

# Inviscid dynamical structures near Couette flow

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## Abstract

Consider inviscid fluids in a channel  $\{-1 < y < 1\}$ . For the Couette flow  $\vec{v}_0 = (y, 0)$ , the vertical velocity of solutions to the linearized Euler equation at  $\vec{v}_0$  decays in time. At the nonlinear level, such inviscid damping is widely open. First, we show that in any (vorticity)  $H^s$  ( $s < \frac{3}{2}$ ) neighborhood of Couette flow, there exist non-parallel steady flows with arbitrary minimal horizontal period. This implies that nonlinear inviscid damping is not true in any (vorticity)  $H^s$  ( $s < \frac{3}{2}$ ) neighborhood of Couette flow and for any horizontal period. Indeed, the long time behavior in such neighborhoods are very rich, including nontrivial steady flows, stable and unstable manifolds of nearby unstable shears. Second, in the (vorticity)  $H^s$  ( $s > \frac{3}{2}$ ) neighborhood of Couette, we show that there exist no non-parallel steadily travelling flows  $\vec{v}(x - ct, y)$ , and no unstable shears. This suggests that the long time dynamics in  $H^s$  ( $s > \frac{3}{2}$ ) neighborhoods of Couette might be much simpler. Such contrasting dynamics in  $H^s$  spaces with the critical power  $s = \frac{3}{2}$  is a truly nonlinear phenomena, since the linear inviscid damping near Couette is true for any initial vorticity in  $L^2$ .

## 1 Introduction

Consider the incompressible inviscid fluid in a channel  $\{(x, y) \mid -1 \leq y \leq 1\}$ , satisfying the 2D Euler equation

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x P \\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y P \end{cases} \quad (1)$$

with the incompressibility condition

$$\partial_x u + \partial_y v = 0 \quad (2)$$

and the boundary conditions

$$v = 0 \text{ on } \{y = -1\} \text{ and } \{y = 1\}. \quad (3)$$

Here,  $\vec{u} = (u, v)$  is the fluid velocity and  $P$  is the pressure. Define the vorticity  $\omega = u_y - v_x$ , then  $\omega$  satisfies the equation

$$\omega_t + u\omega_x + v\omega_y = 0.$$

Any shear flow  $(U(y), 0)$  is a steady solution for (1). The Couette flow  $\vec{u}_0 = (y, 0)$  is among the simplest laminar flows, however, it poses several long-standing puzzles in hydrodynamics. First, for any Reynolds number  $R > 0$ , the Couette flow is also a steady state for Navier-Stokes equations

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x P + \frac{1}{R}\Delta u \\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y P + \frac{1}{R}\Delta v \end{cases} \quad (4)$$

with (3) and the boundary conditions

$$(u, v) = (\pm 1, 0) \text{ on } \{y = \pm 1\}.$$

The so called Sommerfeld paradox ([22]) is that Couette flow is linearly stable for any  $R > 0$  (proved in [19]), but it becomes turbulent when  $R$  is large as revealed in experiments and numerical simulations. We refer to ([10]) and the references therein for attempts to resolve this paradox. In this paper, we are interested in another mystery about Couette flow, namely, the inviscid damping. It is obvious that Couette flow is nonlinearly stable in any  $L^p$  norm of vorticity  $\omega$ , since for Couette flow  $\omega_0 = 1$  and thus the vorticity perturbation is preserved along the perturbed flow trajectory. In 1907, Orr ([17]) observed that for the linearized Euler equation around Couette, the vertical velocity  $v(t)$  tends to zero when  $t$  goes to infinity. We refer to Section 4 for a more detailed study on the linear damping of Couette flow. It is unusual that such damping phenomena can occur for a time reversible system such as the Euler equation. Moreover, the issue of inviscid damping also appears in the study of many other stable flows ([4], [1], [18], [20]), and is believed to play important roles on explaining the appearance of coherent structures in 2D turbulence. To be precise mathematically, the problem of nonlinear inviscid damping near Couette flow is to prove or disprove the following statement: When the initial velocity is close enough to Couette in the sense that

$$\|(u(0), v(0)) - (y, 0)\|_{\mathbf{X}} \text{ is small enough}$$

in some function space  $\mathbf{X}$ , then

$$\|v(t)\|_{L^2} \rightarrow 0 \text{ when } t \rightarrow \infty,$$

that is,  $(u(t), v(t))$  tends asymptotically to a shear flow  $(U_\infty(y), 0)$  near the Couette flow. So far, nonlinear inviscid damping has not been proved for Couette flow or any other stable Euler flows. Our first result shows that the minimal regularity for such nonlinear damping to be true is  $H^{\frac{5}{2}}$ , that is, the velocity space  $\mathbf{X}$  must be at least  $H^{\frac{5}{2}}$ .

**Theorem 1** *Fixed any  $T > 0$  and  $0 \leq s < \frac{3}{2}$ , then for any  $\varepsilon > 0$ , there exists a steady solution  $(u_\varepsilon(x, y), v_\varepsilon(x, y))$  to Euler equation (1) with (2)-(3) such that  $(u_\varepsilon(x, y), v_\varepsilon(x, y))$  has minimal  $x$ -period  $T$ ,*

$$\|\omega_\varepsilon - 1\|_{H^s_{(0,T) \times (-1,1)}} < \varepsilon, \quad \text{where } \omega_\varepsilon = \partial_y u_\varepsilon - \partial_x v_\varepsilon,$$

and  $v_\varepsilon(x, y)$  is not identically zero.

The above Theorem immediately implies that nonlinear inviscid damping is *not* true in any (vorticity)  $H^s$  ( $s < \frac{3}{2}$ ) neighborhood, or equivalently in any (velocity)  $H^s$  ( $s < \frac{5}{2}$ ) neighborhood of Couette flow. As a corollary of the proof of Theorem 1, we also get the following structural instability result for Couette flow.

**Corollary 1** *Fixed any  $T > 0$  and  $0 \leq s < \frac{3}{2}$ , then for any  $\varepsilon > 0$ , there exists a shear flow  $(U_\varepsilon(y), 0)$  such that  $\|U'_\varepsilon(y) - 1\|_{H^s(-1,1)} < \varepsilon$  and  $(U_\varepsilon(y), 0)$  is exponentially unstable to perturbations of  $x$ -period  $T$ .*

The shear flow  $(U_\varepsilon(y), 0)$  is unstable in the sense that unstable eigenvalues exist for the linearized problem in the domain  $\Omega_T = S_T \times (-1, 1)$ , where  $S_T$  is the  $T$ -periodic circle. By our results in [14], there exist stable and unstable manifolds near  $(U_\varepsilon(y), 0)$  for the Euler equation (1) in  $\Omega_T$ . Therefore, Theorem 1 and Corollary 1 imply that the long time dynamics in the (vorticity)  $H^s$  ( $s < \frac{3}{2}$ ) neighborhood of Couette flow is very rich, including nontrivial steady flows, stable and unstable manifolds of nearby unstable shear flows.

Our next theorem shows that there exist *no* nontrivial steadily travelling flows in the (vorticity)  $H^s$  ( $s > \frac{3}{2}$ ) neighborhoods of Couette flow.

**Theorem 2** *Fixed any  $T > 0$ ,  $s > \frac{3}{2}$ , there exists  $\varepsilon_0 > 0$  such that any travelling solution  $(u(x - cy, y), v(x - cy, y))$  ( $c \in \mathbf{R}$ ) to Euler equation (1)-(3) with  $x$ -period  $T$  and satisfying that*

$$\|\omega - 1\|_{H^s_{(0,T) \times (-1,1)}} < \varepsilon_0,$$

must have  $v(x, y) \equiv 0$ , that is,  $(u, v)$  is necessarily a shear flow.

By the proof of Theorem 2, we also have the following

**Corollary 2** *Fixed any  $T > 0$  and  $s > \frac{3}{2}$ , there exists  $\varepsilon_0 > 0$  such that any shear flow  $(U(y), 0)$  satisfying*

$$\|U'(y) - 1\|_{H^s(-1,1)} \leq \varepsilon_0,$$

is linearly stable to perturbations of  $x$ -period  $T$ .

Theorem 2 and Corollary 2 suggests that in the (vorticity)  $H^s$  ( $s > \frac{3}{2}$ ) neighborhoods of Couette flow, the long time dynamical behavior of Euler flows might be much simpler. Particularly, the only steady structures in any reference frame are

nearby stable shear flows. A necessary condition for nonlinear inviscid damping in any space is that there exist no nontrivial invariant structures (time-periodic, quasi-periodic solutions etc.) near Couette flow in this space. Theorem 3 is a first step in this direction.

In Theorem 3 in Section 4, we show that the linear decay holds true for any initial vorticity in  $L^2$  and the optimal decay rate is already achieved for initial vorticity in  $H^1$  (see Remark 2). This indicates that the contrasting dynamics in  $H^s$  neighborhoods of Couette with  $s < \frac{3}{2}$  or  $s > \frac{3}{2}$  is a truly nonlinear phenomena and it can not be traced back to the linear level.

A similar phenomena of collisionless damping for electron plasmas was discovered at the linear level by Landau ([9]) in 1946. In the physical literature, the collisionless damping had been often ([8], [18], [1]) compared with the inviscid damping problem. In [15], we obtained similar results for the nonlinear Landau damping problem. Moreover, in the case of collisionless plasmas we are able to prove a stronger result that  $H^{\frac{3}{2}}$  is the critical regularity for the existence of *any* nontrivial invariant structure near a stable homogeneous state.

This paper is organized as follows. In Section 2, we construct nontrivial steady flows near Couette flow in (vorticity)  $H^s$  ( $s < \frac{3}{2}$ ) neighborhood and for any minimal  $x$ -period. In Section 3, the non-existence of nontrivial travelling flows is proved in (vorticity)  $H^s$  ( $s > \frac{3}{2}$ ) neighborhood. Section 4 is to study the linear damping problem in Sobolev spaces. Throughout this paper, we use  $C$  to denote a generic constant in the estimates and only indicate the dependence of  $C$  when it matters.

## 2 Existence of Cat's-eyes in $H^s$ ( $s < \frac{3}{2}$ )

In this Section, we construct steady flows of Kelvin's cat's eyes structure near Couette flow in the (vorticity)  $H^s$  ( $s < \frac{3}{2}$ ) space. Our strategy is to construct cat's eyes flows by bifurcation at modified shear flows near Couette. We split the proof into several steps.

**Lemma 1** *Assume  $U(y) \in C^5[-1, 1]$ , is odd, monotone in  $[-1, 1]$ , and  $U'(0) > 0$ . Let  $Q(y) = \frac{U''(y)}{U(y)}$  and define the operator*

$$\mathcal{L} := -\frac{d^2}{dy^2} + Q(y), \quad H^2(-1, 1) \rightarrow L^2(-1, 1),$$

*with zero Dirichlet conditions at  $\{y = \pm 1\}$ . If  $\mathcal{L}$  has a negative eigenvalue  $-k_0^2$ , then  $\exists \varepsilon_0 > 0$ , such that for each  $0 < \varepsilon < \varepsilon_0$ , there exist a steady solution  $(u_\varepsilon(x, y), v_\varepsilon(x, y))$  to Euler equations (1)-(3) which has minimal period  $T_\varepsilon$  in  $x$ ,*

$$\|\omega_\varepsilon(x, y) - U'(y)\|_{H^2(0, T_\varepsilon) \times (-1, 1)} = \varepsilon,$$

*and the streamlines of this steady flow near  $y = 0$  have cat's eyes structure, with a leading order expression given by (9). When  $\varepsilon \rightarrow 0$ ,  $T_\varepsilon \rightarrow \frac{2\pi}{k_0}$ .*

**Proof.** The proof is a slight modification of that in [10]. Let  $\psi_0(y)$  to be a stream function associated with the shear  $(U(y), 0)$ , i.e.,  $\psi_0'(y) = U(y)$ . Since  $\psi_0(y), Q(y)$  are even in  $[-1, 1]$ , we let  $\psi_0(y) = G(\frac{1}{2}y^2)$  and  $Q(y) = H(\frac{1}{2}y^2)$ . Then

$$G' \left( \frac{1}{2}y^2 \right) = \frac{\psi_0'(y)}{y} = \frac{U(y)}{y} > 0, \text{ when } y \in [-1, 1],$$

and  $G, H \in C^1$  because  $U(y) \in C^5$ . So we can define a function  $f_0 \in C^2$   $[\min \psi_0, \max \psi_0]$  such that

$$f_0' = H \circ G^{-1} \text{ and } f_0(\psi_0(0)) = \psi_0''(0).$$

Then we extend  $f_0$  to  $f \in C_0^2(\mathbf{R})$  such that  $f = f_0$  in  $[\min \psi_0, \max \psi_0]$ . By our construction,

$$f'(\psi_0(y)) = Q(y), \quad \text{for } y \in [-1, 1], \quad (5)$$

which implies that

$$f'(\psi_0(y))\psi_0'(y) = U''(y) = \psi_0'''(y),$$

and an integration of above yields

$$f(\psi_0(y)) = \psi_0''(y), \quad \text{for } y \in [-1, 1]. \quad (6)$$

We construct steady flows near  $(U(y), 0)$  by solving the elliptic equation

$$\Delta\psi = f(\psi),$$

where  $\psi(x, y)$  is the stream function and  $(u, v) = (\psi_y, -\psi_x)$  is the steady velocity. Let  $\xi = \alpha x$ ,  $\psi(x, y) = \tilde{\psi}(\xi, y)$ , where  $\tilde{\psi}(\xi, y)$  is  $2\pi$ -periodic in  $\xi$ . We use  $\alpha^2$  as the bifurcation parameter. The equation for  $\tilde{\psi}(\xi, y)$  becomes

$$\alpha^2 \frac{\partial^2 \tilde{\psi}}{\partial \xi^2} + \frac{\partial^2 \tilde{\psi}}{\partial y^2} - f(\tilde{\psi}) = 0, \quad (7)$$

with the boundary conditions that  $\tilde{\psi}$  takes constant values on  $\{y = \pm 1\}$ . Define the perturbation of the stream function

$$\phi(\xi, y) = \tilde{\psi}(\xi, y) - \psi_0(y).$$

Then by using (6), we reduce the equation (7) to

$$\alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial y^2} - (f(\phi + \psi_0(y)) - f(\psi_0(y))) = 0. \quad (8)$$

Define the spaces

$$B = \{ \phi(\xi, y) \in H^3([0, 2\pi] \times [-1, 1]), \phi(\xi, -1) = \phi(\xi, 1) = 0, 2\pi\text{-periodic and even in } \xi \}$$

and

$$D = \{ \phi(\xi, y) \in H^1([0, 2\pi] \times [-1, 1]), 2\pi\text{-periodic and even in } \xi \}.$$

Consider the mapping

$$F(\phi, \alpha^2) : B \times \mathbb{R}^+ \mapsto D$$

defined by

$$F(\phi, \alpha^2) = \alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial y^2} - (f(\phi + \psi_0(y)) - f(\psi_0(y))).$$

We study the bifurcation near the trivial solution  $\phi = 0$  of the equation  $F(\phi, \alpha^2) = 0$  in  $B$ , whose solutions give steady flows with  $x$ -period  $\frac{2\pi}{\alpha}$ . The linearized operator of  $F$  around  $(0, k_0^2)$  has the form

$$\begin{aligned} \mathcal{G} &:= F_\psi(0, k_0^2) = k_0^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y} - f'(\psi_0(y)) \\ &= k_0^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y} - Q(y). \end{aligned}$$

By Sturm-Liouville theory, all eigenvalues of  $\mathcal{L}$  are simple. In fact, as proven in Appendix of [10],  $-k_0^2$  is the only negative eigenvalue of  $\mathcal{L}$ . Let  $\phi_0(y)$  be the corresponding positive eigenfunction. So the kernel of  $\mathcal{G} : B \mapsto D$  is given by

$$\ker(\mathcal{G}) = \{\phi_0(y) \cos \xi\},$$

In particular, the dimension of  $\ker(\mathcal{G})$  is 1. Since  $\mathcal{G}$  is self-adjoint,  $\phi_0(y) \cos \xi \notin R(\mathcal{G})$  – the range of  $\mathcal{G}$ . Notice that  $\partial_{\alpha^2} \partial_\phi F(\phi, \alpha^2)$  is continuous and

$$\partial_{\alpha^2} \partial_\phi F(0, k_0^2)(\phi_0(y) \cos \xi) = \frac{\partial^2}{\partial \xi^2} [\phi_0(y) \cos \xi] = -\phi_0(y) \cos \xi \notin R(\mathcal{G}).$$

Therefore by the Crandall-Rabinowitz local bifurcation theorem [6], there exists a local bifurcating curve  $(\phi(\beta), \alpha^2(\beta))$  of  $F(\phi, \alpha^2) = 0$ , which intersects the trivial curve  $(0, \alpha^2)$  at  $\alpha^2 = k_0^2$ , such that

$$\phi(\beta) = \beta \phi_0(y) \cos \xi + o(\beta),$$

$\alpha^2(\beta)$  is a continuous function of  $\beta$ , and  $\alpha^2(0) = k_0^2$ . So the stream functions of the perturbed steady flows in  $(\xi, y)$  coordinates take the form

$$\psi(\xi, y) = \psi_0(y) + \beta \phi_0(y) \cos \xi + o(\beta). \quad (9)$$

Since  $\phi_0(y) > 0$ ,  $\psi'_0(0) = U(0) = 0$ , the streamlines of perturbed flows have cat's eyes structure near  $\{y = 0\}$ , with saddle points near  $(2\pi j, 0)$ . The proof is completed. ■

In the next lemma, we study the eigenvalue problem of  $\mathcal{L}$  for a class of monotone shear flows near Couette flow. Let

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds, \quad -\infty < x < +\infty,$$

be the error function. For  $\gamma > 0$ ,  $a > 0$ , we define the shear profile

$$U_{\gamma,a}(y) = y + a\gamma^2 \operatorname{erf}\left(\frac{y}{\gamma}\right), \quad y \in (-1, 1). \quad (10)$$

Denote

$$Q_{\gamma,a}(y) = \frac{U_{\gamma,a}''(y)}{U_{\gamma,a}(y)}, \quad (11)$$

and

$$\mathcal{L}_{\gamma,a} : H^2(-1, 1) \rightarrow L^2(-1, 1)$$

to be the operator  $-\frac{d^2}{dy^2} + Q_{\gamma,a}(y)$  with the Dirichlet conditions at  $\{y = \pm 1\}$ .

**Lemma 2** *For any fixed  $a > \frac{1}{2}$ , when  $\gamma$  is small enough, the operator  $\mathcal{L}_{\gamma,a}$  has a unique negative eigenvalue  $-\beta_{\gamma,a}^2$ . When  $\gamma \rightarrow 0$ ,  $\beta_{\gamma,a}$  tends to the unique root  $\beta_a$  of the equation*

$$2a = \beta_a \coth \beta_a, \quad (12)$$

with the error estimate  $|\beta_{\gamma,a} - \beta_a| = O(\sqrt{\gamma})$ .

**Proof.** We write the potential function  $Q_{\gamma,a}(y)$  as

$$\begin{aligned} Q_{\gamma,a}(y) &= -\frac{4a}{\gamma\sqrt{\pi}} \frac{ye^{-(\frac{y}{\gamma})^2}}{y + a\gamma^2 \operatorname{erf}\left(\frac{y}{\gamma}\right)} \\ &= -\frac{4a}{\gamma\sqrt{\pi}} e^{-(\frac{y}{\gamma})^2} \frac{1}{1 + \gamma a \operatorname{erf}\left(\frac{y}{\gamma}\right) / \left(\frac{y}{\gamma}\right)} \\ &= -4a \frac{1}{\gamma} \sigma\left(\frac{y}{\gamma}\right) \frac{1}{1 + \gamma a \Lambda\left(\frac{y}{\gamma}\right)}, \end{aligned}$$

where

$$\sigma(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}, \quad \Lambda(y) = \frac{\operatorname{erf}(y)}{y}.$$

Since  $\Lambda(y)$  is positive and bounded, we formally derive that

$$Q_{\gamma,a}(y) \rightarrow -4a\delta(0), \quad \text{when } \gamma \rightarrow 0.$$

Thus, when  $\gamma \rightarrow 0$ , the operator  $\mathcal{L}_{\gamma,a}$  tends to  $-\frac{d^2}{dy^2} - 4a\delta(0)$ , for which the eigenvalue can be calculated by the formula (12). We implement these ideas rigorously below. We divide the proof into several steps.

Step 1: Denote  $\lambda_{\gamma,a}$  to be the lowest eigenvalue of  $\mathcal{L}_{\gamma,a}$  and  $\phi_{\gamma,a}$  the corresponding eigenfunction with  $\|\phi_{\gamma,a}\|_{L^2} = 1$ . We show that for  $\gamma > 0$  small enough,

$$-16a^2 \leq \lambda_{\gamma,a} < 0,$$

and

$$\|\phi_{\gamma,a}\|_{H^1} \leq 8a + 1. \quad (13)$$

Note that

$$\lambda_{\gamma,a} = \min_{\substack{\|\phi\|_{L^2}=1 \\ \phi \in H_0^1}} (\mathcal{L}_{\gamma,a}\phi, \phi) = \min_{\substack{\|\phi\|_{L^2}=1 \\ \phi \in H_0^1}} \left( \|\phi'\|_{L^2(-1,1)}^2 + \int_{-1}^1 Q_{\gamma,a}(y) \phi(y)^2 dy \right).$$

Let  $\phi_1(y) = (1 - |y|)$ , then when  $\gamma$  is small enough,

$$\lambda_{\gamma,a} \leq \frac{(\mathcal{L}_{\gamma,a}\phi_1, \phi_1)}{\|\phi_1\|_{L^2}^2} \leq \frac{3}{4}(2 - 4a) < 0,$$

since  $\|\phi_1\|_{L^2}^2 = \frac{2}{3}$  and

$$\lim_{\gamma \rightarrow 0^+} (\mathcal{L}_{\gamma,a}\phi_1, \phi_1) = \|\phi_1'\|_{L^2(-1,1)}^2 - 4a\phi_1(0)^2 = 2 - 4a < 0.$$

To estimate the lower bound of  $\lambda_{\gamma,a}$ , we take any  $\phi \in H_0^1(-1, 1)$  with  $\|\phi\|_{L^2} = 1$ , then

$$\begin{aligned} (\mathcal{L}_{\gamma,a}\phi, \phi) &\geq \|\phi'\|_{L^2(-1,1)}^2 - \int_{-1}^1 \frac{4a}{\gamma\sqrt{\pi}} e^{-(\frac{y}{\gamma})^2} dy \|\phi\|_{L^\infty}^2 \\ &\geq \|\phi'\|_{L^2(-1,1)}^2 - 4a \|\phi\|_{L^\infty}^2 \\ &\geq \|\phi'\|_{L^2(-1,1)}^2 - 8a \|\phi\|_{L^2} \|\phi'\|_{L^2(-1,1)} \geq -16a^2. \end{aligned} \quad (14)$$

Taking the minimum of above estimate, we get  $\lambda_{\gamma,a} \geq -16a^2$ . Moreover, again from estimate (14),

$$0 > \lambda_{\gamma,a} = (\mathcal{L}_{\gamma,a}\phi_{\gamma,a}, \phi_{\gamma,a}) \geq \|\phi'_{\gamma,a}\|_{L^2}^2 - 8a \|\phi'_{\gamma,a}\|_{L^2}$$

which implies that  $\|\phi'_{\gamma,a}\|_{L^2} \leq 8a$ .

Step 2: Let  $\lambda_a$  be defined by

$$\lambda_a = \min_{\substack{\|\phi\|_{L^2}=1 \\ \phi \in H_0^1}} \|\phi'\|_{L^2}^2 - 4a\phi(0)^2. \quad (15)$$

We show that  $\lambda_a = -\beta_a^2$  where  $\beta_a$  solves the equation (12).

First, we claim that the minimum of (15) is obtained at some function  $\phi_a \in H_0^1$ . To show this claim, we note that by the same estimates as in Step 1,

$$-16a^2 \leq \lambda_a < \frac{3}{2}(2 - 4a) < 0.$$

Let  $\{\phi_n\}_{n=1}^\infty \subset H_0^1$  be a minimizing sequence of (15) with  $\|\phi_n\|_{L^2} = 1$  and

$$\|\phi_n'\|_{L^2}^2 - 4a\phi_n(0)^2 \rightarrow \lambda_a, \text{ when } n \rightarrow \infty.$$

Similar to the estimate (13), when  $n$  is large, we have  $\|\phi_n\|_{H^1} \leq 8a + 1$ . Thus  $\phi_n \rightarrow \phi_a$  weakly  $H^1$ , and strongly in  $L^2 \cap L^\infty$ . Therefore,  $\|\phi_a\|_{L^2} = 1$  and

$$\|\phi_a'\|_{L^2}^2 - 4a\phi_a(0)^2 \leq \lim_{n \rightarrow \infty} \|\phi_n'\|_{L^2}^2 - 4a\phi_n(0)^2 = \lambda_a.$$



Thus  $\phi_a$  is the minimizer of (15).

By taking the variation of (15), one immediately obtains that  $\phi_a \in H_0^1$  satisfies the equation

$$\phi_a'' + 4a\phi_a(0) + \lambda_a\phi_a = 0 \quad (16)$$

in the sense of distribution. In particular,  $\phi_a$  is continuous on  $[-1, 1]$ ,  $\phi_a(\pm 1) = 0$ , and satisfies

$$\phi_a'' + \lambda_a\phi_a = 0, \text{ on } [-1, 1] \setminus \{0\}.$$

Therefore, we have

$$\phi_a(y) = \mp c \sinh\left(\sqrt{-\lambda_a}(y \mp 1)\right), \quad \pm y \in (0, 1],$$

for some constant  $c$ . To satisfy (16), it is easy to check that one must have

$$\phi_a'(0+) - \phi_a'(0-) = -4a\phi_a(0),$$

from which it follows that  $\lambda_a = -\beta_a^2$  and  $\beta_a$  solves the equation (12). Note that since the function

$$f(\beta) = \beta \coth \beta : [0, \infty) \rightarrow \left[\frac{1}{2}, \infty\right)$$

is monotone increasing. So for each  $a > \frac{1}{2}$ , there exists a unique  $\beta_a = f^{-1}(a)$  such that (12) is satisfied.

Step 3: We show that when  $\gamma$  is small enough,

$$|\lambda_{\gamma,a} - \lambda_a| \leq C(a) \sqrt{\gamma}. \quad (17)$$

Denote the quadratic forms

$$H_{\gamma,a}(\phi) = \|\phi'\|_{L^2(-1,1)}^2 + \int_{-1}^1 Q_{\gamma,a}(y) \phi(y)^2 dy$$

and

$$H_a(\phi) = \|\phi'\|_{L^2(-1,1)}^2 - 4a\phi(0)^2$$

in  $H_0^1(-1, 1)$ . Then

$$\begin{aligned} \lambda_a &\leq H_a(\phi_{\gamma,a}) \\ &= H_{\gamma,a}(\phi_{\gamma,a}) + 4a \int_{-1}^1 \frac{1}{\gamma} \sigma\left(\frac{y}{\gamma}\right) \frac{1}{1 + \gamma a \Lambda\left(\frac{y}{\gamma}\right)} \phi_{\gamma,a}^2(y) dy - 4a\phi_{\gamma,a}(0)^2 \\ &= \lambda_{\gamma,a} + 4a \int_{-\frac{1}{\gamma}}^{\frac{1}{\gamma}} \sigma(y) (\phi_{\gamma,a}^2(\gamma y) - \phi_{\gamma,a}^2(0)) dy - 4a \int_{|y| \geq \frac{1}{\gamma}} \sigma(y) dy \phi_{\gamma,a}^2(0) \\ &\quad + 4a \int_{-1}^1 \frac{1}{\gamma} \sigma\left(\frac{y}{\gamma}\right) \frac{\gamma a \Lambda\left(\frac{y}{\gamma}\right)}{1 + \gamma a \Lambda\left(\frac{y}{\gamma}\right)} \phi_{\gamma,a}^2(y) dy \\ &= \lambda_{\gamma,a} + T_1 + T_2 + T_3. \end{aligned}$$

Since

$$\begin{aligned} |\phi_{\gamma,a}^2(\gamma y) - \phi_{\gamma,a}^2(0)| &\leq 2 \|\phi_{\gamma,a}\|_{L^\infty} \left| \int_0^{\gamma y} \phi'_{\gamma,a}(s) ds \right| \\ &\leq C \|\phi_{\gamma,a}\|_{H^1(-1,1)}^2 \sqrt{\gamma} |y|^{\frac{1}{2}} \leq C(8a+1) \sqrt{\gamma} |y|^{\frac{1}{2}}, \end{aligned}$$

so

$$|T_1| \leq C(a) \sqrt{\gamma} \int_{\mathbf{R}} \sigma(y) |y|^{\frac{1}{2}} dy \leq C(a) \sqrt{\gamma}.$$

When  $\gamma$  is small enough, we have

$$|T_2| \leq C(a) \|\phi_{\gamma,a}\|_{L^\infty}^2 \int_{|y| \geq \frac{1}{\gamma}} \sigma(y) dy \leq C(a) \sqrt{\gamma},$$

and

$$|T_3| \leq C(a) \|\phi_{\gamma,a}\|_{L^\infty}^2 \gamma \int_{\mathbf{R}} \sigma(y) dy \leq C(a) \sqrt{\gamma}.$$

Thus

$$\lambda_a - \lambda_{\gamma,a} \leq C(a) \sqrt{\gamma}$$

and similarly

$$\lambda_{\gamma,a} - \lambda_a \leq C(a) \sqrt{\gamma}.$$

This finishes the proof of (17) and thus also the lemma.  $\blacksquare$

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Fixed  $T > 0$ , there exists  $\frac{1}{2} < a_1 < a_2$  such that

$$\beta_{a_1} < \frac{2\pi}{T} < \beta_{a_2}.$$

By Lemma 2, there exists  $\gamma_0 > 0$  small enough, such that when  $0 < \gamma < \gamma_0$ , for all  $a \in (a_1, a_2)$  the operator  $\mathcal{L}_{\gamma,a}$  has a negative eigenvalue  $\lambda_{\gamma,a}$  and

$$\sqrt{-\lambda_{\gamma,a_1}} < \frac{2\pi}{T} < \sqrt{-\lambda_{\gamma,a_2}}. \quad (18)$$

We show that: for  $a \in (a_1, a_2)$ ,  $s \in [0, \frac{3}{2})$ ,

$$\|U'_{\gamma,a}(y) - 1\|_{H^s(-1,1)} \rightarrow 0, \text{ when } \gamma \rightarrow 0. \quad (19)$$

Indeed,

$$U'_{\gamma,a}(y) - 1 = \frac{2a\gamma}{\sqrt{\pi}} e^{-\left(\frac{y}{\gamma}\right)^2},$$

so

$$\|U'_{\gamma,a}(y) - 1\|_{H^s(-1,1)} \leq C \left\| \gamma e^{-\left(\frac{y}{\gamma}\right)^2} \right\|_{H^s(\mathbf{R})}.$$

Using the Fourier transform, one may compute explicitly

$$\left\| \gamma e^{-\left(\frac{y}{\gamma}\right)^2} \right\|_{\dot{H}^s(\mathbf{R})} = C_s \gamma^{\frac{3}{2}-s},$$

which implies (19) by our assumption that  $s < \frac{3}{2}$ . Thus For any  $\varepsilon > 0$ , by choosing  $\gamma_0$  small enough, we can assume that

$$\|U'_{\gamma,a}(y) - 1\|_{H^s(-1,1)} \leq \frac{\varepsilon}{2T}, \text{ when } (\gamma, a) \in (0, \gamma_0) \times (a_1, a_2). \quad (20)$$

By Lemma 1, for any  $(\gamma, \delta) \in (0, \gamma_0) \times (a_1, a_2)$ , there exists local bifurcation of non-parallel steady flows (Cats's eyes) of Euler equation (1)-(3), near the shear flow  $(U_{\gamma,a}(y), 0)$ . For each fixed  $0 < \gamma < \gamma_0$ , we can find  $r_0 > 0$  (independent of  $a \in (a_1, a_2)$ ) such that for any  $0 < r < r_0$ , there exists a nontrivial steady solution

$$(u_{\gamma,a;r}(x, y), v_{\gamma,a;r}(x, y))$$

with vorticity  $\omega_{\gamma,a;r}(x, y)$  which has  $x$ -period  $T(\gamma, a; r)$  and

$$\|\omega_{\gamma,a;r} - U'_{\gamma,a}(y)\|_{H^2(0, T(\gamma, a; r)) \times (-1, 1)} = r.$$

Moreover,

$$\frac{2\pi}{T(\gamma, a; r)} \rightarrow \sqrt{-\lambda_{\gamma,a}}, \text{ when } r \rightarrow 0.$$

By (18), when  $r_0$  is small enough,

$$T(\gamma, a_1; r) < T < T(\gamma, a_2; r), \text{ for } 0 < r < r_0.$$

Since  $T(\gamma, a; r)$  is continuous to  $a$ , for each  $\gamma \in (0, \gamma_0)$  and  $r > 0$  small enough, there exists  $a_T(\gamma, r) \in (a_1, a_2)$ , such that  $T(\gamma, a_T; r) = T$ . Then the flow

$$(u_{\gamma;r}(x, y), v_{\gamma;r}(x, y)) := (u_{\gamma, a_T; r}(x, y), v_{\gamma, a_T; r}(x, y))$$

with the vorticity  $\omega_{\gamma;r} = \omega_{\gamma, a_T; r}$  is a nontrivial steady solution of Euler equation, with  $x$ -period  $T$  and

$$\|\omega_{\gamma;r} - U'_{\gamma, a_T}(y)\|_{H^2(0, T) \times (-1, 1)} = r.$$

Thus for any  $0 < r < \min\{\gamma_0, \frac{\varepsilon}{2}\}$ , combining with (20) we have

$$\|\omega_{\gamma;r}(x, y) - 1\|_{H^s(0, T) \times (-1, 1)} < \varepsilon.$$

This finishes the proof of Theorem 1. ■

For the shear flow  $U_{\gamma,a}(y)$  defined by (10), there is only one inflection point at  $y = 0$ . The following Lemma about linear instability of  $(U_{\gamma,a}(y), 0)$  follows from the result in [11].

**Lemma 3** *If the operator  $\mathcal{L}_{\gamma,a}$  has a negative eigenvalue  $\lambda_{\gamma,a} < 0$ , then the shear flow  $(U_{\gamma,a}(y), 0)$  is linearly exponentially unstable to perturbations of any  $x$ -period greater than  $\frac{2\pi}{\sqrt{-\lambda_{\gamma,a}}}$ .*

From above Lemma, it is easy to prove Corollary 1.

**Proof of Corollary 1.** For any fixed  $T > 0$ , pick  $a > \frac{1}{2}$  such that  $T > \frac{2\pi}{\beta_a}$ . Then there exists  $\gamma$  small enough such that

$$\lambda_{\gamma,a} < 0, \quad T > \frac{2\pi}{\sqrt{-\lambda_{\gamma,a}}}.$$

By Lemma 3, the shear flow  $(U_{\gamma,a}(y), 0)$  is linearly exponentially unstable to perturbations of  $x$ -period  $T$ . For any  $\varepsilon > 0$ , if  $\gamma$  is small enough, by (19) we can let

$$\|U'_{\gamma,a}(y) - 1\|_{H^s(-1,1)} < \varepsilon$$

and this finishes the proof.  $\blacksquare$

**Remark 1** We can use more general shear profiles than  $U_{\gamma,a}(y)$  in (10) to construct cats's eyes flows near Couette. More precisely, define

$$U_{\gamma,a}(y) = y + a\gamma^2 h\left(\frac{y}{\gamma}\right),$$

where  $h \in C^5(\mathbf{R})$  is odd,  $h' \in H^2(\mathbf{R})$ , and

$$\int_{\mathbf{R}} \frac{h''(x)}{x} dx = b_0 > 0, \quad a > \frac{2}{b_0}.$$

By the same proof of Lemma 2, when  $\gamma$  is small enough, the operator

$$\mathcal{L}_{\gamma,a} := -\frac{d^2}{dy^2} + Q_{\gamma,a}(y), \quad \text{with } Q_{\gamma,a}(y) = \frac{U''_{\gamma,a}(y)}{U_{\gamma,a}(y)},$$

has a negative eigenvalue  $-\beta_{\gamma,a}^2$ , where

$$|\beta_{\gamma,a} - \beta_a| = O(\sqrt{\gamma}) \quad \text{and} \quad \frac{b_0 a}{2} = \beta_a \coth \beta_a.$$

Then the same proof of Theorem 1 yields cats's eyes flows bifurcating from  $(U_{\gamma,a}(y), 0)$ . Such shear flows  $(U_{\gamma,a}(y), 0)$  are exponentially unstable for perturbations with  $x$ -period  $T$  near  $\frac{2\pi}{\beta_{\gamma,a}}$ .

### 3 Non-existence of traveling waves in $\mathbf{H}^s$ ( $s > \frac{3}{2}$ )

In this Section, we prove Theorem 2. For the proof, we need a few lemmas. The first lemma is a Hardy type inequality.

**Lemma 4** Let  $s \in (\frac{1}{2}, \frac{3}{2})$ . If  $u(y) \in H^s(-1, 1)$ , and  $u(y_0) = 0$  for some  $y_0 \in [-1, 1]$ , then for any  $1 \leq p < \frac{1}{\frac{3}{2}-s}$ ,

$$\left\| \frac{u(y)}{y - y_0} \right\|_{L^p(-1,1)} \leq C(p) \|u\|_{H^s(-1,1)}.$$

**Proof.** Since  $s > \frac{1}{2}$ , the space  $H^s(-1, 1)$  is embedded to the Hölder space  $C^{0, s-\frac{1}{2}}(-1, 1)$ . So

$$|u(y)| = |u(y) - u(y_0)| \leq |y - y_0|^{s-\frac{1}{2}} \|u\|_{C^{0, \alpha}} \leq C |y - y_0|^{s-\frac{1}{2}} \|u\|_{H^s},$$

and

$$\begin{aligned} \int_{-1}^1 \left| \frac{u(y)}{y - y_0} \right|^p dy &\leq \|u\|_{H^s}^p \int_{-1}^1 \frac{1}{|y - y_0|^{(\frac{3}{2}-s)p}} dy \\ &= \frac{1}{1 - (\frac{3}{2} - s)p} \left( (1 - y_0)^{1 - (\frac{3}{2} - s)p} + (y_0 + 1)^{1 - (\frac{3}{2} - s)p} \right) \|u\|_{H^s}^p \\ &\leq \frac{1}{1 - (\frac{3}{2} - s)p} 2^{1 - (\frac{3}{2} - s)p} \|u\|_{H^s}^p. \end{aligned}$$

This finishes the proof.  $\blacksquare$

**Proof of Theorem 2.** Suppose otherwise, then there exist a sequence  $\varepsilon_n \rightarrow 0$ , and travelling solutions  $(u_n(x - c_n t, y), v_n(x - c_n t, y))$  to Euler equation (1)-(3) which are  $T$ -periodic in  $x$  and such that  $v_n$  is not identically zero,

$$\|\omega_n - 1\|_{H^s_{(0, T) \times (-1, 1)}} < \varepsilon_n, \text{ where } \omega_n(x, y) = \partial_y u_n - \partial_x v_n. \quad (21)$$

We can assume that

$$\int_0^T \int_{-1}^1 u_n(x, y) dy dx = 0, \quad (22)$$

otherwise we consider the travelling wave

$$u_n(x - (c_n + d_n)t, y) - d_n, v_n(x - (c_n + d_n)t, y),$$

with

$$d_n = \frac{1}{2T} \int_0^T \int_{-1}^1 u_n(x, y) dy dx.$$

The travelling wave solutions satisfy the vorticity equation

$$(u_n - c_n) \partial_x \omega_n + v_n \partial_y \omega_n = 0. \quad (23)$$

Because of the condition (22),  $(u_n, v_n)$  is uniquely determined by the vorticity  $\omega_n$  and

$$\|(u_n, v_n) - (y, 0)\|_{H^{s+1}_{(0, T) \times (-1, 1)}} \leq C \|\omega_n - 1\|_{H^s_{(0, T) \times (-1, 1)}} \leq C \varepsilon_n.$$

Since  $s > \frac{3}{2}$ ,

$$\|\partial_y u_n - 1\|_{L^\infty(0, T) \times (-1, 1)} \leq \|u_n - y\|_{H^{s+1}_{(0, T) \times (-1, 1)}} \leq C \varepsilon_n,$$

thus when  $n$  is large,

$$\frac{1}{2} < \partial_y u_n < \frac{3}{2}, \text{ in } (0, T) \times [-1, 1]. \quad (24)$$

Therefore, for each  $x \in (0, T)$ ,  $u_n(x, y)$  is strictly increasing for  $y \in [-1, 1]$ . We divide  $(0, T)$  into three subsets

$$P_n = \{x \mid c_n \leq u_n(x, -1)\}, \quad Q_n = \{x \mid c_n \geq u_n(x, 1)\},$$

and

$$S_n = \{x \mid u_n(x, -1) < c_n < u_n(x, 1)\}.$$

When  $x \in S_n$ , there exists a unique  $y_n(x) \in (-1, 1)$  such that  $u_n(x, y_n(x)) = c_n$ . From (23), it follows that

$$v_n(x, y_n(x)) = 0 \text{ or } \partial_y \omega_n(x, y_n(x)) = 0,$$

and we further divide  $S_n$  into two subsets

$$\begin{aligned} S_n^1 &= \{x \in S_n \mid v_n(x, y_n(x)) = 0\}, \\ S_n^2 &= \{x \in S_n \mid \partial_y \omega_n(x, y_n(x)) = 0\}. \end{aligned}$$

By the incompressible condition (2),

$$\partial_x \omega_n = \partial_x (\partial_y u_n - \partial_x v_n) = -\Delta v_n.$$

Since  $v_n(x, \pm 1) = 0$  by (3), by integration by parts and using (23), we get

$$\begin{aligned} \int_0^T \int_{-1}^1 |\nabla v_n|^2 dy dx &= \int_0^T \int_{-1}^1 v_n \partial_x \omega_n dy dx = - \int_0^T \int_{-1}^1 v_n \frac{\partial_y \omega_n}{u_n - c_n} dy dx \\ &\leq \int_{P_n} \int_{-1}^1 \left| v_n \frac{\partial_y \omega_n}{u_n - c_n} \right| dy dx + \int_{Q_n} \int_{-1}^1 \left| v_n \frac{\partial_y \omega_n}{u_n - c_n} \right| dy dx \\ &\quad + \int_{S_n^1} \int_{-1}^1 \left| v_n \frac{\partial_y \omega_n}{u_n - c_n} \right| dy dx + \int_{S_n^2} \int_{-1}^1 \left| v_n^2 \frac{\partial_y \omega_n}{u_n - c_n} \right| dy dx \\ &= I + II + III + IV. \end{aligned} \tag{25}$$

Since  $(\frac{\pi}{2})^2$  is the lowest eigenvalue of  $-\Delta$  on  $(0, T) \times (-1, 1)$  with periodic boundary condition in  $x$  and Dirichlet boundary condition in  $y$ ,

$$\|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}^2 \geq \left(\frac{\pi}{2}\right)^2 \|v_n\|_{L^2_{(0,T) \times (-1,1)}}^2.$$

Thus by Sobolev embedding, for any  $p > 1$ ,

$$\|v_n\|_{L^p_{(0,T) \times (-1,1)}} \leq C(p) \|v_n\|_{H^1_{(0,T) \times (-1,1)}} \leq C(p) \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}. \tag{26}$$

Since

$$\|\partial_y \omega_n\|_{H^{s-1}_{(0,T) \times (-1,1)}} \leq \|\omega_n - 1\|_{H^s_{(0,T) \times (-1,1)}} < \varepsilon_n, \tag{27}$$

again by Sobolev embedding,

$$\|\partial_y \omega_n\|_{L^p_{(0,T) \times (-1,1)}} \leq C(p) \varepsilon_n \text{ for any } 1 < p < \frac{2}{(2-s)_+},$$

where

$$(2-s)_+ = \max\{2-s, 0\}.$$

So we can always choose  $p_1, p_2, p_3$  such that

$$p_1 > 1, \quad 1 < p_2 < \frac{2}{(2-s)_+}, \quad 1 < p_3 < 2,$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.$$

When  $x \in P_n$ ,

$$|u_n(x, y) - c_n| \geq |u_n(x, y) - u_n(x, -1)|,$$

so

$$\begin{aligned} I &\leq \|v_n\|_{L^{p_1}_{(0,T) \times (-1,1)}} \|\partial_y \omega_n\|_{L^{p_2}_{(0,T) \times (-1,1)}} \left\| \frac{v_n}{y+1} \right\|_{L^{p_3}_{P_n \times (-1,1)}} \left\| \frac{y+1}{u_n(x, y) - u_n(x, -1)} \right\|_{L^\infty_{(0,T) \times (-1,1)}} \\ &\leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}} \left( \int_{P_n} \left\| \frac{v_n}{y+1} \right\|_{L^{p_3}(-1,1)}^{p_3} dx \right)^{\frac{1}{p_3}} \\ &\leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}} \left( \int_{P_n} \|v_n\|_{H^1(-1,1)}^{p_3} dx \right)^{\frac{1}{p_3}} \quad (\text{By Lemma 4}) \\ &\leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}} \|v_n\|_{H^1_{(0,T) \times (-1,1)}} T^{\frac{1}{p_3} - \frac{1}{2}} \leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}^2. \end{aligned}$$

Here in the second inequality above, we use (26), (27) and the estimate

$$\left| \frac{y+1}{u_n(x, y) - u_n(x, -1)} \right| = \frac{1}{|\partial_y u_n(x, \tilde{y})|} \leq 2, \quad \tilde{y} \in (-1, y),$$

due to (24). By similar estimates as that for  $I$ , we get

$$II, III \leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}^2.$$

To estimate  $IV$ , we choose

$$1 < p_1 < \min\left\{2, \frac{1}{\left(\frac{5}{2}-s\right)_+}\right\}, \quad p_2 = 2p'_1,$$

then

$$\begin{aligned} IV &\leq C \|v_n\|_{L^{p_2}_{(0,T) \times (-1,1)}}^2 \left\| \frac{\partial_y \omega_n}{y - y_n(x)} \right\|_{L^{p_1}_{S_n^2 \times (-1,1)}} \\ &\leq C \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}^2 \left( \int_{S_n^2} \|\partial_y \omega_n\|_{H^{s-1}(-1,1)}^{p_1} dx \right)^{\frac{1}{p_1}} \quad (\text{By Lemma 4}) \\ &\leq C \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}^2 \|\omega_n - 1\|_{H^s_{(0,T) \times (-1,1)}} \leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T) \times (-1,1)}}^2. \end{aligned}$$

Thus from (25) and above estimates,

$$\|\nabla v_n\|_{L^2_{(0,T)\times(-1,1)}}^2 \leq C\varepsilon_n \|\nabla v_n\|_{L^2_{(0,T)\times(-1,1)}}^2,$$

When  $n$  is large, this implies that  $\nabla v_n = 0$  and thus  $v_n = 0$ . This is a contradiction. ■

To prove Corollary 2, we use the following Lemma which follows from Theorem 2.7 of [13].

**Lemma 5** *Let  $U(y) \in C^2[-1, 1]$  be a monotone flow. Denote  $U_s^1, \dots, U_s^l$  to be all the inflection values of  $U(y)$ , that is,  $U_s^i = U(y^i)$  for some  $y^i \in [-1, 1]$  satisfying  $U''(y^i) = 0$ . Then the shear flow  $(U(y), 0)$  is linearly stable to perturbations of  $x$ -period  $T$ , if for any  $1 \leq i \leq l$ , the operator*

$$L_i = -\frac{d^2}{dy^2} + \frac{U''}{U - U_s^i}$$

*with Dirichlet boundary conditions in  $[-1, 1]$  has the lowest eigenvalue greater than  $-(\frac{2\pi}{T})^2$ .*

**Proof of Corollary 2.** We use the notations in Lemma 5. We shall show that  $L_i > 0$  for any  $1 \leq i \leq l$ , when  $\|U'(y) - 1\|_{H^s(-1,1)} \leq \varepsilon_0$  ( $s > \frac{3}{2}$ ) is sufficiently small. Then the conclusion of Corollary 2 follows from Lemma 5. Take any nonzero function  $u \in H_0^1(-1, 1)$ , then

$$(L_i u, u) = \|u'\|_{L^2(-1,1)}^2 + \int_{-1}^1 \frac{U''}{U - U_s^i} u(y)^2 dy.$$

Fix

$$1 < p_1 < \min\left\{2, \frac{1}{(\frac{5}{2} - s)_+}\right\}$$

and let  $p_2 = 2p_1'$ . Since  $u(\pm 1) = 0$ ,

$$\|u'\|_{L^2(-1,1)} \geq \frac{\pi}{2} \|u\|_{L^2(-1,1)},$$

and by Sobolev embedding

$$\|u\|_{L^{p_2}(-1,1)} \leq C \|u'\|_{L^2(-1,1)}.$$

When  $\varepsilon_0$  is small enough,

$$\frac{1}{2} < U'(y) < \frac{3}{2}, \text{ for } y \in [-1, 1],$$

thus by Lemma 4,

$$\begin{aligned} (L_i u, u) &\geq \|u'\|_{L^2}^2 - \int_{-1}^1 \left| \frac{U''}{y - y^i} \right| u(y)^2 dy \left\| \frac{y - y^i}{U - U_s^i} \right\|_{L^\infty} \\ &\geq \|u'\|_{L^2}^2 - 2 \left\| \frac{U''}{y - y^i} \right\|_{L^{p_1}} \|u\|_{L^{p_2}}^2 \\ &\geq \|u'\|_{L^2}^2 - C \|U''\|_{H^{s-1}} \|u'\|_{L^2}^2 \\ &\geq (1 - C\varepsilon_0) \|u'\|_{L^2(-1,1)}^2 > 0. \end{aligned}$$



This shows that  $L_i > 0$  when  $\varepsilon_0$  is sufficiently small and the proof is completed. ■

## 4 Linear decay problem

In this Section, we studied the linearized Euler equation around Couette flow. In the vorticity form, the linearized equation becomes

$$\omega_t + y\omega_x = 0, \quad (28)$$

where  $\omega(t, x, y)$  has  $x$ -period  $T$ . If the initial vorticity  $\omega(t=0) = \omega^0(x, y)$ , then

$$\omega(t, x, y) = \omega^0(x - ty, y). \quad (29)$$

Notice that any  $\omega = \omega(y)$  is a steady solution of (28). For a general solution (28), the  $x$ -independent component of  $\omega$  remains steady and does not affect the evolution of the vertical velocity  $v(t)$ . So we only consider  $\omega^0(x, y)$  with  $\int_0^T \omega^0(x, y) dx = 0$ , and for such functions the Fourier series representation is

$$\omega^0(x, y) = \sum_{0 \neq k \in \mathbf{Z}} e^{i\frac{2\pi}{T}kx} \omega_k^0(y).$$

Under this assumption, it is easy to see that such a vorticity field uniquely determines a velocity field satisfying

$$\int_0^T \vec{u}(x, y) dx \equiv 0. \quad (30)$$

To simplify notations, we take  $T = 2\pi$  below. We define the space  $H_x^{s_x} H_y^{s_y}$  by

$$h = \sum_{0 \neq k \in \mathbf{Z}} e^{ikx} h_k(y) \in H_x^{s_x} H_y^{s_y} \text{ iff } \|h\|_{H_x^{s_x} H_y^{s_y}} = \left( \sum_{k \neq 0} |k|^{2s_x} \|h_k\|_{H_y^{s_y}}^2 \right)^{\frac{1}{2}} < \infty.$$

**Theorem 3** Assume  $\int_0^T \omega^0(x, y) dx = 0$ . Let  $\omega(t, x, y)$  be the solution of (28) with  $\omega(t=0) = \omega^0(x, y)$ , and

$$\vec{u}(t, x, y) = (u(t, x, y), v(t, x, y))$$

is the corresponding velocity satisfying (30).

(i) If  $\omega^0(x, y) \in L_{x,y}^2$ , then

$$\|\vec{u}(t, x, y)\|_{L_{x,y}^2} \rightarrow 0, \text{ when } t \rightarrow \infty.$$

(ii) If  $\omega^0(x, y) \in H_x^{-1} H_y^1$ , then

$$\|\vec{u}(t, x, y)\|_{L_{x,y}^2} = O\left(\frac{1}{t}\right), \text{ when } t \rightarrow \infty.$$

(iii) If  $\omega^0(x, y) \in H_x^{-1}H_y^2$ , then

$$\|v(t, x, y)\|_{L_{x,y}^2} = O\left(\frac{1}{t^2}\right), \text{ when } t \rightarrow \infty.$$

(iv) If  $\omega^0(x, y) \in H_x^{-s}H_y^s$  ( $0 < s < 1$ ), then

$$\|\vec{u}(t, x, y)\|_{L_{x,y}^2} = o\left(\frac{1}{t^s}\right), \text{ when } t \rightarrow \infty.$$

(v) If  $\omega^0(x, y) \in H_x^{-1}H_y^s$  ( $1 \leq s \leq 2$ ), then

$$\|v(t, x, y)\|_{L_{x,y}^2} = O\left(\frac{1}{t^{1+s}}\right), \text{ when } t \rightarrow \infty.$$

**Proof.** Proof of (i): We shall show that  $\omega(t, x, y) \rightarrow 0$  weakly in  $L_{x,y}^2$ , then  $\|\vec{u}(t, x, y)\|_{L_{x,y}^2} \rightarrow 0$  because of the compactness of the mapping  $\omega \rightarrow \vec{u}$  in  $L_{x,y}^2$ . To show the weak convergence, we take any test function

$$\phi(x, y) = \sum e^{ikx} \phi_k(y) \in L^2.$$

Then

$$\begin{aligned} & \int_0^{2\pi} \int_{-1}^1 \omega(t, x, y) \phi(x, y) dy dx \\ &= \int_0^{2\pi} \int_{-1}^1 \omega^0(x, y) \phi(x + ty, y) dy dx \text{ (by (29))} \\ &= \sum_{k \neq 0} \int_{-1}^1 \omega_k^0(y) \phi_{-k}(y) e^{-itky} dy \\ &= \sum_{|k| \leq N} \int_{-1}^1 \omega_k^0(y) \phi_{-k}(y) e^{-itky} dy + \sum_{|k| > N} \int_{-1}^1 \omega_k^0(y) \phi_{-k}(y) e^{-itky} dy = I + II. \end{aligned}$$

For any  $\varepsilon > 0$ , we fixed  $N$  large enough such that

$$|II| \leq \left( \sum_{|k| > N} \|\omega_k^0(y)\|_{L_y^2}^2 \right)^{\frac{1}{2}} \|\phi\|_{L_{x,y}^2} < \varepsilon.$$

By Riemann-Lesbegue Theorem,  $|I| \rightarrow 0$  when  $t \rightarrow \infty$ . Since  $\varepsilon$  is arbitrary, this proves that

$$\int_0^{2\pi} \int_{-1}^1 \omega(t, x, y) \phi(x, y) dy dx \rightarrow 0 \text{ when } t \rightarrow \infty.$$

Proof of (ii): Define the space  $\tilde{H}^1$  for the stream function by

$$\tilde{H}^1 = \{ \psi \in H_{x,y}^1 \mid \psi = 0 \text{ on } \{y = \pm 1\}, \psi \text{ is } T\text{-periodic in } x \}.$$

Then

$$\psi = \sum e^{ikx} \psi_k(y) \in H_{x,y}^1$$

implies that  $\psi_k(y) \in H_0^1(-1, 1)$  and  $\sum_k \|\psi_k\|_{H_y^1}^2 < \infty$ . By a duality lemma in [12],

$$\begin{aligned} \|\vec{u}(t, x, y)\|_{L_{x,y}^2} &\leq C \sup_{\psi \in \tilde{H}^1, \|\psi\|_{H^1} \leq 1} \left| \int_0^{2\pi} \int_{-1}^1 \omega(t, x, y) \psi(x, y) dy dx \right| \\ &= C \sup_{\psi \in \tilde{H}^1, \|\psi\|_{H^1} \leq 1} \sum_{k \neq 0} \left| \int_{-1}^1 \omega_k^0(y) \psi_{-k}(y) e^{-itky} dy \right| \\ &= \frac{C}{t} \sup_{\psi} \sum_{k \neq 0} \frac{1}{|k|} \int_{-1}^1 \frac{d}{dy} (\omega_k^0(y) \psi_{-k}(y)) e^{-itky} dy \\ &\leq \frac{C}{t} \sup_{\psi} \left( \sum_{k \neq 0} \frac{1}{|k|^2} \|\omega_k^0(y)\|_{H_y^1}^2 \right)^{\frac{1}{2}} \left( \sum_k \|\psi_k\|_{H_y^1}^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C}{t} \|\omega^0\|_{H_x^{-1} H_y^1}. \end{aligned}$$

Proof of (iii): Note that

$$-\Delta v = \omega_x = \omega_x^0(x - ty, y) \text{ in } \Omega = (0, 2\pi) \times (-1, 1),$$

and  $v = 0$  on  $\{y = \pm 1\}$ . Define the function  $\varphi(t, x, y)$  by solving  $-\Delta \varphi = v$  in  $\Omega$  and  $\varphi = 0$  on  $\{y = \pm 1\}$ . Let

$$v(t, x, y) = \sum_{k \neq 0} e^{ikx} v_k(y, t) \text{ and } \varphi(t, x, y) = \sum_{k \neq 0} e^{ikx} \varphi_k(y, t),$$

where  $\varphi_k$  satisfies that

$$\left( -\frac{d^2}{dy^2} + k^2 \right) \varphi_k = v_k, \quad \varphi_k(\pm 1) = 0.$$

Then

$$\begin{aligned}
\|v\|_{L^2(\Omega)}^2 &= \int \int_{\Omega} \bar{\varphi}(t, x, y) \omega_x^0(x - ty, y) dx dy \\
&= \sum_{k \neq 0} \int_{-1}^1 ik \bar{\varphi}_k(y, t) \omega_k^0(y) e^{-itky} dy \\
&= -\frac{1}{t} \sum_{k \neq 0} \int_{-1}^1 \frac{d}{dy} (\bar{\varphi}_k(y, t) \omega_k^0(y)) e^{-itky} dy \\
&= \frac{1}{it^2} \sum_{k \neq 0} \frac{1}{k} \left( e^{-itky} dy \frac{d}{dy} (\bar{\varphi}_k(y, t) \omega_k^0(y)) \Big|_{-1}^1 - \int_{-1}^1 \frac{d^2}{dy^2} (\bar{\varphi}_k(y, t) \omega_k^0(y)) e^{-itky} dy \right) \\
&\leq \frac{C}{t^2} \sum_{k \neq 0} \frac{1}{|k|} \|\bar{\varphi}_k(y, t) \omega_k^0(y)\|_{H^2(-1,1)} \leq \frac{C}{t^2} \sum_{k \neq 0} \frac{1}{|k|} \|\varphi_k(y, t)\|_{H_y^2} \|\omega_k^0(y)\|_{H_y^2} \\
&\leq \frac{C}{t^2} \sum_{k \neq 0} \frac{1}{|k|} \|v_k(y, t)\|_{L_y^2} \|\omega_k^0(y)\|_{H_y^2} \\
&\leq \frac{C}{t^2} \left( \sum_{k \neq 0} \frac{1}{|k|^2} \|\omega_k^0(y)\|_{H_y^2}^2 \right)^{\frac{1}{2}} \left( \sum_k \|v_k(y, t)\|_{L_y^2}^2 \right)^{\frac{1}{2}} \\
&\leq \frac{C}{t^2} \|\omega^0\|_{H_x^{-1} H_y^2} \|v\|_{L^2(\Omega)},
\end{aligned}$$

therefore

$$\|v\|_{L^2(\Omega)} \leq \frac{C}{t^2} \|\omega^0\|_{H_x^{-1} H_y^2}.$$

The decay rates in (iv) and (v) follow from (i)-(iii) by interpolation. This finishes the proof of Theorem 3.  $\blacksquare$

**Remark 2** *The decay rates  $O(1/t)$  for  $\|u\|_{L^2}$  and  $O(1/t^2)$  for  $\|v\|_{L^2}$  in Theorem 3 (ii), (iii) are optimal. They cannot be improved even for smooth initial vorticity. Consider a single mode solution with  $\omega^0(x, y) = e^{ikx} \phi(y)$  and  $\phi(y) \in C^\infty(-1, 1)$ . Then  $\omega(t, x, y) = e^{ikx} e^{-itky} \phi(y)$  and by Poisson's equation the stream function is  $\psi(t, x, y) = e^{ikx} \psi_k(t, y)$ , where  $\psi_k(t, y)$  satisfies*

$$\left( -\frac{d^2}{dy^2} + k^2 \right) \psi_k(t, y) = e^{-itky} \phi(y), \quad \psi_k(t, \pm 1) = 0.$$

Denote  $G(y, y_0)$  to be the Green's function given by

$$G(y, y_0) = \frac{1}{k \sinh k} \sinh k(y_{<} + 1) \sinh k(1 - y_{>}),$$

where  $y_{<}$  and  $y_{>}$  are the lesser and greater of  $y$  and  $y_0$  respectively. Then we have

$$\psi_k(t, y) = \int_{-1}^1 G(y, y_0) e^{-ikt y_0} \phi(y_0) dy_0, \quad (31)$$

and the  $1/t^2$  decay of  $\psi_k(t, y)$  follows from integration by parts because  $G(y, y_0)$  is  $C^1$  and its derivative is piecewise differentiable. Moreover, by explicit evaluation of the integral in (31), it can be shown that

$$\psi_k(t, y) = \frac{1}{t^2} f_k(y) e^{-ikt y} + O\left(\frac{1}{t^3}\right)$$

where  $f_k(y)$  is not identically zero. Thus

$$\|u(t, x, y)\|_{L^2_{x,y}} = \|\psi'_k\|_{L^2_y} \sim \frac{1}{t}$$

and

$$\|v(t, x, y)\|_{L^2_{x,y}} = k \|\psi_k\|_{L^2_y} \sim \frac{1}{t^2}.$$

The same decay rate  $O(\frac{1}{t^2})$  for  $v(t, x, y)$  was obtained in ([3], [2], [17]). Our main purpose in this section is to get the linear decay for most general perturbations. We note that the calculations in [5] contain mistakes and only yield the estimate

$$\psi_k(t, y) = \frac{1}{t} g_k(y) e^{-ikt y} + O\left(\frac{1}{t^2}\right),$$

from which only  $O(\frac{1}{t})$  decay is obtained for  $v(t, x, y)$  and no decay is obtained for  $u(t, x, y) = \psi'_k(t, y)$ .

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