed exist near (in* $W^{s,p}$ *space,* $s < 1 + \frac{1}{p}$*) any homogeneous equilibrium including Maxwellian. So the claim of the critical travel speed based on (20) is not true.*

Proof of Corollary 1. From the proof of Theorem 1 and Proposition 1, it follows that: Fixed $T > 0$, for any $\varepsilon > 0$, there exists a homogeneous profile $f_\varepsilon(v) \in C^\infty(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$, such that $f_\varepsilon(v) \geq 0$, $\int_{\mathbf{R}} f_\varepsilon(v)\ dv = 1$,

$$\left(\frac{2\pi}{T}\right)^2 = k_0^2 = \int_{\mathbf{R}} \frac{f_\varepsilon'(v)}{v - v_\varepsilon} dv \quad \text{with} \quad f_\varepsilon'(v_\varepsilon) = 0,$$

and

$$\|f_\varepsilon(v) - f_0(v)\|_{L^1(\mathbf{R})} + T \int_{\mathbf{R}} v^2 \left| f_\varepsilon(v) - f_0(v) \right| dv + \|f_\varepsilon(v) - f_0(v)\|_{W^{s,p}(\mathbf{R})} < \frac{\varepsilon}{2}. \tag{21}$$

Define $f_\delta(v) \in C^\infty(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$ by

$$f_\delta(v) = \frac{1}{\delta} f_\varepsilon\left(v_\varepsilon + \frac{v - v_\varepsilon}{\delta}\right),$$

Then $f_\delta(v) \geq 0$, $\int_{\mathbf{R}} f_\delta(v)\ dv = 1$ and

$$k_0(\delta)^2 = \int \frac{f_\delta'(v)}{v - v_\varepsilon} dv = \frac{1}{\delta^2}\left(\frac{2\pi}{T}\right)^2.$$

We consider two cases below.

Case 1: $f_{\varepsilon}''(v_{\varepsilon}) > 0$. By Lemma 7 and Remark 6 thereafter, there exist unstable modes of the linearized VP equation around $(f_{\varepsilon}(v), 0)$, for wave numbers k in the internal (k_1, k_0). Here k_1 is defined by

$$k_1^2 = \int_{\mathbf{R}} \frac{f_{\varepsilon}'(v)}{v - c_1} dv,$$

and c_1 is a maximum point $f_{\varepsilon}(v)$. If there is no maximum point c_1 of f_{ε} such that

$$\int_{\mathbf{R}} \frac{f_{\varepsilon}'(v)}{v - c_1} dv < k_0^2,$$

then $k_1 = 0$. Choose $\delta < 1$ such that

$$\|f_{\varepsilon}(v) - f_{\delta}(v)\|_{L^1(\mathbf{R})} + T \int_{\mathbf{R}} v^2 |f_{\varepsilon} - f_{\delta}| \, dv + \|f_{\varepsilon} - f_{\delta}\|_{W^{s,p}(\mathbf{R})} < \frac{\varepsilon}{2}. \tag{22}$$

Then again by Lemma 7 and Remark 6, there exist unstable modes of the linearized VP equation around $(f_{\delta}(v), 0)$, for wave numbers k in the internal $(k_1(\delta), k_0(\delta))$. Since $k_0(\delta) > k_0$ and $k_0(\delta) - k_1(\delta) \to k_0 - k_1 > 0$ when $\delta \to 1-$, we have $k_0 \in (k_1(\delta), k_0(\delta))$ when δ is close enough to 1. This implies that $(f_{\delta}(v), 0)$ is linearly unstable under perturbations of period T. Moreover, the inequalities (21) and (22) imply that

$$\|f_{\delta}(v) - f_0(v)\|_{L^1(\mathbf{R})} + T \int_{\mathbf{R}} v^2 |f_{\delta}(v) - f_0(v)| \, dv + \|f_{\delta}(v) - f_0(v)\|_{W^{s,p}(\mathbf{R})} < \varepsilon.$$

Case 2: $f_{\varepsilon}''(v_{\varepsilon}) < 0$. Choose $\delta > 1$ sufficiently close to 1, then by the same argument as in Case 1, $(f_{\delta}(v), 0)$ is linearly unstable under perturbations of period T and

$$\|f_{\delta}(v) - f_0(v)\|_{L^1(\mathbf{R})} + T \int_{\mathbf{R}} v^2 |f_{\delta}(v) - f_0(v)| \, dv + \|f_{\delta}(v) - f_0(v)\|_{W^{s,p}(\mathbf{R})} < \varepsilon.$$

This finishes the proof of Corollary 1. ■

3 Nonexistence of BGK waves in $W^{s,p}\left(s > 1 + \frac{1}{p}\right)$

In this Section, we prove Theorem 2. The next lemma is a Hardy type inequality.

Lemma 4 *If $u(v) \in W^{s,p}(\mathbf{R})$ $\left(p > 1, s > \frac{1}{p}\right)$, and $u(0) = 0$, then*

$$\int_{\mathbf{R}} \left| \frac{u(v)}{v} \right| dv \le C \|u\|_{W^{s,p}(\mathbf{R})},$$

for some constant C.

Proof. Since $s > \frac{1}{p}$, the space $W^{s,p}(\mathbf{R})$ is embedded to the Hölder space $C^{0,\alpha}$ with $\alpha \in \left(0, s - \frac{1}{p}\right)$. So

$$|u(v)| = |u(v) - u(0)| \leq |v|^{\alpha} \|u\|_{C^{0,\alpha}} \leq C \|u\|_{W^{s,p}} |v|^{\alpha}$$

and thus

$$\begin{aligned}
\int_{\mathbf{R}} \left| \frac{u(v)}{v} \right| dv &\leq \int_{-1}^{1} \left| \frac{u(v)}{v} \right| dv + \int_{|v|\geq 1} \left| \frac{u(v)}{v} \right| dv \\
&\leq C \|u\|_{W^{s,p}} \int_{-1}^{1} |v|^{-1+\alpha} dv + \left(\int_{|v|\geq 1} \frac{1}{|v|^{p'}} dv \right)^{\frac{1}{p'}} \|u\|_{L^p} \\
&\leq C \|u\|_{W^{s,p}(\mathbf{R})}.
\end{aligned}$$

■

Proof of Theorem 2. Suppose otherwise, then there exist a sequence $\varepsilon_n \to 0$, and nontrivial travelling wave solutions

$$(f_n(x - c_n t, v), E_n(x - c_n t))$$

to (1) such that $E_n(x)$ is not identically zero, $\int_0^T E_n(x)\,dx = 0$, $f_n(x,v)$ and $\beta_n(x)$ are $T-$periodic in x,

$$\int_0^T \int_{\mathbf{R}} v^2 f_n(x,v)\,dvdx < \infty \text{ and } \|f_n(x,v) - f_0(v)\|_{W^{s,p}_{x,v}} < \varepsilon_n.$$

The travelling BGK waves satisfy

$$(v - c_n)\,\partial_x f_n - E_n \partial_v f_n = 0, \tag{23}$$

and

$$\frac{\partial E_n}{\partial x} = -\int_{-\infty}^{+\infty} f_n dv + 1. \tag{24}$$

Because $f_n \in W^{s,p}_{x,v}$ with $s > 1 + \frac{1}{p} > \frac{2}{p}$, so by Sobolev embedding

$$\|f_n\|_{L^{\infty}_{x,v}} \leq C \|f_n\|_{W^{s,p}_{x,v}} < \infty.$$

By a standard estimate in kinetic theory,

$$\rho_n = \int f_n dv \leq \|f_n\|_{L^{\infty}_{x,v}}^{\frac{2}{3}} \left(\int v^2 f_n dv \right)^{\frac{1}{3}} \tag{25}$$

and thus $\rho_n \in L^3(0,T)$. So $E_n(x) \in W^{1,3}(0,T)$ which implies that $E_n(x) \in H^1(0,T)$ and $E_n(x)$ is absolutely continuous. Define two sets $\mathbf{P}_n = \{E_n \neq 0\}$ and $\mathbf{Q}_n = \{E_n = 0\}$. Then $\mathbf{P}_n$ is of non-zero measure and $E_n' = 0$ a.e. on $\mathbf{Q}_n$. Thus we have

$$\int_0^T |E_n'(x)|^2 dx = -\int_0^T \rho_n(x) E_n'(x)\,dx = -\int_{\mathbf{P}_n} \rho_n(x) E_n'(x)\,dx. \tag{26}$$

Since $s-1>\frac{1}{p}$, by the trace theorem for fractional Sobolev Space,

$$\partial_x f_n|_{v=c_n}, \quad \partial_v f_n|_{v=c_n} \in L^p(0,T).$$

So from equation (23), $\partial_v f_n|_{v=c_n}=0$ for a.e. $x \in \mathbf{P}_n$. By Lemma 4, for a.e. $x \in \mathbf{P}_n$,

$$\left|\int \frac{\partial_v f_n}{v-c_n} dv\right|(x) \leq C \|f_n(x,v)\|_{W_v^{s,p}} \in L^p(\mathbf{P}_n).$$

From (23), when $x \in \mathbf{P}_n$,

$$\rho_n'(x) = \int \frac{\partial_v f_n}{v-c_n} dv E_n(x) \in L^p(\mathbf{P}_n)$$

and it follows from (26) that

$$\int_0^T |E_n'(x)|^2 dx - \int_{\mathbf{P}_n} \int \frac{\partial_v f_n}{v-c_n} dv E_n(x)^2 dx = 0. \tag{27}$$

Denote $|\mathbf{P}_n|$ to be the measure of the set $\mathbf{P}_n$. We consider two cases.

Case 1: $|\mathbf{P}_n| \to 0$ when $n \to \infty$. Since

$$\|E_n\|_{L^\infty(0,T)} \leq \|E_n'\|_{L^1(0,T)} \leq \sqrt{T}\|E_n'\|_{L^2(0,T)},$$

so from (27),

$$\begin{aligned}
\|E_n'\|_{L^2(0,T)}^2 &\leq T\|E_n'\|_{L^2(0,T)}^2 \int_{\mathbf{P}_n}\int \left|\frac{\partial_v f_n}{v-c_n}\right| dvdx \\
&\leq T\|E_n'\|_{L^2(0,T)}^2 \int_{\mathbf{P}_n} \|f_n(x,v)\|_{W_v^{s,p}} dx \\
&\leq T\|E_n'\|_{L^2(0,T)}^2 \left(\int_{\mathbf{P}_n} \|f_n(x,v)-f_0\|_{W_v^{s,p}} dx + |\mathbf{P}_n| \|f_0\|_{W^{s,p}}\right) \\
&\leq T\|E_n'\|_{L^2(0,T)}^2 \left(C\|f_n(x,v)-f_0\|_{W_{x,v}^{s,p}} + |\mathbf{P}_n| \|f_0\|_{W^{s,p}}\right) \\
&< \|E_n'\|_{L^2(0,T)}^2,
\end{aligned}$$

when n is large enough. Thus for large n, $\|E_n'\|_{L^2(0,T)}=0$ and thus $E_n(x)\equiv 0$, which is a contradiction.

Case 2: $|\mathbf{P}_n| \to d > 0$ when $n \to \infty$. When n is large enough, we have $|\mathbf{P}_n| \geq \frac{d}{2}$. By the trace Theorem,

$$\begin{aligned}
\|\partial_v f_n(x,c_n) - \partial_v f_0(c_n)\|_{L^p(\mathbf{P}_n)} &\leq \|\partial_v f_n(x,c_n) - \partial_v f_0(c_n)\|_{L^p(0,T)} \\
&\leq C\|f_n - f_0\|_{W^{s,p}} \leq C\varepsilon_n.
\end{aligned}$$

Since $\partial_v f_n(x,c_n) = 0$ for a.e. $x \in \mathbf{P}_n$, so $\|\partial_v f_0(c_n)\|_{L^p(\mathbf{P}_n)} \leq C\varepsilon_n$ which implies that

$$|\partial_v f_0(c_n)| \leq \frac{C\varepsilon_n}{\left(\frac{d}{2}\right)^{\frac{1}{p}}}.$$

Thus $\partial_v f_0(c_n) \to 0$ when $n \to +\infty$. Therefore there exist a subsequence of $\{c_n\}$, such that either it converges to one of the critical points of f_0, say $v_i \in S$ or it diverges. We discuss these two cases separately below. To simplify notations, we still denote the subsequence by $\{c_n\}$.

Case 2.1: $c_n \to v_i \in S$. Rewrite (27) as

$$\int_0^T |E_n'(x)|^2 \, dx = \int_{\mathbf{R}} \frac{\partial_v f_0}{v - v_i} dv \int_{\mathbf{P}_n} E_n(x)^2 \, dx + \int_{\mathbf{P}_n} V_n(x) E_n(x)^2 \, dx, \quad (28)$$

where

$$V_n(x) = \int_{\mathbf{R}} \frac{\partial_v f_n}{v - c_n} dv - \int_{\mathbf{R}} \frac{\partial_v f_0}{v - v_i} dv = \int_{\mathbf{R}} \frac{\partial_v (f_n(x, v + c_n) - f_0(v + v_i))}{v} dv.$$

Note that $\partial_v (f_n(x, v + c_n) - f_0(v + v_i))|_{v=0} = 0$ for $x \in \mathbf{P}_n$, so by Lemma 4, we have

$$\begin{aligned}
\int_{\mathbf{P}_n} |V_n(x)| \, dx &\leq C \int_{\mathbf{P}_n} \|f_n(x, v + c_n) - f_0(v + v_i)\|_{W_v^{s,p}} \, dx \\
&\leq C \int_0^T \left(\|f_n - f_0\|_{W_v^{s,p}} + \|f_0(v + c_n) - f_0(v + v_i)\|_{W^{s,p}} \right) dx \\
&\leq C \left(\|f_n - f_0\|_{W_{x,v}^{s,p}} + \|f_0(v + c_n) - f_0(v + v_i)\|_{W^{s,p}} \right)
\end{aligned}$$

So $\int_{\mathbf{P}_n} |V_n(x)| \, dx \to 0$ when $n \to \infty$. Since $\int_0^T E_n(x) \, dx = 0$ and $E_n \in H^1(0,T)$ is $T-$periodic, we have

$$\|E_n'\|_{L^2(0,T)} \geq \frac{2\pi}{T} \|E_n\|_{L^2(0,T)}.$$

Also by the assumption of Theorem 2,

$$a_i = \int_{\mathbf{R}} \frac{\partial_v f_0}{v - v_i} < \left(\frac{2\pi}{T} \right)^2.$$

Combining above, from (28), we get

$$\begin{aligned}
\|E_n'\|_{L^2(0,T)}^2 &\leq \frac{\max\{a_i, 0\}}{\left(\frac{2\pi}{T}\right)^2} \|E_n\|_{L^2(0,T)}^2 + \int_{\mathbf{P}_n} |V_n(x)| \, dx \, \|E_n\|_{L^\infty}^2 \\
&\leq \|E_n'\|_{L^2(0,T)}^2 \left(\frac{\max\{a_i, 0\}}{\left(\frac{2\pi}{T}\right)^2} + T \int_{\mathbf{P}_n} |V_n(x)| \, dx \right) \\
&< \|E_n'\|_{L^2(0,T)}^2,
\end{aligned}$$

when n is large enough. A contradiction again.

Case 2.2: $\{c_n\}$ diverges. We assume $c_n \to +\infty$, and the case when $c_n \to -\infty$ is similar. Again, for a.e. $x \in \mathbf{P}_n$, $\partial_v f_n(x, c_n) = 0$. Let $\chi_n(v)$ be a cut-off function such that: $0 \leq \chi_n \leq 1$, $\chi_n(v) = 1$ when $v \in \left[\frac{c_n}{2}, \frac{3c_n}{2}\right]$;

$\chi_n(v) = 0$ when $v \notin \left[\frac{c_n}{2} - 1, \frac{3c_n}{2} + 1\right]$ and $|\chi_n|_{C^1} \le M$ (independent of n). Since $W^{s_1,p} \hookrightarrow W^{s_2,p}$ when $s_1 > s_2$, we can assume $\frac{1}{p} < s - 1 \le 1$. Then

$$\begin{aligned}
&\int_{\mathbf{P}_n} \left| \int_{\mathbf{R}} \frac{\partial_v f_n}{v - c_n} dv \right| dx \le \int_{\mathbf{P}_n} \left(\int_{\mathbf{R}} \left| \frac{\chi_n \partial_v f_n}{v - c_n} \right| dv + \int_{\mathbf{R}} \left| \frac{(1 - \chi_n) \partial_v f_n}{v - c_n} \right| dv \right) dx \\
&\le C \int_{\mathbf{P}_n} \left(\|\chi_n \partial_v f_n\|_{W_v^{s-1,p}} + \int_{|v - c_n| \ge \frac{c_n}{2}} \left| \frac{\partial_v f_n}{v - c_n} \right| dv \right) dx \\
&\le C \int_0^T \left(\|\chi_n \partial_v (f_n - f_0)\|_{W_v^{s-1,p}} + \|\chi_n \partial_v f_0\|_{W^{s-1,p}} + c_n^{-1+\frac{1}{p'}} \|f_n\|_{W_v^{1,p}} \right) dx \\
&\le C(M) \|f_n - f_0\|_{W_{x,v}^{s,p}} + CT \|\chi_n \partial_v f_0\|_{W^{s-1,p}} + CT c_n^{-1+\frac{1}{p'}} \|f_n\|_{W_{x,v}^{1,p}} \\
&\to 0, \text{ when } n \to \infty,
\end{aligned}$$

and this again leads to a contradiction as in Case 1. In the above, we use two estimates:

i)

$$\|\chi_n \partial_v (f_n - f_0)\|_{W_v^{s-1,p}} \le C(M) \|\partial_v (f_n - f_0)\|_{W_v^{s-1,p}}.$$

ii)

$$\|\chi_n \partial_v f_0\|_{W^{s-1,p}} \to 0, \text{ when } n \to \infty. \tag{29}$$

We prove them below. Estimate i) follows from the following general estimate:

Given $u(v) \in C_0^1(\mathbf{R})$, then for any $g \in W^{\alpha,p}(\mathbf{R})$ $(p > 1, 0 \le \alpha \le 1)$, we have

$$\|ug\|_{W^{\alpha,p}(\mathbf{R})} \le C(\|u\|_{C^1}) \|g\|_{W^{\alpha,p}(\mathbf{R})}. \tag{30}$$

This estimate is obvious for $\alpha = 0$ and $\alpha = 1$, and the case $\alpha \in (0,1)$ then follows from the interpolation theorem. To show estimate ii), we first note that for any $h \in C_0^\infty(\mathbf{R})$, obviously

$$\|\chi_n h\|_{W^{s-1,p}} \le C \|\chi_n h\|_{W^{1,p}} \to 0, \text{ when } n \to \infty.$$

Then the estimate (29) follows by using the fact that $C_0^\infty(\mathbf{R})$ is dense in $W^{s-1,p}$ and the estimate (30). This finishes the proof of Theorem 2. ■

In the above proof of Theorem 2, we do not assume that the possible BGK waves to have the form (18) or the electric field to vanish only at finitely many points. So we can exclude any traveling structures which might have the form of a nontrivial wave profile plus a homogeneous part.

The following Lemma shows that the condition $0 < T < T_0$ in Theorem 2 is necessary.

Lemma 5 *Assume $f_0(v) \in C^4(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$ $(p > 1)$. Let $S = \{v_i\}_{i=1}^l$ be the set of all extrema points of f_0 and $0 < T_0 < +\infty$ be defined by*

$$\left(\frac{2\pi}{T_0}\right)^2 = \max_{v_i \in S} \int \frac{f_0'(v)}{v - v_i} dv = \int \frac{f_0'(v)}{v - v_m} dv \tag{31}$$

Then $\exists\, \varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$ there exist nontrivial travelling wave solutions $(f_\varepsilon(x - v_m t, v), E_\varepsilon(x - v_m t))$ to (1), such that $(f_\varepsilon(x, v), E_\varepsilon(x))$ has period T_0 in x, $E_\varepsilon(x)$ not identically zero, and

$$\|f_\varepsilon - f_0\|_{L^1_{x,v}} + \int_0^T \int_{\mathbf{R}} v^2 |f_0 - f_\varepsilon| \; dxdv + \|f_\varepsilon - f_0\|_{W^{2,p}_{x,v}} < \varepsilon \qquad (32)$$

Proof. To simplify notations, we assume $v_m = 0$. As in the proof of Lemma 2, for $\delta_1 > 0$ we define

$$\begin{aligned} f_{\delta_1}(v) &= f_0(v)\left(1 - \sigma\left(\frac{v}{\delta_1}\right)\right) + \left(\frac{f_0(v) + f_0(-v)}{2}\right)\sigma\left(\frac{v}{\delta_1}\right) \\ &= f_0(v) - \left(\frac{f_0(v) - f_0(-v)}{2}\right)\sigma\left(\frac{v}{\delta_1}\right), \end{aligned}$$

where $\sigma(v)$ is the cut-off function defined by (7). Then we have:

$$\text{i)} \quad f_{\delta_1}'(0) = 0, \; \int \frac{f_0'(v)}{v} dv = \int \frac{f_{\delta_1}'(v)}{v} dv = \left(\frac{2\pi}{T_0}\right)^2,$$

and

$$\text{ii)} \quad f_{\delta_1}(v) \in C^4(\mathbf{R}) \cap W^{2,p}(\mathbf{R}); \; \|f_{\delta_1} - f_0\|_{W^{2,p}(\mathbf{R})} \to 0, \text{when} \quad \delta_1 \to 0.$$

Property i) follows since $\sigma(v)$ is even. To prove property ii), we only need to show that $\|\partial_{vv}(f_{\delta_1} - f_0)\|_{L^p(\mathbf{R})} \to 0$ when $\delta_1 \to 0$. Since in the proof of Lemma 2, it is already shown that $\|f_{\delta_1} - f_0\|_{W^{1,p}(\mathbf{R})} \to 0$ when $\delta_1 \to 0$. Note that

$$\begin{aligned} \partial_{vv}(f_{\delta_1} - f_0) &= \frac{1}{2\delta_1^2}\sigma''\left(\frac{v}{\delta_1}\right)(f_0(v) - f_0(-v)) + \frac{1}{\delta_1}\sigma'\left(\frac{v}{\delta_1}\right)(f_0'(v) + f_0'(-v)) \\ &\quad + \frac{1}{2}\sigma\left(\frac{v}{\delta_1}\right)(f_0''(v) - f_0''(-v)) \\ &= I + II + III. \end{aligned}$$

Since

$$\begin{aligned} f_0(v) - f_0(-v) &= \int_{-v}^{v} f_0'(s)\, ds = \int_{-v}^{v}\int_0^s f_0''(\tau)\, d\tau ds \\ &= \int_0^v (v - \tau) f_0''(\tau)\, d\tau + \int_{-v}^0 (-v - \tau) f_0''(\tau)\, d\tau, \end{aligned}$$

and

$$\begin{aligned} & |f_0(v) - f_0(-v)|^p \\ &\le C\left(\left|\int_0^v (v - \tau) f_0''(\tau)\, d\tau\right|^p + \left|\int_{-v}^0 (-v - \tau) f_0''(\tau)\, d\tau\right|^p\right) \\ &\le C\left(\int_0^v |f_0''(\tau)|^p\, d\tau \left(\int_0^v (v - \tau)^{\frac{p}{p-1}}\, d\tau\right)^{p-1} + \int_{-v}^0 |f_0''(\tau)|^p\, d\tau \left(\int_{-v}^0 (v + \tau)^{\frac{p}{p-1}}\, d\tau\right)^{p-1}\right) \\ &\le Cv^{2p-1}\|f_0''\|_{L^p(-v,v)}^p, \end{aligned}$$

so

$$\int_{\mathbf{R}} |I|^p \ dv \leq \frac{C}{\delta_1^{2p}} \int_{\delta_1}^{2\delta_1} |f_0(v) - f_0(-v)|^p \, dv \leq \frac{C}{\delta_1^{2p}} \int_{\delta_1}^{2\delta_1} v^{2p-1} \|f_0''\|_{L^p(-v,v)}^p \, dv$$
$$\leq C \|f_0''\|_{L^p(-2\delta_1, 2\delta_1)}^p .$$

Similarly,

$$\int_{\mathbf{R}} |II|^p \ dv \leq \frac{C}{\delta_1^p} \int_{\delta_1}^{2\delta_1} \left(\left| \int_0^v f_0''(\tau)\, d\tau \right|^p + \left| \int_{-v}^0 f_0''(\tau)\, d\tau \right|^p \right) dv$$
$$\leq \frac{C}{\delta_1^p} \int_{\delta_1}^{2\delta_1} v^{p-1} \|f_0''\|_{L^p(-v,v)}^p \, dv \leq C \|f_0''\|_{L^p(-2\delta_1, 2\delta_1)}^p$$

and

$$\int_{\mathbf{R}} |III|^p \ dv \leq C \|f_0''\|_{L^p(-2\delta_1, 2\delta_1)}^p ,$$

thus when $\delta_1 \to 0$, $\|\partial_{vv}(f_{\delta_1} - f_0)\|_{L^p(\mathbf{R})} \to 0$. Choose $\delta_1 > 0$ such that

$$\|f_{\delta_1} - f_0\|_{W^{2,p}(\mathbf{R})} < \varepsilon/2.$$

Since $f_{\delta_1} \in C^4(\mathbf{R}) \cap W^{2,p}(\mathbf{R})$ and

$$\int \frac{f_{\delta_1}'(v)}{v} dv = \left(\frac{2\pi}{T_0}\right)^2 ,$$

this is exactly the Case 3 treated in the proof of Proposition 1, so we can construct a nontrivial BGK solution $(f_\varepsilon, E_\varepsilon)$ near $(f_{\delta_1}(v), 0)$ satisfying

$$\|f_\varepsilon - f_{\delta_1}\|_{L^1_{x,v}} + \int_0^T \int_{\mathbf{R}} v^2 |f_\varepsilon - f_{\delta_1}| \ dxdv + \|f_\varepsilon - f_{\delta_1}\|_{W^{2,p}_{x,v}} < \frac{\varepsilon}{2}.$$

Thus $(f_\varepsilon, E_\varepsilon)$ is a BGK solution satisfying (32). ■

From the proof of Theorem 2, it is easy to get Corollary 2.

Proof of Corollary 2. Suppose otherwise, then there exists a sequence $\varepsilon_n \to 0$, and homogeneous states $\{f_n(v)\}$ which are linear unstable with $x-$period T and $\|f_n - f_0\|_{W^{s,p}(\mathbf{R})} < \varepsilon_n$. By Lemma 7, for each n, there exists a critical point v_n of $f_n(v)$ such that

$$\int \frac{f_n'(v)}{v - v_n} dv > \left(\frac{2\pi}{T}\right)^2 .$$

Since

$$|f_0'(v_n)| \leq \|\partial_v(f_n - f_0)\|_{C(\mathbf{R})} \leq C \|f_n - f_0\|_{W^{s,p}(\mathbf{R})} \leq C\varepsilon_n,$$

either $\{v_n\}$ converges to one of the critical point of $f_0(v)$, say v_0, or $\{v_n\}$ diverges. As in the proof of Theorem 2, in the first case, we have

$$\int \frac{f_n'(v)}{v - v_n} dv \to \int \frac{f_0'(v)}{v - v_0} dv, \text{ when } n \to \infty.$$

This implies that

$$\int \frac{f_0'(v)}{v-v_0}dv \geq \left(\frac{2\pi}{T}\right)^2 > \left(\frac{2\pi}{T_0}\right)^2,$$

a contradiction. For the second case, we have

$$\int \frac{f_n'(v)}{v-v_n}dv \to 0, \text{when } n \to \infty,$$

a contradiction again. ■

4 Linear damping

In this section, we study in details the linear damping problem in Sobolev spaces. First, the linear decay estimates derived here are used in Section 5 to show that all invariant structures in H^s $\left(s > \frac{3}{2}\right)$ neighborhood of stable homogeneous states are trivial. Second, the linear decay holds true for initial data as rough as $f(t=0) \in L^2$, and this suggests that Theorems 1, 2 and 3 about nonlinear dynamics have no analogues at the linear level. We refer to Remark 5 for more discussions.

The linearized Vlasov-Poisson around a homogeneous state $(f_0(v), 0)$ is the following

$$\begin{cases} \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f_0}{\partial v} = 0, \\ \frac{\partial E}{\partial x} = -\int_{-\infty}^{+\infty} f \, dv, \end{cases} \tag{33}$$

where f and E are $T-$periodic in x and the neutralizing condition becomes $\int_0^T \int_{\mathbf{R}} f \, dvdx = 0$. Notice that any $(f, E) = (g(v), 0)$ with $\int g(v)\, dv = 0$ is a steady solution of the linear system (33). For a general solution (f, E) of (33), the homogeneous component of f remains steady and does not affect the evolution of E. So we can consider a function $h(x, v)$ which is $T-$periodic in x and $\int_0^T h(x, v)\, dx = 0$. Denote its Fourier series representation by

$$h(x, v) = \sum_{0 \neq k \in \mathbf{Z}} e^{i\frac{2\pi}{T}kx} h_k(v).$$

We define the space $H_x^{s_x} H_v^{s_v}$ by

$$h \in H_x^{s_x} H_v^{s_v} \text{ if } \|h\|_{H_x^{s_x} H_v^{s_v}} = \left(\sum_{k \neq 0} |k|^{2s_x} \|h_k\|_{H_v^{s_v}}^2\right)^{\frac{1}{2}} < \infty.$$

Proposition 2 *Assume $f_0(v) \in H^{s_0}(\mathbf{R})$ $\left(s_0 > \frac{3}{2}\right)$ and let $0 < T_0 \leq +\infty$ be defined by (3). Let $(f(x, v, t), E(x, t))$ be a solution of (33) with $x-$period $T < T_0$ and $g(x, v) = f(x, v, 0) - \frac{1}{T}\int_0^T f(x, v, 0)\, dx$. If $g \in H_x^{s_x} H_v^{s_v}$ with $|s_v| \leq s_0 - 1$, then*

$$\left\|t^{s_v} E(x, t)\right\|_{L_t^2 H_x^{\frac{3}{2}+s_x+s_v}} \leq C_0 \|g\|_{H_x^{s_x} H_v^{s_v}} \leq C_0 \|f(x, v, 0)\|_{H_x^{s_x} H_v^{s_v}}, \tag{34}$$

for some constant C_0.

One may compare this proposition with other smoothing estimates in PDEs. Here based on the most naive estimate, the initial value $g \in H_x^{s_x} H_v^{s_v} \subset H_{x,v}^{s_x+s_v}$ only implies $E(0) \in H_x^{s_x+1}$ which is much weaker than $H_x^{\frac{3}{2}+s_v+s_x}$ in the above proposition. However, this improved regularity of E may blow up as $t \to 0$.

Proof. To simplify notations, we assume $T = 2\pi$. Let

$$g(x,v) = \sum_{0 \neq k \in \mathbf{Z}} e^{ikx} g_k(v),$$

then by assumption

$$\|g\|_{H_x^{s_x} H_v^{s_v}} = \sum_{0 \neq k \in \mathbf{Z}} |k|^{2s_x} \|g_k\|_{H^{s_v}}^2 < \infty.$$

Let

$$f(x,v,t) = \sum_{k \neq 0} e^{ikx} h_k(v,t), \ E(x,t) = \sum_{k \neq 0 \in \mathbf{Z}} e^{ikx} E_k(t),$$

then

$$E_k(t) = -\frac{1}{ik} \int_{\mathbf{R}} h_k(v,t) \ dv, \ E_k(0) = -\frac{1}{ik} \int_{\mathbf{R}} g_k(v) \, dv.$$

Below we denote C to be a generic constant depending only on f_0.When $k > 0$, we use the the well-known formula for $E_k(t)$

$$E_k(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{G_k(-p/ik)}{k^2 - F(-p/ik)} e^{pt} dp, \tag{35}$$

where

$$G_k(z) = \int_{-\infty}^{+\infty} \frac{g_k(v)}{v - z} dv, \ F(z) = \int_{-\infty}^{+\infty} \frac{f_0'(v)}{v - z} dv, \ \operatorname{Im} z > 0$$

and σ is chosen so that the integrand in (35) has no poles for $\operatorname{Re} p > \sigma$. The formula (35) was derived in Landau's original 1946 paper ([29]) by using Laplace transforms. Here we follow the notations in ([48]). By using the new variable $z = -p/ik$, we get

$$E_k(t) = \frac{k}{2\pi} \int_{\frac{i\sigma}{k} - \infty}^{\frac{i\sigma}{k} + \infty} \frac{G_k(z)}{k^2 - F(z)} e^{-ikzt} dz. \tag{36}$$

By assumption $k \geq 1 = \frac{2\pi}{T} > \frac{2\pi}{T_0}$, so by Penrose's criterion (Lemma 7), there exist no unstable modes to the linearized equation with $x-$period $2\pi/k$. Therefore, $k^2 - F(z) \neq 0$ when $\operatorname{Im} z > 0$. Moreover, by the proof of Lemma 7, under the condition $k > \frac{2\pi}{T_0}$, $k^2 - F(x + i0) \neq 0$ for any $x \in \mathbf{R}$. It is also easy to see that $F(x + i0) \to 0$ when $x \to \infty$. So there exists $c_0 > 0$, such that

$$\left|k^2 - F(x + i0)\right| \geq c_0 k^2, \ \text{for any } x \in \mathbf{R} \text{ and } k. \tag{37}$$

Note that for $z = i\sigma + x$, when $\sigma \to 0+$, by (50),

$$G_k(z) \to G_k(x+i0) = P\int_{\mathbf{R}} \frac{g_k(v)}{v-x}dv + i\pi g_k(x) = \mathcal{H}g_k + i\pi g_k,$$

and

$$F(z) \to F(x+i0) = P\int_{\mathbf{R}} \frac{f_0'(v)}{v-x}dv + i\pi f_0'(x) = \mathcal{H}f_0' + i\pi f_0',$$

where $\mathcal{H}$ is the Hilbert transform. So letting $\sigma \to 0+$, from (36), we have

$$E_k(t) = \frac{k}{2\pi}\int_{\mathbf{R}} \frac{G_k(x+i0)}{k^2 - F(x+i0)} e^{-ikxt}dx. \tag{38}$$

Let

$$A_k(t) = \frac{1}{2\pi}\int_{\mathbf{R}} \frac{G_k(x+i0)}{k^2 - F(x+i0)} e^{-ixt}dx$$

be the Fourier transform of

$$H_k(x) = \frac{G_k(x+i0)}{k^2 - F(x+i0)},$$

then $E_k(t) = kA_k(kt)$. Since $\mathcal{H} : H^s \to H^s$ is bounded for any $s \in \mathbf{R}$,

$$\|G_k(x+i0)\|_{H^{s_v}} \leq C\|g_k\|_{H_v^{s_v}}, \qquad |F(x+i0)\|_{H^{s_0-1}} \leq C\|f_0\|_{H_v^{s_0}}.$$

By (37) and the inequality

$$\|f_1 f_2\|_{H^s} \leq C_{s,s_1}\|f_1\|_{H^{s_1}}\|f_2\|_{H^s}, \text{ if } s_1 > \frac{1}{2}, \ |s| \leq s_1,$$

we have

$$\begin{aligned}\|H_k\|_{H^{s_v}} &\leq \frac{1}{k^2}\|G_k(x+i0)\|_{H^{s_v}} + \frac{1}{k^4}\left\|G_k(x+i0)\frac{F(x+i0)}{1-F(x+i0)/k^2}\right\|_{H^{s_v}} \\ &\leq \frac{C}{k^2}\|g_k\|_{H_v^{s_v}}\left(1 + \frac{1}{k^2}\left\|\frac{F(x+i0)}{1-F(x+i0)/k^2}\right\|_{H^{s_0-1}}\right) \leq \frac{C'}{k^2}\|g_k\|_{H_v^{s_v}}.\end{aligned}$$

where C' depends on f_0 but not k. In the above, the second inequality holds since the estimates

$$\left|1 - F(x+i0)/k^2\right| \geq c_0 \text{ and } \|F(x+i0)\|_{H^{s_0-1}} \leq C\|f_0\|_{H^{s_0}},$$

imply

$$\left\|\frac{F(x+i0)}{1-F(x+i0)/k^2}\right\|_{H^{s_0-1}} < C\|f_0\|_{H^{s_0}} \tag{39}$$

through direct verification where, for $0 < s_0 - 1 < 1$, one needs to use the equivalent characterization of $W^{s,p}(\mathbf{R}^n)$ when $0 < s < 1$, $p > 1$ (See [45, Lemma 35.2]):

$$W^{s,p}(\mathbf{R}^n) = \left\{u \in L^p(\mathbf{R}^n) \ \middle| \ \int\int_{\mathbf{R}^n \times \mathbf{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}dxdy < \infty\right\}.$$

So

$$\int_{\mathbf{R}} |t|^{2s_v} |A_k(t)|^2 \, dt \le \|H_k\|^2_{H^{s_v}} \le \frac{C}{k^4} \|g_k\|^2_{H_v^{s_v}}$$

and

$$\begin{aligned} \|t^{s_v} E_k(t)\|^2_{L^2} &= \int_{\mathbf{R}} |t|^{2s_v} |E_k(t)|^2 \, dt = \int |t|^{2s_v} k^2 |A_k(kt)|^2 \, dt \\ &= k^{1-2s_v} \int_{\mathbf{R}} |t|^{2s_v} |A_k(t)|^2 \, dt \le C k^{-3-2s_v} \|g_k\|^2_{H_v^{s_v}} . \end{aligned}$$

For $k < 0$, the same estimate

$$\|t^{s_v} E_k(t)\|^2_{L^2} \le C |k|^{-3-2s_v} \|g_k\|^2_{H_v^{s_v}} ,$$

follows by taking the complex conjugate of the $k > 0$ case. Thus

$$\begin{aligned} \|t^{s_v} E(x,t)\|^2_{L_t^2 H_x^{\frac{3}{2}+s_x+s_v}} &= \sum_{k \neq 0} |k|^{3+2s_v+2s_x} \|t^{s_v} E_k(t)\|^2_{L^2} \\ &\le C \sum_{k\neq 0} |k|^{2s_x} \|g_k\|^2_{H_v^{s_v}} = C \|g\|^2_{H_x^{s_x} H_v^{s_v}} . \end{aligned}$$

This finishes the proof. ■

The decay estimate in Proposition 4 is in the integral form. With some additional assumption on the initial data, we can obtain the pointwise decay estimate.

Proposition 3 *Assume $f_0(v) \in H^{s_0}(\mathbf{R})$ $\left(s_0 > \frac{3}{2}\right)$ and let $0 < T_0 \le +\infty$ be defined by (3). Let $(f(x,v,t), E(x,t))$ be a solution of (33) with $x-$period $T < T_0$ and*

$$g(x,v) = f(x,v,0) - \frac{1}{T} \int_0^T f(x,v,0) \, dx.$$

If $g \in H_x^{s_x} H_v^{s_v}$, $vg \in H_x^{s'_x} H_v^{s'_v}$ with $s_v > -\frac{1}{2}$, $s_v + s'_v \ge 0$ and $\max\{|s_v|, |s'_v|\} \le s_0 - 1$, then

$$\|E\|_{H^s}(t) = o\left(t^{-\frac{s_v+s'_v}{2}}\right), \text{ when } t \to \infty,$$

where

$$s = \min\left\{\frac{3}{2} + s_x + s_v, \frac{1}{2} + s'_x + s'_v\right\}. \tag{40}$$

Corollary 3 *Assume $f_0(v) \in H^{s_0}(\mathbf{R})$ $\left(s_0 > \frac{3}{2}\right)$ and $T < T_0$.*

(i) If $g \in H_x^{-\frac{3}{2}} L_v^2$ and $vg \in H_x^{-\frac{1}{2}} L_v^2$, then $\|E\|_{L_x^2}(t) \to 0$ when $t \to \infty$.

(ii) If $g, vg \in H_{x,v}^k$ with $k \le s_0 - 1$, then $\|E\|_{H^{k+\frac{1}{2}}}(t) = o\left(t^{-k}\right)$ when $t \to \infty$.

Proposition 3 and its Corollary shows that linear damping is true for initial data of very low regularity, even in certain negative Sobolev spaces. It also shows that the decay rate is mainly determined by the regularity in v, although the regularity in x affects the norm of electrical field that decays.

Proof of Proposition 3. First we derive a formula for $E_t(t)$. We notice that (f_t, E_t) satisfies the linear system (33) and

$$\begin{aligned} f_t(x,v,0) &= -v\partial_x f(x,v,0) + E(x,0) f_0'(v) \\ &= \sum_{k\neq 0} e^{ikx}\left(-ikvg_k(v) - \frac{1}{ik}\int_{\mathbf{R}} g_k(v)\,dv\ f_0'(v)\right) = \sum_{j=1}^{3} \tilde{g}^j(x,v), \end{aligned}$$

where

$$\begin{aligned} \tilde{g}^1(x,v) &= -\partial_x(vg(x,v)), \\ \tilde{g}^2(x,v) &= -\frac{1}{ik}\sum_{k\neq 0} e^{ikx}\int_{\mathbf{R}} g_k(v)\,\sigma(v)\,dv\ f_0'(v) = \sum_{k\neq 0} e^{ikx}\tilde{g}_k^2(v), \\ \tilde{g}^3(x,v) &= -\frac{1}{ik}\sum_{k\neq 0} e^{ikx}\int_{\mathbf{R}} g_k(v)(1-\sigma(v))\,dv\ f_0'(v) = \sum_{k\neq 0} e^{ikx}\tilde{g}_k^3(v), \end{aligned}$$

and $\sigma(v)$ is the cut-off function defined by (7). Then $\tilde{g}_1 \in H_x^{s_x'-1}H_v^{s_v'}$, $\tilde{g}_2 \in H_x^{s_x+1}H_v^{s_0-1}$, $\tilde{g}_3 \in H_x^{s_x'+1}H_v^{s_0-1}$, and

$$\|\tilde{g}_1\|_{H_x^{s_x'-1}H_v^{s_v'}} = \|vg\|_{H_x^{s_x'}H_v^{s_v'}}$$

$$\begin{aligned} \|\tilde{g}_2\|^2_{H_x^{s_x+1}H_v^{s_0-1}} &= \sum_k |k|^{2s_x}\left\|\int_{\mathbf{R}} g_k(v)\,\sigma(v)\,dv\ f_0'\right\|^2_{H_v^{s_0-1}} \\ &\leq \sum_k |k|^{2s_x}\|g_k\|^2_{H_v^{s_v}}\|\sigma(v)\|^2_{H_v^{-s_v}}\|f_0\|^2_{H_v^{s_0}} \leq C\|g\|^2_{H_x^{s_x}H_v^{s_v}}, \end{aligned}$$

$$\|\tilde{g}_3\|^2_{H_x^{s_x'+1}H_v^{s_0-1}} \leq \sum_k |k|^{2s_x'}\|vg_k\|^2_{H_v^{s_v'}}\left\|\frac{1-\sigma(v)}{v}\right\|^2_{H_v^{-s_v'}}\|f_0\|^2_{H_v^{s_0}} \leq C\|vg\|^2_{H_x^{s_x'}H_v^{s_v'}}.$$

Correspondingly, we decompose

$$(f_t, E_t) = \sum_{i=1}^{3}\left(f_t^i, E_t^i\right)$$

with $\left(f_t^i, E_t^i\right)$ being the solution of (33) with initial data $f_t^i(t=0) = \tilde{g}_i(x,v)$. Then by Proposition we have

$$\left\|t^{s_v'}E_t^1\right\|_{L_t^2H_x^{\frac{1}{2}+s_x'+s_v'}} \leq C\|vg\|_{H_x^{s_x'}H_v^{s_v'}}, \quad \left\|t^{s_0-1}E_t^2\right\|_{L_t^2H_x^{\frac{5}{2}+s_x+s_0-1}} \leq C\|g\|_{H_x^{s_x}H_v^{s_v}}$$

and

$$\left\|t^{s_0-1}E_t^3\right\|_{L_t^2H_x^{\frac{5}{2}+s_x'+s_0-1}} \leq C\|vg\|_{H_x^{s_x'}H_v^{s_v'}}.$$

For any $t_2 > t_1$ sufficiently large and s defined by (40), we have

$$\begin{aligned}
&\left|\|E\|_{H_x^s}^2(t_2) - \|E\|_{H_x^s}^2(t_1)\right| \\
&= \left|\int_{t_1}^{t_2} \langle E(t), E_t(t)\rangle_{H_x^s} dt\right| \le \int_{t_1}^{t_2} \|E(t)\|_{H_x^s} \left(\sum_{i=1}^{3} \left\|E_t^i(t)\right\|_{H_x^s}\right) dt \\
&\le t_1^{-s_v - s_v'} \int_{t_1}^{t_2} \|t^{s_v} E(t)\|_{H_x^s} \left\|t^{s_v'} E_t^1\right\|_{H_x^{\frac{1}{2}+s_x'+s_v'}} dt \\
&\qquad + t_1^{-s_v-(s_0-1)} \int_{t_1}^{t_2} \|t^{s_v} E(t)\|_{H_x^s} \left\|t^{s_0-1} E_t^2\right\|_{H_x^{\frac{5}{2}+s_x+s_0-1}} dt \\
&\qquad + t_1^{-s_v-(s_0-1)} \int_{t_1}^{t_2} \|t^{s_v} E(t)\|_{H_x^s} \left\|t^{s_0-1} E_t^3\right\| dt \\
&\le t_1^{-s_v - s_v'} \|t^{s_v} E(x,t)\|_{L_t^2(t_1,t_2) H_x^{\frac{3}{2}+s_x+s_v}} \\
&\cdot \left(\left\|t^{s_v'} E_t^1\right\|_{L_t^2(t_1,t_2) H_x^{\frac{1}{2}+s_x'+s_v'}} + \left\|t^{s_0-1} E_t^2\right\|_{L_t^2(t_1,t_2) H_x^{\frac{5}{2}+s_x+s_0-1}} + \left\|t^{s_0-1} E_t^3\right\|_{L_t^2(t_1,t_2) H_x^{\frac{5}{2}+s_x'+s_0-1}}\right).
\end{aligned}$$

So $\left\{\|E\|_{H_x^s}^2(t)\right\}_{t\ge 0}$ is a Cauchy sequence, thus $\lim_{t\to\infty} \|E\|_{H_x^s}^2(t)$ exists and must be zero since $\|t^{s_v} E\|_{L_t^2 H_x^s}^2 < \infty$ with $s_v > -\frac{1}{2}$. By fixing t_1 and letting $t_2 \to \infty$ in the above computation, it follows that

$$\|E\|_{H_x^s}^2(t_1) = o\left(t_1^{-s_v - s_v'}\right).$$

This finishes the proof. ■

Remark 3 *The integral decay estimate in Proposition 4 is optimal and the pointwise decay estimate in Proposition 3 is close to be optimal. Intuitively, the integral estimate (34) suggests that*

$$\|E(x,t)\|_{H_x^{\frac{3}{2}+s_x+s_v}} = o\left(t^{-\left(s_v+\frac{1}{2}\right)}\right). \tag{41}$$

In [48], the single-mode solution $e^{ikx}(f(v,t), E(t))$ with initial profile

$$f(v,0) = g(v) = \begin{cases} (v-\alpha)^2 e^{-(v-\alpha)^2} & v \ge \alpha \\ 0 & v \le \alpha \end{cases}, \ \alpha \text{ is arbitrary constant,}$$

was calculated explicitly for the linearized problem at Maxwellian, and the decay rate for $|E(t)|$ was found to be $O\left(t^{-3}\right)$. Note that $g(v), vg \in H^2$ and $g''', (vg)'''$ are delta functions which belong to $H^{-\left(\frac{1}{2}+\varepsilon\right)}$ for any $\varepsilon > 0$, and thus $g(v), vg \in H^{\frac{5}{2}-\varepsilon}$. So Proposition 4 suggests a decay rate $o\left(t^{-(3-\varepsilon)}\right)$ in the integral form and Corollary 3 (ii) yields a pointwise decay rate $o\left(t^{-\frac{5}{2}+\varepsilon}\right)$. In [2, pp. 188-189], the authors made a more general claim about the decay rate of single mode *solutions: for initial profile* $g(v)$ *with* $(n+1)-th$ *derivative being* $\delta-$*function like, the decay rate of* $|E(t)|$ *is* $O\left(t^{-(n+1)}\right)$. *In such cases, our results give the decay rates* $o\left(t^{-(n-\varepsilon)}\right)$ *in the integral form and* $o\left(t^{-\left(n+\frac{1}{2}-\varepsilon\right)}\right)$ *pointwise.*

In Theorem 3, we use the integral estimate (34) to prove that $H^{\frac{3}{2}}$ *is the critical regularity for existence or nonexistence of nontrivial invariant structures near stable homogeneous states. This again suggests that the decay estimate in Proposition 4 is optimal.*

Remark 4 *The linear decay result is also true for initial data in* L^p *space. For simplicity, we consider a single mode solution*

$$(f(x,v,t), E(x,t)) = e^{ikx}(h(v,t), E(t)) \tag{42}$$

to (33) with $h(v,0) = g(v)$. *Assume* $f_0(v) \in L^1(\mathbf{R}) \cap W^{2,p_0}(\mathbf{R})$ $(p_0 > 1)$ *and* $0 < T_0 \leq +\infty$ *be defined by (3). We have the following result: If* $T = \frac{2\pi}{k} < T_0$ *and* $g(v) \in L^p$ $(p > 1)$, $v^2 g \in L^1$, *then* $|E(t)| \to 0$ *when* $t \to +\infty$. *We prove it briefly below. Since* $g \in L^p, v^2 g \in L^1$, *so*

$$\|g\|_{L^1(\mathbf{R})} \leq \int_{|v|\leq 1} |g| \, dv + \int_{|v|\geq 1} |g| \, dv \leq 2^{1/p'} \|g\|_{L^p} + \left\|v^2 g\right\|_{L^1} < \infty,$$

and

$$\|vg\|_{L^q} \leq \left\|v\,|g|^{\frac{1}{2}}\right\|_{L^2} \left\||g|^{\frac{1}{2}}\right\|_{L^{2p}} \leq \left\|v^2 g\right\|_{L^1}^{\frac{1}{2}} \|g\|_{L^p}^{\frac{1}{2}},$$

for $1 < q < 2$ *satisfying* $\frac{1}{q} = \frac{1}{2} + \frac{1}{2p}$. *Since* $q < p$, *for any* $1 < q_1 < q$, *letting* $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{q}$, *we have*

$$\begin{aligned} \|g\|_{L^{q_1}(\mathbf{R})} &\leq \left(\|g\|_{L^{q_1}(|v|\leq 1)} + \|g\|_{L^{q_1}(|v|\geq 1)}\right) \\ &\leq C\left(\|g\|_{L^p} + \left\|\frac{1}{v}\right\|_{L^{q_2}(|v|\geq 1)} \|vg\|_{L^q}\right) < \infty. \end{aligned}$$

Since $\mathcal{H}$ *is bounded* $L^p \to L^p$ *for any* $p > 1$ *and the Fourier transform is bounded* $L^p \to L^{p'}$ *for any* $1 < p \leq 2$, *so from (38),*

$$\|E(t)\|_{L^{q_1 \prime}} \leq C \|g\|_{L^{q_1}(\mathbf{R})} < \infty. \tag{43}$$

As in the proof of Proposition 3, (f_t, E_t) *satisfies (33)with*

$$f_t(t=0) = e^{ikx}\left(-ikvg(v) - \frac{1}{ik}\int_{\mathbf{R}} g(v)\,dv\; f_0'(v)\right) = e^{ikx}\tilde{g}(v).$$

Since

$$\|\tilde{g}(v)\|_{L^q} \leq C\left(\|vg\|_{L^q} + \|g\|_{L^1(\mathbf{R})} \|f_0'\|_{L^q}\right) < \infty,$$

by using the estimate for $E(t)$, *we get*

$$\|E'(t)\|_{L^{q'}} \leq C \|\tilde{g}(v)\|_{L^q} < \infty. \tag{44}$$

The decay of $|E(t)|$ *follows from the estimates (43) and (44).*

Remark 5 *In Proposition 3, we prove that the linear decay of electrical field E in L^2 norm holds true for initial data as rough as*

$$f(t=0) \in H_x^{-\frac{3}{2}} L_v^2,\ vf(t=0) \in H_x^{-\frac{1}{2}} L_v^2.$$

In particular, it is not necessary to have any assumption on derivatives of $f(t=0)$ to get linear decay of E. The linear decay result implies that there exist no nontrivial invariant structures even in $H_x^{-\frac{3}{2}} L_v^2$ space for the linearized problem. So our result on existence of BGK waves in $W^{s,p}$ $\left(s < 1 + \frac{1}{p}\right)$ neighborhood (Theorem 1) can not be traced back to the linearized level. Also, the contrasting nonlinear dynamics in $W^{s,p}$ $\left(s > 1 + \frac{1}{p}\right)$ and particularly in H^s $\left(s > \frac{3}{2}\right)$ spaces (Theorems 2 and 3) have no analogue on the linearized level. These again are due to the fact that particle trapping effects are completely ignored on the linear level, but instead they play an important role on nonlinear dynamics.

5 Invariant structures in H^s $\left(s > \frac{3}{2}\right)$

We define invariant structures near a homogeneous state $(f_0(v), 0)$ in $H_{x,v}^s$ $(s \geq 0)$ space to be the solutions $(f(t), E(t))$ of nonlinear VP equation (1a)-(1b), satisfying that for all $t \in \mathbf{R}$,

$$\|f(t) - f_0\|_{H^s((0,T)\times\mathbf{R})} < \varepsilon_0,$$

for some constant $\varepsilon_0 > 0$. The above defined invariant structures include the well known structures such as travelling waves, time-periodic, quasi-periodic or almost periodic solutions. In Sections 2 and 3, we prove that $W^{1+\frac{1}{p},p}$ is the critical regularity for existence of nontrivial travelling waves near a stable homogeneous state. For $p = 2$, this critical regularity is $H^{\frac{3}{2}}$. In this section, we prove a much stronger result that $H^{\frac{3}{2}}$ is also the critical regularity for existence of any nontrivial invariant structure near a stable homogeneous state. In the proof, we use the linear decay estimate in Proposition 4.

Lemma 6 *Assume $f_0(v) \in H^{s_0}(\mathbf{R})$ $\left(s_0 > \frac{3}{2}\right)$ and let $0 < T_0 \leq +\infty$ be defined by (3). Let $(f(x,v,t), E(x,t))$ be a solution of (1a)-(1b) with $x-$period $T < T_0$, satisfying that: For some $\frac{3}{2} < s \leq s_0$ and sufficiently small ε_0,*

$$\|f(t) - f_0\|_{L_x^2 H_v^s((0,T)\times\mathbf{R})} < \varepsilon_0,\ \textit{for all } t \geq 0.$$

Then

$$\left\|(1+t)^{s-1} E(x,t)\right\|_{L^2_{\{t\geq 0\}} H_x^{\frac{3}{2}}} \leq C\varepsilon_0, \tag{45}$$

for some constant C.

Proof. Denote L_0 to be the linearized operator corresponding to the linearized Vlasov-Poisson equation at $(f_0(v), 0)$, and $\mathcal{E}$ is the mapping from $f(x,v)$

to $E(x)$ by the Poisson equation

$$E_x = -\int f\ dv,$$

where f satisfies the neutral condition $\int_0^T \int_{\mathbf{R}} f(x,v)\, dvdx = 0$. It follows from Proposition 4 that: For any $0 \leq s_v \leq s_0 - 1$, if $h(x,v) \in L_x^2 H_v^{s_v}$,then

$$\left\| (1+t)^{s_v} \mathcal{E}\left(e^{tL_0} h\right) \right\|_{L_t^2 H_x^{\frac{3}{2}}} \leq C \left\| h(x,v) \right\|_{L_x^2 H_v^{s_v}}. \tag{46}$$

Denote $f_1(t) = f(t) - f_0$, then

$$\partial_t f_1 = L_0 f_1 + E \partial_v f_1.$$

Thus

$$f_1(t) = e^{tL_0} f_1(0) + \int_0^t e^{(t-u)L_0} \left(E\partial_v f_1\right)(u)\, du = f_{\text{lin}}(t) + f_{\text{non}}(t),$$

and correspondingly

$$E(t) = \mathcal{E}\left(f_{\text{lin}}(t)\right) + \mathcal{E}\left(f_{\text{non}}(t)\right) = E_{\text{lin}}(t) + E_{\text{non}}(t).$$

By the linear estimate (46),

$$\left\| (1+t)^{s-1} E_{\text{lin}}(x,t) \right\|_{L^2_{\{t\geq 0\}} H_x^{\frac{3}{2}}} \leq C \left\| f_1(0) \right\|_{L_x^2 H_v^{s-1}},$$

and

$$\begin{aligned}
&\left\| (1+t)^{s-1} E_{\text{non}}(x,t) \right\|^2_{L^2_{\{t\geq 0\}} H_x^{\frac{3}{2}}} \\
&= \int_0^\infty (1+t)^{2(s-1)} \left\| E_{\text{non}}(x,t) \right\|^2_{H_x^{\frac{3}{2}}} dt \\
&\leq \int_0^\infty (1+t)^{2(s-1)} \left(\int_0^t \left\| \mathcal{E}\left[e^{(t-u)L_0} \left(E\partial_v f_1\right)(u) \right] \right\|_{H_x^{\frac{3}{2}}} du \right)^2 dt \\
&\leq \int_0^\infty (1+t)^{2(s-1)} \int_0^t (1+(t-u))^{-2(s-1)} (1+u)^{-2(s-1)}\, du \\
&\qquad \cdot \int_0^t (1+u)^{2(s-1)} (1+(t-u))^{2(s-1)} \left\| \mathcal{E}\left[e^{(t-u)L_0} \left(E\partial_v f_1\right)(u) \right] \right\|^2_{H_x^{\frac{3}{2}}}\ dudt \\
&\leq C \int_0^\infty \int_0^t (1+u)^{2(s-1)} (1+(t-u))^{2(s-1)} \left\| \mathcal{E}\left[e^{(t-u)L_0} \left(E\partial_v f_1\right)(u) \right] \right\|^2_{H_x^{\frac{3}{2}}}\ dudt \\
&= C \int_0^\infty (1+u)^{2(s-1)} \int_u^\infty (1+(t-u))^{2(s-1)} \left\| \mathcal{E}\left[e^{(t-u)L_0} \left(E\partial_v f_1\right)(u) \right] \right\|^2_{H_x^{\frac{3}{2}}}\ dtdu \\
&\leq C \int_0^\infty (1+u)^{2(s-1)} \left\| \left(E\partial_v f_1\right)(u) \right\|^2_{L_x^2 H_v^{s-1}} du \\
&\leq C \int_0^\infty (1+u)^{2(s-1)} \left\| E(u) \right\|^2_{H_x^{\frac{3}{2}}} \left\| f_1(u) \right\|^2_{L_x^2 H_v^s} du \\
&\leq C \varepsilon_0^2 \left\| (1+t)^{s-1} E(x,t) \right\|^2_{L^2_{\{t\geq 0\}} H_x^{\frac{3}{2}}}.
\end{aligned}$$

In the above estimate, we use the fact that

$$\int_0^t (1+(t-u))^{-2(s-1)} (1+u)^{-2(s-1)}\, du \le C\,(1+t)^{-2(s-1)}$$

because $2(s-1) > 1$ by our assumption that $s > \frac{3}{2}$, and the inequality

$$\|E\partial_v f_1\|_{L_x^2 H_v^{s-1}} \le C\,\|E\|_{H_x^{\frac{3}{2}}}\,\|f_1\|_{L_x^2 H_v^s}.$$

Thus

$$\begin{aligned}
&\left\|(1+t)^{s-1} E(x,t)\right\|_{L^2_{\{t\ge 0\}} H_x^{\frac{3}{2}}} \\
&\le \left\|(1+t)^{s-1} E_{\mathrm{lin}}(x,t)\right\|_{L^2_{\{t\ge 0\}} H_x^{\frac{3}{2}}} + \left\|(1+t)^{s-1} E_{\mathrm{non}}(x,t)\right\|_{L^2_{\{t\ge 0\}} H_x^{\frac{3}{2}}} \\
&\le C\,\|f_1(0)\|_{L_x^2 H_v^s} + C\varepsilon_0 \left\|(1+t)^{s-1} E(x,t)\right\|_{L^2_{\{t\ge 0\}} H_x^{\frac{3}{2}}}.
\end{aligned}$$

By taking $\varepsilon_0 = \frac{1}{2C}$, we get the estimate (45). ■

Proof of Theorem 3. For any $t_0 > 0$, let $\left(\tilde f(t), \tilde E(t)\right)$ be the solution of nonlinear VP equation (1a)-(1b) with the initial data

$$\left(\tilde f(0), \tilde E(0)\right) = \left(f(-t_0), E(-t_0)\right).$$

Then

$$\left(f(t), E(t)\right) = \left(\tilde f(t+t_0), \tilde E(t+t_0)\right).$$

The assumption (4) implies that

$$\left\|\tilde f(t) - f_0\right\|_{L_x^2 H_v^s} < \varepsilon_0, \text{ for all } t \in \mathbf{R}.$$

Thus by Lemma 6,

$$\left\|(1+t)^{s-1} \tilde E(x,t)\right\|_{L^2_{\{t\ge 0\}} H_x^{\frac{3}{2}}} \le C\varepsilon_0.$$

So

$$\begin{aligned}
\int_0^1 \|E(x,t)\|^2_{H_x^{\frac{3}{2}}}\, dt &= \int_{t_0}^{t_0+1} \left\|\tilde E(x,t)\right\|^2_{H_x^{\frac{3}{2}}}\, dt \\
&\le \frac{1}{(1+t_0)^{2(s-1)}} \int_{t_0}^{t_0+1} (1+t)^{2(s-1)} \left\|\tilde E(x,t)\right\|^2_{H_x^{\frac{3}{2}}}\, dt \\
&\le \frac{(C\varepsilon_0)^2}{(1+t_0)^{2(s-1)}}.
\end{aligned}$$

Since t_0 can be arbitrarily large, we have

$$\int_0^1 \|E(x,t)\|^2_{H_x^{\frac{3}{2}}}\, dt = 0$$

and thus $E(x,t) \equiv 0$ when $t \in [0,1]$. Repeating the above argument for any finite time interval $I \subset \mathbf{R}$, we get $E(x,t) \equiv 0$ when $t \in I$. Thus $E(x,t) \equiv 0$ for any $t \in \mathbf{R}$. ■

The following nonlinear instability result follows immediately from Theorem 3.

Corollary 4 *Assume the homogeneous profile $f_0(v) \in H^s(\mathbf{R})$ $\left(s > \frac{3}{2}\right)$. For any $T < T_0$ (defined by (3)), there exists $\varepsilon_0 > 0$, such that for any solution $(f(t), E(t))$ to the nonlinear VP equation (1a)-(1b) with nonzero $E(0)$, there exists $T \in \mathbf{R}$ such that $\|f(T) - f_0\|_{L_x^2 H_v^s} \geq \varepsilon_0$.*

The invariant structures studied in Theorem 3 stay in the $L_x^2 H_v^s$ $\left(s > \frac{3}{2}\right)$ neighborhood of a stable homogeneous state $(f_0(v), 0)$ for all time $t \in \mathbf{R}$. We can also study the positive (or negative) invariant structures near $(f_0(v), 0)$, which are solutions $(f(t), E(t))$ to nonlinear VP equation satisfying that $\|f(t) - f_0\|_{L_x^2 H_v^s} < \varepsilon_0$, for all $t \geq 0$ (or $t \leq 0$). The next theorem shows that the electric field of these semi-invarint structures must decay when $t \to +\infty$ (or $t \to -\infty$).

Theorem 4 *Assume the homogeneous profile $f_0(v) \in H^s(\mathbf{R})$ $\left(s > \frac{3}{2}\right)$. For any $T < T_0$ (defined by (3)), there exists $\varepsilon_0 > 0$ sufficiently small, such that if*

$$\|f(t) - f_0\|_{L_x^2 H_v^s} < \varepsilon_0, \text{ for all } t \geq 0 \ \ (\text{or } t \leq 0),$$

and

$$\|f(0)\|_{L^\infty_{x,v}} < \infty, \ \int_0^T \int_{\mathbf{R}} v^2 f(0,x,v)\, dv dx < \infty,$$

then $\|E(t,x)\|_{L_x^2} \to 0$ when $t \to +\infty$ $(\text{or } t \to -\infty)$.

Proof. We only consider the positive invariant case, since the proof is the same for the negative invariant case. First, there exists a constant C depending on $M_1 = \|f(0)\|_{L^\infty}$ and $M_2 = \int_0^T \int_{\mathbf{R}} \frac{1}{2} v^2 f(0)\, dv dx$, such that

$$\|E(x,t)\|_{H_x^1} \leq C, \text{ for all } t.$$

Indeed, by the same estimate as in (25),

$$\begin{aligned}\|\rho(x,0)\|_{L^3} &= \left\| \int f(x,v,0)\, dv \right\|_{L^3} \leq \|f(0)\|_{L^\infty}^{\frac{2}{3}} \left(\int_0^T \int_{\mathbf{R}} v^2 f(x,v,0)\, dv dx \right)^{1/3} \\ &= M_1^{\frac{2}{3}} M_2^{\frac{1}{3}}\end{aligned}$$

and

$$\|E(x,0)\|_{H^1} \leq C\,\|1-\rho(x,0)\|_{L^2} \leq C\left(T^{/2} + T^{1/6}\,\|\rho(x,0)\|_{L^3}\right) \leq C. \quad (47)$$

By the energy conservation,

$$\int_0^T \int_{\mathbf{R}} v^2 f(x,v,t)\,dvdx + \|E(x,t)\|_{L^2_x}^2 = \int_0^T \int_{\mathbf{R}} v^2 f(x,v,0)\,dvdx + \|E(x,0)\|_{L^2}^2 < C.$$

Let $j = \int vf\ dv$, then

$$|j(t)| = \left|\int vf\ \ (t)\,dv\right| \leq \|f(t)\|_{L^\infty}^{1/3} \left(\int_{\mathbf{R}} v^2 f(x,v,t)\,dvdx\right)^{2/3},$$

and thus

$$\|j(x,t)\|_{L_x^{\frac{3}{2}}} \leq M_1^{\frac{1}{3}} M_2^{\frac{3}{2}} \leq C.$$

Since

$$\begin{aligned} \frac{d}{dt}\|E(x,t)\|_{L^2_x}^2 &= \int_0^T j(x,t)\,E(x,t)\,dx \\ &\leq \|j(x,t)\|_{L_x^{\frac{3}{2}}} \|E(x,t)\|_{L^3_x} \leq C\,\|E(x,t)\|_{H^1_x}, \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \|E(x,t)\|_{H^1_x}\,dt &\leq \left(\int_0^\infty (1+t)^{-2(s-1)}\,dt\right)^{\frac{1}{2}} \left(\int_0^\infty (1+t)^{2(s-1)}\,\|E(x,t)\|_{H_x^{\frac{3}{2}}}^2\,dt\right)^{\frac{3}{2}} \\ &\leq C\varepsilon_0, \end{aligned}$$

thus $\lim_{t\to\infty}\|E(x,t)\|_{L^2_x}$ exists and this limit must be zero. This finishes the proof. ■

6 Appendix

In this appendix, we reformulate Penrose's linear stability criterion. The main purpose is to clarify the intervals of wave numbers (periods) for which linear instability can be found. In the original paper of Penrose [41], a necessary and sufficient condition was given for linear instability of a homogeneous state at certain wave number. However, the precise range of unstable wave numbers was not given in [41].

Lemma 7 *Assume $f_0(v) \in W^{2,p}(\mathbf{R})$ $(p>1)$. Let $S = \{v_i\}_{i=1}^l$ be the set of all extrema points of f_0. If for some $1 \leq i \leq l$,*

$$\int \frac{f_0'(v)}{v - v_i}dv = \left(\frac{2\pi}{T_i}\right)^2 > 0, \quad (48)$$

then there exists linearly growing mode with $x-$period T near T_i. More precisely, when v_i is a minimum (maximum) point of f_0, unstable modes exist for T slightly greater (smaller) than T. Let $0 < T_0 \leq +\infty$ be defined by

$$\left(\frac{2\pi}{T_0}\right)^2 = \max\left\{0, \max_{v_i \in S} \int \frac{f_0'(v)}{v - v_i} dv\right\}.$$

Then for $T < T_0$, there exist no unstable modes with $x-$Period T.

Proof. Plugging the normal mode solution

$$(f(x,v,t), E(x,t)) = e^{ik(x-ct)}(f_k(v), E_k)$$

into the linearized Vlasov-Poisson equation, we obtain the standard dispersion relation

$$k^2 - \int \frac{f_0'(v)}{v-c} dv = 0. \tag{49}$$

Linear instability with $x-$period T corresponds to a solution of (49) with $k = \frac{2\pi}{T}$ and $\operatorname{Im} c > 0$. When the condition (48) is satisfied, we have a neutral mode of stability with $k_0 = \left(\frac{2\pi}{T_i}\right)^2$ and $c_0 = v_i$. Then local bifurcation of unstable modes near (k_0, c_0) can be shown, for example, by the arguments used in [30] for the shear flow instability. The bifurcation direction can be seem from the following computation. Let (k, c) be an unstable mode near (k_0, c_0) .Then

$$k^2 - k_0^2 = \int \frac{f_0'(v)}{v-c} dv - \int \frac{f_0'(v)}{v - v_i} dv = (c - v_i) \int \frac{f_0'(v)}{(v-v_i)(v-c)} dv$$

and by Plemelj formula when $\operatorname{Im} c \to 0+$,

$$\frac{k^2 - k_0^2}{c - v_i} = \int \frac{f_0'(v)}{(v-v_i)(v-c)} dv \to P \int \frac{f_0'(v)}{(v-v_i)^2} dv + i\pi f_0''(v_i),$$

where $P\int$ is the Cauchy principal value. So when $f_0''(v_i) > 0$ (< 0), we have to let $k^2 < k_0^2$ $\left(k^2 > k_0^2\right)$ to ensure $\operatorname{Im} c > 0$. The linear stability when $T < T_0$ can be seem most easily from the following Nyquist graph (see [41]) in the complex plane

$$Z(\xi + i0) = \lim_{\eta \to 0+} \int \frac{f_0'(v)}{v - (\xi + i\eta)} dv = P \int \frac{f_0'(v)}{v - \xi} dv + i\pi f_0'(\xi),\ \xi \in \mathbf{R}. \tag{50}$$

The unstable wave numbers consist of the part on the positive real axis enclosed by the graph of $Z(\xi + i0)$. So the maximal unstable wave number correspond to the right-most intersection point of the graph of $Z(\xi + i0)$ with the positive real axis. Therefore if one of the integral $\int \frac{f_0'(v)}{v - v_i} dv$ is positive, the maximal unstable wave number $k_{\max}$ is

$$k_{\max}^2 = \max_{v_i \in S} \int \frac{f_0'(v)}{v - v_i} dv = \left(\frac{2\pi}{T_0}\right)^2,$$

and all perturbations with $k > k_{\max}$ or equivalently $T < T_0$ are linearly stable. For homogeneous states with all $\int \frac{f_0'(v)}{v-v_i}dv$ to be non-positive, such as Maxwellian $e^{-\frac{1}{2}v^2}$, perturbations of any period (wave number) are linearly stable and thus $T_0 = +\infty$. ■

Remark 6 *1) The assumption $f_0(v) \in W^{2,p}(\mathbf{R})$ $(p > 1)$ is used to ensure that $f_0'(v)$ is locally Hölder continuous and thus the function $Z(\xi + i0)$ is well defined, continuous and bounded. Lemma 7 is still true for $f_0 \in W^{1,p}$ and f_0' locally Hölder continuous, particularly for $f_0(v) \in W^{s,p}(\mathbf{R})$ $\left(p > 1, s > 1 + \frac{1}{p}\right)$.*

2) The local bifurcation of unstable modes near a neutral mode (k_0, v_i) can be extended globally in the following way. Let v_i be an extrema point of $f_0(v)$,

$$k_0^2 = \left(\frac{2\pi}{T_i}\right)^2 = \int \frac{f_0'(v)}{v - v_i}dv > 0.$$

Suppose $f_0''(v_i) > 0$, then the unstable modes with $\operatorname{Im} c > 0$ exist when k is slightly less than k_0. This unstable mode can be continuated by decreasing k as long as the growth rate is not zero. This continuation process can only stop at another neutral mode (k_1, c_1) with $k_1 < k_0$, $c_1 \in \mathbf{R}$,. By (50), we must have

$$f_0'(c_1) = 0,\ k_1^2 = \left(\frac{2\pi}{T_1}\right)^2 = \int \frac{f_0'(v)}{v - c_1}dv > 0.$$

For any wave number $k \in (k_1, k_0)$, there exists an unstable mode. Moreover, since the local bifurcation of unstable modes near k_1 is only for slightly larger wave number, we must have $f_0''(c_1) < 0$. Similarly, when $f_0''(v_i) < 0$, the unstable modes exist for wave numbers $k \in (k_0, k_2)$, where

$$k_2^2 = \left(\frac{2\pi}{T_2}\right)^2 = \int \frac{f_0'(v)}{v - c_2}dv,\ \text{with } f_0'(c_2) = 0,\ f_0''(c_2) > 0.$$

From the above continuation argument, it is also easy to see the linear stability for $k > k_{\max}$ without using the Nyquist graph. Suppose at some $k' > k_{\max}$ there exists an unstable mode. Then we can extend this unstable mode for $k > k'$ until it stops at a neutral mode (k'', c'') with

$$(k'')^2 = \int \frac{f_0'(v)}{v - c''}dv > 0,\ f_0'(c'') = 0.$$

But $k'' > k' > k_{\max}$, this is a contradiction with the definition of $k_{\max}$. We also note that $k_{\max}$ must occur at a minimal point of f_0, since the unstable modes only bifurcate for wave numbers less than $k_{\max}$.

3) Finally, we point out that there could exist "stability gaps" of wave numbers in $(0, k_{\max})$. By our discussions above in 2), such stability gap must be of the form $\left(\bar{k}, \tilde{k}\right)$ where

$$\bar{k}^2 = \int \frac{f_0'(v)}{v - \bar{c}}dv > 0,\ \tilde{k}^2 = \int \frac{f_0'(v)}{v - \tilde{c}}dv > 0,$$

and $\bar{c}$, $\tilde{c}$ are minimum and maximum points of f_0 respectively. From the Nyquist graph of $Z(\xi + i0)$, it is easy to see that these stability gaps correspond to positive intervals in the real axis not enclosed by the Nyquist curve.

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