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Journal of Functional Analysis 255 (2008) 1191–1224
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Instability of nonlinear dispersive solitary waves

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Received 9 October 2007; accepted 2 June 2008

Available online 11 July 2008

Communicated by I. Rodnianski

Abstract

We consider linear instability of solitary waves of several classes of dispersive long wave models. They include generalizations of KDV, BBM, regularized Boussinesq equations, with general dispersive operators and nonlinear terms. We obtain criteria for the existence of exponentially growing solutions to the linearized problem. The novelty is that we dealt with models with nonlocal dispersive terms, for which the spectra problem is out of reach by the Evans function technique. For the proof, we reduce the linearized problem to study a family of nonlocal operators, which are closely related to properties of solitary waves. A continuation argument with a moving kernel formula is used to find the instability criteria. These techniques have also been extended to study instability of periodic waves and of the full water wave problem.

Keywords: Instability; Solitary waves; Dispersive long waves

1. Introduction

We consider the stability and instability of solitary wave solutions of several classes of equations modeling weakly nonlinear, dispersive long waves. More specifically, we establish criteria for the linear exponential instability of solitary waves of BBM, KDV, and regularized Boussinesq type equations. These equations respectively have the forms:

1. BBM type

$$\partial_t u + \partial_x u + \partial_x f(u) + \partial_t \mathcal{M} u = 0; \tag{1.1}$$

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.06.003

2. KDV type

$$\partial_t u + \partial_x f(u) - \partial_x \mathcal{M} u = 0; \tag{1.2}$$

3. Regularized Boussinesq (RBou) type

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2 f(u) + \partial_t^2 \mathcal{M} u = 0. \tag{1.3}$$

Here, the pseudo-differential operator $\mathcal{M}$ is defined as

$$(\mathcal{M}g)^{\wedge}(k) = \alpha(k)\hat{g}(k),$$

where $\hat{g}$ is the Fourier transformation of g. Throughout this paper, we assume:

(i) f is C^1 with $f(0) = f'(0) = 0$, and $f(u)/u \to \infty$.
(ii) $a|k|^m \leqslant \alpha(k) \leqslant b|k|^m$ for large k, where $m \geqslant 1$ and $a, b > 0$.

If $f(u) = u^2$ and $\mathcal{M} = -\partial_x^2$, the above equations recover the original BBM [11], KDV [28], and regularized Boussinesq [49] equations, which have been used to model the unidirectional propagation of water waves of long wavelengths and small amplitude. As explained in [11], the nonlinear term $f(u)$ is related to nonlinear effects suffered by the waves being modeled, while the form of the symbol α is related to dispersive and possibly, dissipative effects. If $\alpha(k)$ is a polynomial function of k, then $\mathcal{M}$ is a differential operator and in particular is a local operator. On the other hand, in many situations in fluid dynamics and mathematical physics, equations of the above types arise in which $\alpha(k)$ is not a polynomial and hence the operator $\mathcal{M}$ is non-local. Some examples include: Benjamin–Ono equation [8], Smith equation [45] and intermediate long-wave equation [29], which are of KDV types with $\alpha(k) = |k|$, $\sqrt{1+k^2} - 1$ and $k\coth(kH) - H^{-1}$, respectively.

Below we assume $\alpha(k) \geqslant 0$, since the results and proofs can be easily modified for cases of sign-changing symbols (see Section 5, (b)). Each of Eqs. (1.1)–(1.3) admits solitary-wave solutions of the form $u(x,t) = u_c(x - ct)$ for $c > 1$, $c > 0$ and $c^2 > 1$, respectively, where $u_c(x) \to 0$ as $|x| \to \infty$. For example, the KDV solitary-wave solutions have the form [28]

$$u_c(x) = 3c\,\mathrm{sech}^2(\sqrt{c}x/2)$$

and for the Benjamin–Ono equation [8]

$$u_c(x) = \frac{4c}{1 + c^2x^2}.$$

For a broad class of symbols α, the existence of solitary-wave solutions has been established [10,12]. For many equations such as the classical KDV and BBM equations, the solitary waves are positive, symmetric and single-humped. But the oscillatory solitary waves with multiple humps are not uncommon [5,7], especially for the sign changing $\alpha(k)$. In our study, we do not assume any additional property of solitary waves besides their decay at infinity. We consider the linearized equations around solitary waves in the traveling frame $(x - ct, t)$ and seek a growing mode solution of the form $e^{\lambda t}u(x)$ with $\mathrm{Re}\,\lambda > 0$. Define the operator $\mathcal{L}_0$ by (2.2), (4.4), and

(3.5), and the momentum function $P(c)$ by (2.23), (4.8), and (3.6), for BBM, KDV and RBou type equations, respectively.

Theorem 1. *For solitary waves $u_c(x-ct)$ of Eqs.* (1.1)–(1.3)*, we assume*

$$\ker \mathcal{L}_0 = \{u_{cx}\}. \tag{1.4}$$

Denote by $n^-(\mathcal{L}_0)$ the number (counting multiplicity) of negative eigenvalues of the operators $\mathcal{L}_0$. Then there exists a purely growing mode $e^{\lambda t}u(x)$ with $\lambda > 0$, $u \in H^m(\mathbf{R})$ to the linearized equations (2.3), (4.2) *and* (3.2)*, if one of the following two conditions is true*:

(i) $n^-(\mathcal{L}_0)$ *is even and* $dP/dc > 0$.
(ii) $n^-(\mathcal{L}_0)$ *is odd and* $dP/dc < 0$.

Note that the operators $\mathcal{L}_0$ are obtained from the linearization of elliptic type equations satisfied by solitary waves, and $P(c) = Q(u_c)$ where $Q(u)$ is the momentum invariant due to the translation symmetry of the evolution equations (1.1)–(1.3). For example, for KDV type equations $Q(u) = \frac{1}{2}\int u^2\,dx$. The assumption (1.4) can be proved for $\mathcal{M} = -\partial_x^2$ and for some non-local dispersive operators [1,3]. It implies that the solitary wave branch $u_c(x)$ is unique for the parameter c. More discussions about the spectrum assumptions for $\mathcal{L}_0$ are in Section 5, part (a).

Let us relate our result to the literature on stability and instability of solitary waves. The first rigorous proof of stability of solitary waves was obtained by Benjamin [9], for the original KDV equation. Benjamin's idea is to show that stable solitary waves are local energy minimizers under the constraint of constant momentum. This idea was already anticipated by Boussinesq [16] and had been extended to prove stability results in more general settings [2,4,13,22,48]. In particular, it was shown in [15,46] that for KDV and BBM type equations, the solitary waves are orbitally stable in the energy norm if and only if $dP/dc > 0$, under the hypothesis

$$\ker \mathcal{L}_0 = \{u_{cx}\}, \quad \text{and} \quad n^-(\mathcal{L}_0) = 1. \tag{1.5}$$

For power like nonlinear terms and dispersive operators with symbols $\alpha(k) = |k|^\mu$, the function $P(c)$ can be computed by scaling and thus the more explicit stability criteria was obtained (see [15,46]). The stability criterion $dP/dc > 0$ in [15,46] was proved by a straight application of the abstract theory of [22], and this was also proved in [48]. The instability proof of [22] cannot apply directly to KDV and BBM cases. In [15,46], the proof of [22] was modified to yield the instability criterion $dP/dc < 0$ under the assumption (1.5), by estimating the sublinear growth of the anti-derivative of the solution. A less technical way of modification (introduced in [31]) is described in Appendix A for general settings. Applying Theorem 1 to the KDV and BBM cases under the assumption (1.5), we recover the instability criterion $dP/dc < 0$ in [15,46], and furthermore it helps to clarify the mechanism of this instability by finding a non-oscillatory and exponentially growing solution to the linearized problem. We note that the nonlinear instability proved in [15,46] is in the energy norm $H^{m/2}$ and there is no estimate of the time scale for the growth of instability. The linear instability result might be the first step toward proving a stronger nonlinear instability result in the L^2 norm with an exponential growth.

When $\mathcal{M} = -\partial_x^2$, Pego and Weinstein [41] studied the spectral problem for solitary waves of BBM, KDV and RBou equations by the Evans function technique [6,21], and a purely growing mode was shown to exist when $dP/dc < 0$. Since for the case $\mathcal{M} = -\partial_x^2$ it can be shown that $\ker \mathcal{L}_0 = \{u_{cx}\}$ and $n^-(\mathcal{L}_0) = 1$ (see Section 5(a)), the result of [41] is a special case of Theorem 1. The novelty of our result is to allow general dispersive operators $\mathcal{M}$, particularly the non-local operators, for which the spectral problem cannot be studied via the Evans functions. More comparisons with the Evans function technique are found in Section 5 (c). Moreover, our instability criteria for cases when $n^-(\mathcal{L}_0) \geqslant 2$ appear to be new, even for the relatively well-studied BBM and KDV type equations. The situation $n^-(\mathcal{L}_0) \geqslant 2$ might arise for highly oscillatory solitary waves (i.e. [5,7]). Even for single-humped and positive solitary waves, it is not necessarily true that $n^-(\mathcal{L}_0) = 1$ since there is no Sturm theory for the operator $\mathcal{L}_0$ with a general dispersive term $\mathcal{M}$. One important example is the large solitary waves of the full water wave problem. In [34], a similar instability criterion is derived for solitary water waves, in terms of an operator $\mathcal{L}_0$ with $\alpha(k) = k\coth(kH)$, for which $n^-(\mathcal{L}_0)$ grows without bound [43] as the solitary wave approaches the highest wave even if all solitary water waves are known to be single-humped and positive.

Let us also discuss some implications of our result for solitary wave stability. The solitary waves of regularized Boussinesq equations are known [41,44] to be highly indefinite energy saddles, and therefore their stability cannot be pursued by showing energy minimizers as in the BBM and KDV cases. More interestingly, solitary waves of the full water wave are also indefinite energy saddles [14,27] and thus the study of stability of RBou solitary waves might shed some light on the full water wave problem. We note that energy saddles are not necessarily unstable. Indeed, it was shown in [42] that small solitary waves of the regularized Boussinesq equation are spectrally stable, that is, there are no growing modes to the linearized equation. So far, we do not know any method of proving nonlinear stability for solitary waves of energy saddle type. Their spectral stability is naturally the first step and our next theorem might be useful in such a study.

Theorem 2. *Consider solitary waves $u_c(x-ct)$ of Eqs.* (1.1)–(1.3)*, and assume* $\ker \mathcal{L}_0 = \{u_{cx}\}$. *Suppose all possible growing modes are purely growing and the spectral stability exchanges at c_0, then $P'(c_0) = 0$.*

For the original regularized Boussinesq equation, it was shown in [41, p. 79] that $P'(c) > 0$ for any $c^2 > 1$. By Theorem 2 and the spectral stability of small solitary waves [42], it follows that either all solitary waves are spectrally stable or there is oscillatory instability for some solitary waves. So the spectral stability of large solitary waves will follow if one could exclude the oscillatory instability, namely, show that any growing mode must be purely growing. For BBM and KDV type equations, when $n^-(\mathcal{L}_0) \geqslant 2$, the solitary waves are also of energy saddle type and their stability cannot be studied by the usual energy arguments. Above remarks also apply to these cases. We note that for KDV and BBM equations, under the hypothesis (1.5) the oscillatory instability can be excluded as in the case $\mathcal{M} = -\partial_x^2$ [41, p. 79], by adapting the arguments for finite-dimensional Hamiltonian systems [39].

We briefly discuss the proof of Theorem 1. The growing mode equations (2.4), (3.3) and (4.3) are non-self-adjoint eigenvalue problems for variable coefficient operators and rather few systematic techniques are available to study them. Our key step is to reformulate the eigenvalue problems in terms of a family of operators $\mathcal{A}^\lambda$, which has the form of $\mathcal{M}$ plus some non-local but bounded terms. The idea is to try to relate the eigenvalue problems to the elliptic type problems for solitary waves. The existence of a purely growing mode is equivalent to find some $\lambda > 0$ such that $\mathcal{A}^\lambda$ has a nontrivial kernel. This is achieved by a continuation strategy to exploit the difference of the spectra of $\mathcal{A}^\lambda$ for λ near infinity and zero. First, we show that the essential spectrum of $\mathcal{A}^\lambda$ lies in the right-half complex plane and away from the imaginary axis. For large λ, the spectra of the operator $\mathcal{A}^\lambda$ is shown to lie entirely in the right-half complex plane. So if for small λ, the operator $\mathcal{A}^\lambda$ has an odd number of eigenvalues in the left-half plane, then the spectrum of $\mathcal{A}^\lambda$ must go across the origin at some $\lambda > 0$ where a purely growing mode is found. The zero-limit operator $\mathcal{A}^0$ is exactly the operator $\mathcal{L}_0$. Since the convergence of $\mathcal{A}^\lambda$ to $\mathcal{L}_0$ is rather weak, the usual perturbation theories do not apply and we use the asymptotic perturbation theory of Vock and Hunziker [47] to study perturbations of the eigenvalues of $\mathcal{L}_0$. To count the number of eigenvalues of $\mathcal{A}^\lambda$ (λ small) in the left-half plane, we need to know how the zero eigenvalue of $\mathcal{L}_0$ is perturbed, for which we derive a moving kernel formula. The instability criteria in Theorem 1 and the transition point formula in Theorem 2 follow from this moving kernel formula. One important technical issue in the proof is to use the decay of solitary waves to obtain a priori estimates and gain certain compactness.

The approach of using non-local dispersion operators $\mathcal{A}^\lambda$ with a continuation argument to find instability criteria originated from our previous works [30,32,33] on 2D ideal fluid and 1D electrostatic plasma, which have been extended to study instability of galaxies [24] and 3D electromagnetic plasmas [36,37]. The consideration of the movement of $\ker \mathcal{A}^0$ was suggested in [33, Remark 3.2]. The techniques developed in this paper have also been extended to get stability criteria for periodic dispersive waves [35], and to prove instability of large solitary waves for the full water wave problem [34]. This rather general approach might also be useful for studying instability for higher-dimensional problems or coupled systems of dispersive waves, which have been much less understood.

This paper is organized as follows. In Section 2, we give details of the proof of Theorem 1 for the BBM case. Section 3 treats the RBou case, whose proof is rather similar to the BBM case. The KDV case has some subtle difference with the previous two cases and is discussed in Section 4. In Section 5, we discuss some extensions and open issues. Appendix A gives an alternative way of modifying the nonlinear instability proof of [22] for general dispersive long wave models.

2. The BBM type equations

Consider a traveling solution $u(x,t) = u_c(x - ct)$ $(c > 1)$ of the BBM type equation (1.1). Then u_c satisfies the equation

$$\mathcal{M}u_c + \left(1 - \frac{1}{c}\right)u_c - \frac{1}{c} f(u_c) = 0. \tag{2.1}$$

We define the following operator $\mathcal{L}_0 : H^m \to L^2$ by the linearization of (2.1)

$$\mathcal{L}_0 = \mathcal{M} + \left(1 - \frac{1}{c}\right) - \frac{1}{c} f'(u_c). \tag{2.2}$$

The linearized equation of (1.1) in the traveling frame $(x - ct, t)$ is

$$(\partial_t - c\partial_x)(u + \mathcal{M}u) + \partial_x\big(u + f'(u_c)u\big) = 0. \tag{2.3}$$

For a growing mode solution $e^{\lambda t}u(x)$ $(\operatorname{Re}\lambda > 0)$ of (2.3), $u(x)$ satisfies

$$(\lambda - c\partial_x)(u + \mathcal{M}u) + \partial_x\big(u + f'(u_c)u\big) = 0, \tag{2.4}$$

which can be written as

$$\mathcal{M}u + u + \frac{\partial_x}{\lambda - c\partial_x}\big(u + f'(u_c)u\big) = 0.$$

This motivates us to define a family of operators $\mathcal{A}^\lambda : H^m \to L^2$ by

$$\mathcal{A}^\lambda u = \mathcal{M}u + u + \frac{\partial_x}{\lambda - c\partial_x}\big(u + f'(u_c)u\big).$$

The existence of a growing mode is reduced to find $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ such that the operator $\mathcal{A}^\lambda$ has a nontrivial kernel. Below, we seek a purely growing mode with $\lambda > 0$. We use a continuation strategy, by exploiting the difference of the spectra of the operators $\mathcal{A}^\lambda$ for λ near infinity and zero. We divide the proof into several steps.

2.1. The properties of $\mathcal{A}^\lambda$

Define the following operators

$$\mathcal{D} = c\partial_x \quad \text{and} \quad \mathcal{E}^{\lambda,\pm} = \frac{\lambda}{\lambda \pm \mathcal{D}}.$$

Then the operator $\mathcal{A}^\lambda$ $(\lambda > 0)$ can be written as

$$\mathcal{A}^\lambda = \mathcal{M} + 1 - \frac{1}{c}\big(1 - \mathcal{E}^{\lambda,-}\big)\big(1 + f'(u_c)\big).$$

Throughout this paper, for a sequence of operators P_n and $P : L^2(R) \to L^2(R)$ we say that P_n converge to P strongly in L^2 when $n \to +\infty$, provided that P_n converge to P in the strong topology of operators in $L^2(\mathbf{R})$. That is, for any $u \in L^2(\mathbf{R})$, $\|P_n u - Pu\|_{L^2(\mathbf{R})} \to 0$ when $n \to +\infty$.

Lemma 2.1.

(a) *For $\lambda > 0$, the operators $\mathcal{E}^{\lambda,\pm}$ are continuous in λ and*

$$\big\|\mathcal{E}^{\lambda,\pm}\big\|_{L^2 \to L^2} \leqslant 1, \tag{2.5}$$

$$\big\|1 - \mathcal{E}^{\lambda,\pm}\big\|_{L^2 \to L^2} \leqslant 1. \tag{2.6}$$

(b) *When $\lambda \to 0+$, $\mathcal{E}^{\lambda,\pm}$ converges to* 0 *strongly in L^2.*
(c) *When $\lambda \to +\infty$, $\mathcal{E}^{\lambda,\pm}$ converges to* 1 (*the identity operator*) *strongly in L^2.*

Proof. We have

$$\|\mathcal{E}^{\lambda,\pm}\phi\|_{L^2}^2 = \int_{\mathbb{R}} \left|\frac{\lambda}{\lambda \pm ick}\right|^2 |\hat{\phi}(k)|^2\, dk \leqslant \int_{\mathbb{R}} |\hat{\phi}(k)|^2\, dk = \|\phi\|_{L^2}^2$$

and (2.5) follows. Similarly, we get (2.6). By the dominated convergence theorem,

$$\|\mathcal{E}^{\lambda,\pm}\phi\|_{L^2}^2 = \int_{\mathbb{R}} \left|\frac{\lambda}{\lambda \pm ick}\right|^2 |\hat{\phi}(k)|^2\, dk \to 0,$$

when $\lambda \to 0+$. Thus $\mathcal{E}^{\lambda,\pm} \to 0$ in the strong topology of operators in $L^2(\mathbf{R})$. The proof of (c) is similar and we skip it. □

Corollary 1. *For $\lambda > 0$, the operator $\mathcal{A}^\lambda$ converges to $\mathcal{L}_0$ strongly in L^2 when $\lambda \to 0+$, and converges to $\mathcal{M} + 1$ strongly in L^2 when $\lambda \to +\infty$.*

The following theorem states that the essential spectrum of $\mathcal{A}^\lambda$ is to the right and away from the imaginary axis.

Proposition 1. *For any $\lambda > 0$, we have*

$$\sigma_{\text{ess}}(\mathcal{A}^\lambda) \subset \left\{ z \;\middle|\; \operatorname{Re} z \geqslant \frac{1}{2}\left(1 - \frac{1}{c}\right) > 0 \right\}. \tag{2.7}$$

The proof of Proposition 1 is based on the following lemmas.

Lemma 2.2. *Consider any sequence*

$$\{u_n\} \in H^m(\mathbf{R}), \quad \|u_n\|_2 = 1, \quad \operatorname{supp} u_n \subset \{x \mid |x| \geqslant n\}.$$

Then for any complex number z with $\operatorname{Re} z < \frac{1}{2}(1 - \frac{1}{c})$, we have

$$\operatorname{Re}\big((\mathcal{A}^\lambda - z)u_n, u_n\big) \geqslant \frac{1}{4}\left(1 - \frac{1}{c}\right),$$

when n is large enough.

Proof. We have

$$\begin{aligned}
&\operatorname{Re}\big((\mathcal{A}^\lambda - z)u_n, u_n\big) \\
&\quad = \big((\mathcal{M}+1)u_n, u_n\big) - \operatorname{Re} z - \operatorname{Re}\left(\frac{1}{c}\big(1 - \mathcal{E}^{\lambda,-}\big)\big(1 + f'(u_c)\big)u_n, u_n\right) \\
&\quad = \big((\mathcal{M}+1)u_n, u_n\big) - \operatorname{Re} z - \frac{1}{c}\operatorname{Re}\big(\big(1 + f'(u_c)\big)u_n, \big(1 - \mathcal{E}^{\lambda,+}\big)u_n\big)
\end{aligned}$$

$$\geqslant 1-\frac{1}{2}\left(1-\frac{1}{c}\right)-\frac{1}{c}\left(1+\max_{|x|\geqslant n}\left|f'(u_c)\right|\right)\left\|\left(1-\mathcal{E}^{\lambda,+}\right)u_n\right\|_2$$

$$\geqslant \frac{1}{2}\left(1-\frac{1}{c}\right)-\frac{1}{c}\max_{|x|\geqslant n}\left|f'(u_c)\right| \quad \big(\text{by Lemma 2.1(a)}\big)$$

$$\geqslant \frac{1}{4}\left(1-\frac{1}{c}\right), \quad \text{when } n \text{ is big enough.} \qquad \square$$

To study the essential spectrum of $\mathcal{A}^\lambda$, first we introduce the Zhislin spectrum $Z(\mathcal{A}^\lambda)$ [25]. A Zhislin sequence for $\mathcal{A}^\lambda$ and $z \in \mathbb{C}$ is a sequence

$$\{u_n\} \in H^m, \quad \|u_n\|_2 = 1, \quad \operatorname{supp} u_n \subset \left\{x \mid |x| \geqslant n\right\}$$

and $\|(\mathcal{A}^\lambda - z)u_n\|_2 \to 0$ as $n \to \infty$. The set of all z such that a Zhislin sequence exists for $\mathcal{A}^\lambda$ and z is denoted $Z(\mathcal{A}^\lambda)$. From the above definition and Lemma 2.2, we readily have

$$Z\big(\mathcal{A}^\lambda\big) \subset \left\{z \in \mathbb{C} \,\middle|\, \operatorname{Re} z \geqslant \frac{1}{2}\left(1-\frac{1}{c}\right)\right\}. \tag{2.8}$$

Another related spectrum is the Weyl spectrum $W(\mathcal{A}^\lambda)$ [25]. A Weyl sequence for $\mathcal{A}^\lambda$ and $z \in \mathbb{C}$ is a sequence $\{u_n\} \in H^m$, $\|u_n\|_2 = 1$, $u_n \to 0$ weakly in L^2 and $\|(\mathcal{A}^\lambda - z)u_n\|_2 \to 0$ as $n \to \infty$. The set $W(\mathcal{A}^\lambda)$ is all $z \in \mathbb{C}$ such that a Weyl sequence exists for $\mathcal{A}^\lambda$ and z. By [25, Theorem 10.10], $W(\mathcal{A}^\lambda) \subset \sigma_{\mathrm{ess}}(\mathcal{A}^\lambda)$ and the boundary of $\sigma_{\mathrm{ess}}(\mathcal{A}^\lambda)$ is contained in $W(\mathcal{A}^\lambda)$. So it suffices to show that $W(\mathcal{A}^\lambda) = Z(\mathcal{A}^\lambda)$, which together with (2.8) implies (2.7). By [25, Theorem 10.12], the proof of $W(\mathcal{A}^\lambda) = Z(\mathcal{A}^\lambda)$ is reduced to prove the following lemma.

Lemma 2.3. *Given $\lambda > 0$. Let $\chi \in C_0^\infty(\mathbf{R})$ be a cut-off function such that $\chi|_{\{|x|\leqslant R_0\}} = 1$, for some $R_0 > 0$. Define $\chi_d = \chi(x/d)$, $d > 0$. Then for each d, $\chi_d(\mathcal{A}^\lambda - z)^{-1}$ is compact for some $z \in \rho(\mathcal{A}^\lambda)$, and that there exists $C(d) \to 0$ as $d \to \infty$ such that for any $u \in C_0^\infty(\mathbf{R})$,*

$$\left\|\left[\mathcal{A}^\lambda, \chi_d\right]u\right\|_2 \leqslant C(d)\left(\left\|\mathcal{A}^\lambda u\right\|_2 + \|u\|_2\right). \tag{2.9}$$

Proof. We write $\mathcal{A}^\lambda = \mathcal{M} + 1 + \mathcal{K}^\lambda$, where

$$\mathcal{K}^\lambda = \frac{1}{c}\left(1-\mathcal{E}^{\lambda,-}\right)\left(1+f'(u_c)\right) : L^2 \to L^2 \tag{2.10}$$

is bounded. So $-k \in \rho(\mathcal{A}^\lambda)$ when $k > 0$ is sufficiently large. The compactness of $\chi_d(\mathcal{A}^\lambda + k)^{-1}$ is a corollary of the local compactness of $H^m(\mathbf{R}) \hookrightarrow L^2(\mathbf{R})$. To show (2.9), we note that the graph norm of $\mathcal{A}^\lambda$ is equivalent to $\|\cdot\|_{H^m}$. Below, we use C to denote a generic constant. First, we have

$$\left[\mathcal{K}^\lambda, \chi_d\right] = -\frac{1}{c}\left[\mathcal{E}^{\lambda,-}, \chi_d\right]\left(1+f'(u_c)\right) = -\frac{1}{c}\frac{\lambda}{\lambda-\mathcal{D}}[\mathcal{D}, \chi_d]\frac{1}{\lambda-\mathcal{D}}\left(1+f'(u_c)\right)$$

$$= \frac{1}{\lambda c d}\mathcal{E}^{\lambda,-}\chi'(x/d)\mathcal{E}^{\lambda,-}\left(1+f'(u_c)\right)$$

and thus

$$\big\|[\mathcal{K}^{\lambda}, \chi_d]\big\|_{L^2\to L^2} \leqslant \frac{C}{\lambda d}. \tag{2.11}$$

Let $l = [m]$ to be the largest integer no greater than m and $\delta = m - [m] \in [0, 1)$. Define the following two operators

$$\mathcal{M}_1 = \begin{cases} 1 + (\frac{d}{dx})^l & \text{if } l \neq 2 \bmod 4, \\ 1 - (\frac{d}{dx})^l & \text{if } l = 2 \bmod 4. \end{cases} \tag{2.12}$$

and $\mathcal{M}_2$ is the Fourier multiplier operator with the symbol

$$n(k) = \begin{cases} \frac{\alpha(k)}{1+(ik)^l} & \text{if } l \neq 2 \bmod 4, \\ \frac{\alpha(k)}{1-(ik)^l} & \text{if } l = 2 \bmod 4. \end{cases} \tag{2.13}$$

Then $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1$ and

$$[\mathcal{M}, \chi_d] = \mathcal{M}_2[\mathcal{M}_1, \chi_d] + [\mathcal{M}_2, \chi_d]\mathcal{M}_1.$$

We study $[\mathcal{M}_2, \chi_d]$ in two cases. When $\delta = 0$, that is, when m is an integer, for any $v \in C_0^\infty(\mathbf{R})$ we follow [19, pp. 127, 128] to write

$$\begin{aligned}
[\mathcal{M}_2, \chi_d]v &= -(2\pi)^{-\frac{1}{2}} \int \check{n}(x-y)\big(\chi_d(x) - \chi_d(y)\big)v(y)\,dy \\
&= -\int_0^1\!\!\int (2\pi)^{-\frac{1}{2}}(x-y)\check{n}(x-y)\chi_d'\big(\rho(x-y)+y\big)v(y)\,dy\,d\rho \\
&= \int_0^1 A_\rho v\,d\rho,
\end{aligned}$$

where A_ρ is the integral operator with the kernel function

$$K_\rho(x,y) = -(2\pi)^{-\frac{1}{2}}(x-y)\check{n}(x-y)\chi_d'\big(\rho(x-y)+y\big).$$

Note that $\beta(x) = x\check{n}(x)$ is the inverse Fourier transformation of $in'(k)$ and $n'(k) \in L^2$ when $l = m$, so $\beta(x) \in L^2$. Thus

$$\begin{aligned}
\iint \big|K_\rho(x,y)\big|^2\,dx\,dy &= 2\pi \iint |\beta|^2(x-y)\big|\chi_d'\big|^2\big(\rho(x-y)+y\big)\,dx\,dy \\
&= 2\pi \iint |\beta|^2(x)\big|\chi_d'\big|^2(y)\,dx\,dy = 2\pi\|\beta\|_{L_2}^2\big\|\chi_d'\big\|_{L^2}^2 \\
&= \frac{2\pi}{d}\|\beta\|_{L_2}^2\|\chi'\|_{L^2}^2.
\end{aligned}$$

So

$$\left\|[\mathcal{M}_2, \chi_d]\right\|_{L^2\to L^2} \leqslant \frac{C}{d^{\frac{1}{2}}}$$

and

$$\left\|[\mathcal{M}_2, \chi_d]\mathcal{M}_1 u\right\|_{L^2} \leqslant \frac{C}{d^{\frac{1}{2}}}\|\mathcal{M}_1 u\|_{L^2} \leqslant \frac{C}{d^{\frac{1}{2}}}\|u\|_{H^m}.$$

When $\delta > 0$, we define two Fourier multiplier operators $\mathcal{M}_3$ and $\mathcal{M}_4$ with symbols $1 + |k|^\delta$ and $n_1(k) = n(k)/(1 + |k|^\delta)$, respectively. Then $\mathcal{M}_2 = \mathcal{M}_3\mathcal{M}_4$ and

$$[\mathcal{M}_2, \chi_d] = \mathcal{M}_4[\mathcal{M}_3, \chi_d] + [\mathcal{M}_4, \chi_d]\mathcal{M}_3.$$

Since $n_1'(k) \in L^2$, by the same argument as above, we have

$$\left\|[\mathcal{M}_4, \chi_d]\right\|_{L^2\to L^2} \leqslant \frac{C}{d^{\frac{1}{2}}}.$$

By [40, Theorem 3.3, p. 213],

$$\left\|[\mathcal{M}_3, \chi_d]\right\|_{L^2\to L^2} \leqslant C(\delta)\left\||D|^\delta \chi_d\right\|_*,$$

where $|D|^\delta$ is the fractional differentiation operator with the symbol $|k|^\delta$ and $\|\cdot\|_*$ is the BMO norm. By using Fourier transformations, it is easy to check that

$$\left(|D|^\delta \chi_d\right)(x) = \frac{1}{d^\delta}\left(|D|^\delta \chi\right)\left(\frac{x}{d}\right).$$

So

$$\left\||D|^\delta \chi_d\right\|_* \leqslant 2\left\||D|^\delta \chi_d\right\|_{L^\infty} \leqslant \frac{2}{d^\delta}\left\||D|^\delta \chi\right\|_{L^\infty}$$

and therefore

$$\left\|[\mathcal{M}_3, \chi_d]\right\|_{L^2\to L^2} \leqslant \frac{C}{d^\delta}.$$

Since $\|\mathcal{M}_4\|_{L^2\to L^2}$ is bounded, we have

$$\begin{aligned}\left\|[\mathcal{M}_2, \chi_d]\mathcal{M}_1 u\right\|_{L^2} &\leqslant \left\|\mathcal{M}_4[\mathcal{M}_3, \chi_d]\mathcal{M}_1 u\right\|_{L^2} + \left\|[\mathcal{M}_4, \chi_d]\mathcal{M}_3\mathcal{M}_1 u\right\|_{L^2} \\ &\leqslant \frac{C}{d^\delta}\|\mathcal{M}_1 u\|_{L^2} + \frac{C}{d^{\frac{1}{2}}}\|\mathcal{M}_3\mathcal{M}_1 u\|_{L^2} \leqslant C\left(\frac{1}{d^\delta} + \frac{1}{d^{\frac{1}{2}}}\right)\|u\|_{H^m}.\end{aligned}$$

So in both cases,

$$\big\|[\mathcal{M}_2,\chi_d]\mathcal{M}_1 u\big\|_{L^2} \leqslant C(d)\|u\|_{H^m}, \quad \text{with } C(d) \to 0 \text{ as } d \to \infty. \tag{2.14}$$

Since

$$[\mathcal{M}_1,\chi_d] = \sum_{j=1}^{l} C_j^l \frac{d^j\chi_d}{dx^j}\frac{d^{l-j}}{dx^{l-j}} \text{ or } -\sum_{j=1}^{l} C_j^l \frac{d^j\chi_d}{dx^j}\frac{d^{l-j}}{dx^{l-j}},$$

and

$$\frac{d^j\chi_d}{dx^j}(x) = \frac{1}{d^j}\chi^{(j)}\left(\frac{x}{d}\right) =: \frac{1}{d^j}\chi_d^{(j)},$$

we have

$$\big\|\mathcal{M}_2[\mathcal{M}_1,\chi_d]u\big\|_2 \leqslant \sum_{j=1}^{l}\frac{C_j^l}{d^j}\left(\big\|[\mathcal{M}_2,\chi_d^{(j)}]u^{(l-j)}\big\|_{L^2} + \|\chi\|_{C^l}\big\|\mathcal{M}_2 u^{(l-j)}\big\|_{L^2}\right).$$

By similar estimates as above, when $\delta = 0$,

$$\big\|[\mathcal{M}_2,\chi_d^{(j)}]u^{(l-j)}\big\|_{L^2} \leqslant \frac{C}{d^{\frac{1}{2}}}\big\|u^{(l-j)}\big\|_{L^2} \leqslant \frac{C}{d^{\frac{1}{2}}}\|u\|_{H^m}$$

and when $\delta > 0$,

$$\big\|[\mathcal{M}_2,\chi_d^{(j)}]u^{(l-j)}\big\|_{L^2} \leqslant \frac{C}{d^{\delta}}\big\|u^{(l-j)}\big\|_{L^2} + \frac{C}{d^{\frac{1}{2}}}\big\|\mathcal{M}_3 u^{(l-j)}\big\|_{L^2} \leqslant C\left(\frac{1}{d^{\delta}} + \frac{1}{d^{\frac{1}{2}}}\right)\|u\|_{H^m}.$$

Thus

$$\big\|\mathcal{M}_2[\mathcal{M}_1,\chi_d]u\big\|_2 \leqslant C(d)\|u\|_{H^m}, \quad \text{with } C(d) \to 0 \text{ as } d \to \infty.$$

Combining above with (2.11) and (2.14), we get the estimate (2.9). This finishes the proof of the lemma and Proposition 1. □

To show the existence of growing modes, we need to find some $\lambda > 0$ such that $\mathcal{A}^\lambda$ has a nontrivial kernel. We use a continuation strategy, by comparing the behavior of $\mathcal{A}^\lambda$ near 0 and infinity. First, we study the case near infinity.

Lemma 2.4. *There exists $\Lambda > 0$, such that when $\lambda > \Lambda$, $\mathcal{A}^\lambda$ has no eigenvalues in $\{z \mid \operatorname{Re} z \leqslant 0\}$.*

Proof. Suppose otherwise, then there exists a sequence $\{\lambda_n\} \to \infty$, and $\{k_n\} \in \mathbb{C}$, $\{u_n\} \in H^m(\mathbf{R})$, such that $\operatorname{Re} k_n \leqslant 0$ and $(\mathcal{A}^{\lambda_n} - k_n)u_n = 0$. Since $\|\mathcal{A}^\lambda - \mathcal{M} - 1\| = \|\mathcal{K}^\lambda\| \leqslant M$ for some constant M independent of λ and $\mathcal{M}$ is a self-adjoint positive operator, all discrete eigenvalues of $\mathcal{A}^\lambda$ lie in

$$D_M = \left\{z \,\middle|\, \operatorname{Re} z \geqslant -M \text{ and } |\operatorname{Im} z| \leqslant M\right\}.$$

Therefore, $k_n \to k_\infty \in D_M$ with $\operatorname{Re} k_\infty \leqslant 0$. Denote $e(x) = (f'(u_c))^2$, then $e(x) \to 0$ when $|x| \to \infty$. We normalize u_n by setting $\|u_n\|_{L^2_e} = 1$, where

$$\|u\|_{L^2_e} = \left(\int e(x)|u|^2\,dx \right)^{\frac{1}{2}}. \tag{2.15}$$

We claim that

$$\|u_n\|_{H^{\frac{m}{2}}} \leqslant C, \quad \text{for a constant } C \text{ independent of } n. \tag{2.16}$$

Assuming (2.16), we have $u_n \to u_\infty$ weakly in H^m. Moreover, we claim that $u_\infty \neq 0$. To show this, we choose $R > 0$ large enough such that $\max_{|x| \geqslant R} e(x) \leqslant \frac{1}{2C}$. Then

$$\int\limits_{|x| \geqslant R} e(x)|u_n|^2\,dx \leqslant \frac{1}{2C}\|u_n\|_{L^2} \leqslant \frac{1}{2}.$$

Since $u_n \to u_\infty$ strongly in $L^2(\{|x| \leqslant R\})$, we have

$$\int\limits_{|x| \leqslant R} e(x)|u_\infty|^2\,dx = \lim_{n\to\infty} \int\limits_{|x| \leqslant R} e(x)|u_n|^2\,dx \geqslant \frac{1}{2}$$

and thus $u_\infty \neq 0$. By Corollary 1, $\mathcal{A}^{\lambda_n} \to \mathcal{M} + 1$ strongly in L^2, therefore $\mathcal{A}^{\lambda_n} u_n \to (\mathcal{M}+1)u_\infty$ weakly in L^2 and $(\mathcal{M}+1)u_\infty = k_\infty u_\infty$. Since $\operatorname{Re} k_\infty \leqslant 0$, this is a contradiction. It remains to show (2.16). From $(\mathcal{A}^{\lambda_n} - k_n)u_n = 0$, we get

$$\begin{aligned} 0 \geqslant \operatorname{Re} k_n \|u_n\|_2^2 &= \big((\mathcal{M}+1)u_n, u_n\big) - \frac{1}{c}\operatorname{Re}\big(u_n, \big(1 - \mathcal{E}^{\lambda,+}\big)u_n\big) \\ &\quad - \frac{1}{c}\operatorname{Re}\big(f'(u_c)u_n, \big(1 - \mathcal{E}^{\lambda,+}\big)u_n\big). \end{aligned}$$

By our assumption on the symbol $\alpha(k)$ of $\mathcal{M}$, there exists $K > 0$ such that $\alpha(k) \geqslant a|k|^m$ when $|k| \geqslant K$. So for any $\varepsilon, \delta > 0$, from above and Lemma 2.1, we have

$$\begin{aligned} 0 &\geqslant (1-\delta)\|u_n\|_{L^2}^2 + a \int\limits_{|k| \geqslant K} |k|^m \big|\hat{u}(k)\big|^2\,dk + \delta \int\limits_{|k| \leqslant K} \big|\hat{u}(k)\big|^2\,dk \\ &\quad - \frac{1}{c}\|u_n\|_{L^2}^2 - \frac{1}{c}\|u_n\|_{L^2}\|u_n\|_{L^2_e} \\ &\geqslant (1-\delta)\|u_n\|_{L^2}^2 + \min\left\{\frac{\delta}{K^m}, a\right\} \int |k|^m \big|\hat{u}(k)\big|^2\,dk - \frac{1}{c}\|u_n\|_{L^2}^2 \\ &\quad - \varepsilon\|u_n\|_{L^2}^2 - \frac{\varepsilon}{4c^2}\|u_n\|_{L^2_e}^2 \\ &\geqslant \min\left\{1 - \frac{1}{c} - \delta - \varepsilon, \frac{\delta}{K^m}, a\right\}\|u_n\|_{H^{\frac{m}{2}}}^2 - \frac{\varepsilon}{4c^2}\|u_n\|_{L^2_e}^2. \end{aligned}$$

The bound (2.16) follows by choosing $\delta, \varepsilon > 0$ small. □

2.2. Asymptotic perturbations near $\lambda = 0$

In this subsection, we study the spectra of $\mathcal{A}^\lambda$ for small λ. When $\lambda \to 0+$, $\mathcal{A}^\lambda \to \mathcal{L}_0$ strongly in L^2, where $\mathcal{L}_0$ is defined by (2.2). Since the convergence of $\mathcal{A}^\lambda \to \mathcal{L}_0$ is rather weak, we could not use the regular perturbation theory. Instead, we use the asymptotic perturbation theory developed by Vock and Hunziker [47], see also [25,26]. To apply this theory, we need some preliminary lemmas.

Lemma 2.5. *Given $F \in C_0^\infty(\mathbf{R})$. Consider any sequence $\lambda_n \to 0+$ and $\{u_n\} \in H^m(\mathbf{R})$ satisfying*

$$\left\|\mathcal{A}^{\lambda_n} u_n\right\|_2 + \|u_n\|_2 \leqslant M_1 < \infty \tag{2.17}$$

for some constant M_1. Then if w-$\lim_{n\to\infty} u_n = 0$*, we have*

$$\lim_{n\to\infty} \|F u_n\|_2 = 0 \tag{2.18}$$

and

$$\lim_{n\to\infty} \left\|\left[\mathcal{A}^{\lambda_n}, F\right] u_n\right\|_2 = 0. \tag{2.19}$$

Proof. Since (2.17) implies that $\|u_n\|_{H^m} \leqslant C$, (2.18) follows from the local compactness of $H^m \hookrightarrow L^2$. To prove (2.19), we use the notations in the proof of Lemma 2.3. We write $\mathcal{A}^{\lambda_n} = \mathcal{M} + 1 + \mathcal{K}^{\lambda_n}$. Note that

$$[\mathcal{M}, F] = [\mathcal{M}_2\mathcal{M}_1, F] = \mathcal{M}_2[\mathcal{M}_1, F] + [\mathcal{M}_2, F]\mathcal{M}_1,$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined in (2.12) and (2.13). Let $G \in C_0^\infty(\mathbf{R})$ satisfying $G = 1$ on the support of F. For any $\varepsilon > 0$, we have

$$\begin{aligned}\left\|[\mathcal{M}_1, F]u_n\right\|_2 = \left\|[\mathcal{M}_1, F]Gu_n\right\|_2 &= \left\|\sum_{j=1}^{l} C_j^l \frac{d^j F}{dx^j}\frac{d^{l-j}(Gu_n)}{dx^{l-j}}\right\|_2 \\ &\leqslant C\|u_n\|_{H^{l-1}} \leqslant \varepsilon\|u_n\|_{H^m} + C_\varepsilon\|Gu_n\|_2.\end{aligned}$$

Since ε is arbitrarily small and the second term tends to zero by the local compactness, it follows that $\|[\mathcal{M}_1, F]u_n\|_2 \to 0$ when $n \to \infty$. Since $n'(k) \to 0$ when $|k| \to \infty$, by [19, Theorem C] the commutator $[\mathcal{M}_2, F] : L^2 \to L^2$ is compact. Since $\|\mathcal{M}_1 u_n\|_2 \leqslant \|u_n\|_{H^m} \leqslant C$ and $u_n \to 0$ weakly in L^2, we have $\mathcal{M}_1 u_n \to 0$ weakly in L^2. So $\|[\mathcal{M}_2, F]\mathcal{M}_1 u_n\|_2 \to 0$ and thus $\|[\mathcal{M}, F]u_n\|_2 \to 0$. We write

$$\begin{aligned}\left[\mathcal{K}^{\lambda_n}, F\right]u_n &= -\frac{1}{c}\left[\mathcal{E}^{\lambda_n,-}, F\right]\left(1 + f'(u_c)\right)u_n \\ &= -\frac{1}{c}\mathcal{E}^{\lambda_n,-}F\left(1 + f'(u_c)\right)u_n + \frac{1}{c}F\mathcal{E}^{\lambda_n,-}\left(1 + f'(u_c)\right)u_n = p_n + q_n\end{aligned}$$

and denote $v_n = (1 + f'(u_c))u_n$. From the uniform bound of $\|u_n\|_{H^m}$, we get the uniform bound for $\|v_n\|_{H^m}$. Therefore, by the local compactness,

$$\|p_n\|_2 \leqslant C\|Fv_n\|_2 \to 0, \quad \text{when } n \to \infty.$$

Since the operator $\|\mathcal{E}^{\lambda,-}\|_{L^2 \to L^2} \leqslant 1$ and for any $\lambda > 0$, $\mathcal{E}^{\lambda,-}$ is commutable with $(1 - \frac{d^2}{dx^2})^{\frac{m}{2}}$, we have $\|\mathcal{E}^{\lambda,-}\|_{H^m \to H^m} \leqslant 1$. So denoting $\tilde{v}_n = \mathcal{E}^{\lambda_n,-} v_n$, we have the uniform bound for $\|\tilde{v}_n\|_{H^m}$ and thus

$$\|p_n\|_2 \leqslant C\|F\tilde{v}_n\|_2 \to 0, \quad \text{when } n \to \infty.$$

This finishes the proof of (2.19). □

Lemma 2.6. *Let $z \in \mathbb{C}$ with $\operatorname{Re} z \leqslant \frac{1}{2}(1 - \frac{1}{c})$, then for some $n > 0$ and all $u \in C_0^\infty(|x| \geqslant n)$, we have*

$$\left\|(\mathcal{A}^\lambda - z)u\right\|_2 \geqslant \frac{1}{4}\left(1 - \frac{1}{c}\right)\|u\|_2, \tag{2.20}$$

when λ is sufficiently small.

Proof. The estimate (2.20) follows from

$$\operatorname{Re}\left((\mathcal{A}^\lambda - z)u, u\right) \geqslant \frac{1}{4}\left(1 - \frac{1}{c}\right)\|u\|_2^2, \tag{2.21}$$

which can be obtained as in the proof of Lemma 2.2. □

With above two lemmas, we can apply the asymptotic perturbation theory [25,47] to get the eigenvalue perturbations of $\mathcal{A}^0$ to $\mathcal{A}^\lambda$ with small λ.

Proposition 2. *Each discrete eigenvalue k_0 of $\mathcal{A}^0$ with $k_0 \leqslant \frac{1}{2}(1 - \frac{1}{c})$ is stable with respect to the family $\mathcal{A}^\lambda$ in the following sense: there exists $\lambda_1, \delta > 0$, such that for $0 < \lambda < \lambda_1$, we have*

(i) $$B(k_0; \delta) = \{z \mid 0 < |z - k_0| < \delta\} \subset P(\mathcal{A}^\lambda),$$
where

$$P(\mathcal{A}^\lambda) = \left\{z \,\middle|\, R^\lambda(z) = (\mathcal{A}^\lambda - z)^{-1} \text{ exists and is uniformly bounded for } \lambda \in (0, \lambda_1)\right\}.$$

(ii) *Denote*

$$P_\lambda = \oint_{\{|z-k_0|=\delta\}} R^\lambda(z)\,dz \quad \text{and} \quad P_0 = \oint_{\{|z-k_0|=\delta\}} R^0(z)\,dz$$

to be the perturbed and unperturbed spectral projection. Then $\dim P_\lambda = \dim P_0$ and $\lim_{\lambda \to 0} \|P_\lambda - P_0\| = 0$.

It follows from above that for λ small, the operators $\mathcal{A}^\lambda$ have discrete eigenvalues inside $B(k_0; \delta)$ with the total algebraic multiplicity equal to that of k_0.

2.3. The moving kernel formula and proof of Theorem 1

To understand the entire spectrum of $\mathcal{A}^\lambda$ for small λ, we need to know precisely how the zero eigenvalue of $\mathcal{A}^0 = \mathcal{L}^0$ is perturbed. For this, we derive a moving kernel formula. Let $\lambda_1, \delta > 0$ be as in Proposition 2 for $k_0 = 0$. By our assumptions $\ker \mathcal{A}^0 = \{u_{cx}\}$, $\dim P_0 = 1$ and thus $\dim P_\lambda = 1$ for $\lambda < \lambda_1$. Since the eigenvalues of $\mathcal{A}^\lambda$ appear in conjugate pairs, there can only exist one real eigenvalue k_λ of $\mathcal{A}^\lambda$ inside $B(0;\delta)$. The following lemma determines the sign of k_λ, when λ is sufficiently small.

Lemma 2.7. *Assume* $\ker \mathcal{L}^0 = \{u_{cx}\}$. *For* $\lambda > 0$ *small enough, let* $k_\lambda \in \mathbf{R}$ *to be the only eigenvalue of* $\mathcal{A}^\lambda$ *near the origin. Then*

$$\lim_{\lambda \to 0+} \frac{k_\lambda}{\lambda^2} = -\frac{1}{c}\frac{dP}{dc} \Big/ \|u_{cx}\|_{L^2}^2, \tag{2.22}$$

where the momentum function

$$P(c) = \frac{1}{2}\big((\mathcal{M}+1)u_c, u_c\big). \tag{2.23}$$

By the same proof of (2.16), we get the following a priori estimate which is used in the later proof.

Lemma 2.8. *For* $\lambda > 0$ *small enough, consider* $u \in H^m(\mathbf{R})$ *satisfying the equation*

$$\big(\mathcal{A}^\lambda - z\big)u = v,$$

where $z \in \mathbb{C}$ *with* $\operatorname{Re} z \leqslant \frac{1}{2}(1 - \frac{1}{c})$ *and* $v \in L^2$. *Then we have the estimate*

$$\|u\|_{H^{\frac{m}{2}}} \leqslant C\big(\|u\|_{L^2_e} + \|v\|_{L^2}\big), \tag{2.24}$$

for some constant C independent of λ. *Here, the weighted norm* $\|\cdot\|_{L^2_e}$ *is defined in* (2.15).

Assuming Lemma 2.7, we prove Theorem 1 for BBM type equations.

Proof of Theorem 1 (BBM). We only prove (ii) since the proof of (i) is the same. Assume that $n^-(\mathcal{L}_0)$ is odd and $dP/dc < 0$. Let $k_1^-, \dots, k_l^-$ be all the distinct negative eigenvalues of $\mathcal{L}_0$. Choose $\delta > 0$ small such that the l disks $B(k_i^-;\delta)$ are disjoint and still lie in the left-half plane. By Proposition 2, there exists $\lambda_1 > 0$ and δ small enough, such that for $0 < \lambda < \lambda_1$, $\mathcal{A}^\lambda$ has $n^-(\mathcal{L}_0)$ eigenvalues (counting multiplicity) in $\bigcup_{i=1}^l B(k_i^-;\delta)$. By Lemma 2.7, if $dP/dc < 0$, then the zero eigenvalue of $\mathcal{A}^0$ is perturbed to a positive eigenvalue $0 < k_\lambda < \delta$ of $\mathcal{A}^\lambda$ for small λ. Consider the region

$$\Omega = \big\{z \,\big|\, 0 > \operatorname{Re} z > -2M \text{ and } |\operatorname{Im} z| < 2M\big\},$$

where M is the uniform bound for $\|\mathcal{K}^\lambda\| = \|\mathcal{A}^\lambda - \mathcal{M} - 1\|$. We claim that: for λ small enough, $\mathcal{A}^\lambda$ has exactly $n^-(\mathcal{L}_0) + 1$ eigenvalues (with multiplicity) in

$$\Omega_\delta = \{ z \mid 2\delta > \operatorname{Re} z > -2M \text{ and } |\operatorname{Im} z| < 2M \}.$$

That is, all eigenvalues of $\mathcal{A}^\lambda$ with real parts no greater than 2δ lie in $\bigcup_{i=1}^{l} B(k_i^-;\delta) \cup B(0;\delta)$. Suppose otherwise, there exists a sequence $\lambda_n \to 0$ and

$$\{u_n\} \in H^m(\mathbf{R}), \qquad z_n \in \Omega \setminus \left(\bigcup_{i=1}^{l} B(k_i^-;\delta) \cup B(0;\delta) \right)$$

such that $(\mathcal{A}^{\lambda_n} - z_n)u_n = 0$. We normalize u_n by setting $\|u_n\|_{L_e^2} = 1$. Then by Lemma 2.8, we have $\|u_n\|_{H^{\frac{m}{2}}} \leqslant C$. By the same argument as in the proof of Lemma 2.4, $u_n \to u_\infty \neq 0$ weakly in $H^{\frac{m}{2}}$. Let

$$\lim_{n\to\infty} z_n = z_\infty \in \bar{\Omega} \setminus \left(\bigcup_{i=1}^{l} B(k_i^-;\delta) \cup B(0;\delta) \right)$$

then $\mathcal{L}^0 u_\infty = z_\infty u_\infty$, which is a contradiction. The claim is proved and thus for λ small enough, $\mathcal{A}^\lambda$ has exactly $n^-(\mathcal{L}_0)$ eigenvalues in Ω.

Suppose Theorem 1(ii) is not true, then $\mathcal{A}^\lambda$ has no kernel for any $\lambda > 0$. Define $n_\Omega(\lambda)$ to be the number of eigenvalues (with multiplicity) of $\mathcal{A}^\lambda$ in Ω. Since by 2.7, the region Ω is away from the essential spectrum of $\mathcal{A}^\lambda$, $n_\Omega(\lambda)$ is always a finite integer. In the above, we have proved that $n_\Omega(\lambda) = n^-(\mathcal{L}_0)$ is odd, for λ small enough. By Lemma 2.4, there exists $\Lambda > 0$ such that $n_\Omega(\lambda) = 0$ for $\lambda > \Lambda$. Define two sets

$$S_{\text{odd}} = \{ \lambda > 0 \mid n_\Omega(\lambda) \text{ is odd} \}, \qquad S_{\text{even}} = \{ \lambda > 0 \mid n_\Omega(\lambda) \text{ is even} \}.$$

Then both sets are non-empty. Below, we show that both S_{odd} and S_{even} are open sets. Let $\lambda_0 \in S_{\text{odd}}$ and denote $k_1, \dots, k_l$ $(l \leqslant n_\Omega(\lambda_0))$ to be all distinct eigenvalues of $\mathcal{A}^{\lambda_0}$ in Ω. Denote $ih_1, \dots, ih_p$ to be all eigenvalues of $\mathcal{A}^{\lambda_0}$ on the imaginary axis. Then $|h_j| \leqslant M$, $1 \leqslant j \leqslant p$. Choose $\delta > 0$ sufficiently small such that the disks $B(k_i;\delta)$ $(1 \leqslant i \leqslant l)$ and $B(ih_j;\delta)$ $(1 \leqslant j \leqslant p)$ are disjoint, $B(k_i;\delta) \subset \Omega$ and $B(ih_j;\delta)$ does not contain 0. Note that $\mathcal{A}^\lambda$ is analytic in λ for $\lambda \in (0, +\infty)$. By the analytic perturbation theory [25], if $|\lambda - \lambda_0|$ is sufficiently small, any eigenvalue of $\mathcal{A}^\lambda$ in Ω_δ lies in one of the disks $B(k_i;\delta)$ or $B(ih_j;\delta)$. So $n_\Omega(\lambda)$ is the number $n_\Omega(\lambda_0)$ plus the number of eigenvalues in $\bigcup_{i=1}^{p} B(ih_j;\delta)$ with the negative real part. The second number is even, since the complex eigenvalues of $\mathcal{A}^\lambda$ appears in conjugate pairs. Thus, $n_\Omega(\lambda)$ is odd when $|\lambda - \lambda_0|$ is small enough. This shows that S_{odd} is open. Similarly, S_{even} is open. Thus, $(0, +\infty)$ is the union of two non-empty, disjoint open sets S_{odd} and S_{even}. This is a contradiction.

So there exists $\lambda > 0$ and $0 \neq u \in H^m(\mathbf{R})$ such that $\mathcal{A}^\lambda u = 0$. Then $e^{\lambda t}u(x)$ is a purely growing solution to (2.3). □

It remains to prove the moving kernel formula (2.22).

Proof of Lemma 2.7. We use C to denote a generic constant in our estimates below. As described at the beginning of this subsection, for $\lambda > 0$ small enough, there exists $u_\lambda \in H^m(\mathbf{R})$, such that $(\mathcal{A}^\lambda - k_\lambda)u_\lambda = 0$ with $k_\lambda \in \mathbf{R}$ and $\lim_{\lambda\to 0+} k_\lambda = 0$. We normalize u_λ by setting $\|u_\lambda\|_{L_e^2} = 1$. Then by Lemma 2.8, we have $\|u_\lambda\|_{H^{\frac{m}{2}}} \leqslant C$ and as in the proof of Lemma 2.4,

$u_\lambda \to u_0 \neq 0$ weakly in $H^{\frac{m}{2}}$. Since $\mathcal{A}^0 u_0 = \mathcal{L}_0 u_0 = 0$ and $\ker \mathcal{L}_0 = \{u_{cx}\}$, we have $u_0 = c_0 u_{cx}$ for some $c_0 \neq 0$. Moreover, we have $\|u_\lambda - u_0\|_{H^{\frac{m}{2}}} = 0$. To show this, first we note that $\|u_\lambda - u_0\|_{L^2_e} \to 0$, since

$$\|u_\lambda - u_0\|^2_{L^2_e} \leqslant \int\limits_{|x|\leqslant R} e(x)|u_\lambda - u_0|^2\,dx + \max_{|x|\geqslant R} e(x)\|u_\lambda - u_0\|^2_{L^2},$$

and the second term is arbitrarily small for large R while the first term tends to zero by the local compactness. Since

$$\left(\mathcal{A}^\lambda - k_\lambda\right)(u_\lambda - u_0) = k_\lambda u_0 + \left(\mathcal{A}^0 - \mathcal{A}^\lambda\right)u_0,$$

by Lemma 2.8 we have

$$\|u_\lambda - u_0\|_{H^{\frac{m}{2}}} \leqslant C\left(\|u_\lambda - u_0\|^2_{L^2_e} + |k_\lambda|\|u_0\|^2_{L^2} + \left\|\left(\mathcal{A}^0 - \mathcal{A}^\lambda\right)u_0\right\|^2_{L^2}\right) \to 0,$$

when $\lambda \to 0+$. We can assume $c_0 = 1$ by renormalizing the sequence.

Next, we show that $\lim_{\lambda\to 0+} \frac{k_\lambda}{\lambda} = 0$. From $(\mathcal{A}^\lambda - k_\lambda)u_\lambda = 0$, we have

$$\frac{k_\lambda}{\lambda} u_\lambda = \mathcal{A}^0 \frac{u_\lambda}{\lambda} + \frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} u_\lambda. \tag{2.25}$$

Taking the inner product of above with u_{cx}, we get

$$\frac{k_\lambda}{\lambda}(u_\lambda, u_{cx}) = \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} u_\lambda, u_{cx}\right) =: m(\lambda).$$

We have

$$\begin{aligned} m(\lambda) &= \left(\frac{1}{c}\frac{1}{\lambda - \mathcal{D}}\left(1 + f'(u_c)\right)u_\lambda, u_{cx}\right) = \frac{1}{c}\left(\left(1 + f'(u_c)\right)u_\lambda, \frac{1}{\lambda + \mathcal{D}} u_{cx}\right) \\ &= \frac{1}{c^2}\left(\left(1 + f'(u_c)\right)u_\lambda, \left(1 - \mathcal{E}^{\lambda,+}\right)u_c\right) \to \frac{1}{c^2}\left(\left(1 + f'(u_c)\right)u_{cx}, u_c\right) \\ &= \frac{1}{c^2}\int \frac{d}{dx}\left(\frac{1}{2}u_c^2 + F(u_c)\right)dx = 0, \end{aligned}$$

where $F(u) = \int_0^u f'(s)s\,ds$ and in the above $\lim_{\lambda\to 0+} \mathcal{E}^{\lambda,+} = 0$ is used. So

$$\lim_{\lambda\to 0+} \frac{k_\lambda}{\lambda} = \lim_{\lambda\to 0+} \frac{m(\lambda)}{(u_\lambda, u_{cx})} = 0.$$

We write $u_\lambda = c_\lambda u_{cx} + \lambda v_\lambda$, where $c_\lambda = (u_\lambda, u_{cx})/(u_{cx}, u_{cx})$. Then $(v_\lambda, u_{cx}) = 0$ and $c_\lambda \to 1$ when $\lambda \to 0+$. We claim that $\|v_\lambda\|_{L^2_e} \leqslant C$ (independent of λ). Suppose otherwise, there exists a sequence $\lambda_n \to 0+$ such that $\|v_{\lambda_n}\|_{L^2_e} \geqslant n$. Denote $\tilde{v}_{\lambda_n} = v_{\lambda_n}/\|v_{\lambda_n}\|_{L^2_e}$. Then $\|\tilde{v}_{\lambda_n}\|_{L^2_e} = 1$ and $\tilde{v}_{\lambda_n}$ satisfies the equation

$$\mathcal{A}^{\lambda_n}\tilde{v}_{\lambda_n} = \frac{1}{\|\tilde{v}_{\lambda_n}\|_{L_e^2}}\left(\frac{k_{\lambda_n}}{\lambda_n}u_{\lambda_n} - c_{\lambda_n}\frac{\mathcal{A}^{\lambda_n} - \mathcal{A}^0}{\lambda_n}u_{cx}\right). \tag{2.26}$$

Denote

$$w_\lambda(x) = \frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}u_{cx},$$

then

$$\begin{aligned} w_\lambda(x) &= \frac{1}{c}\frac{1}{\lambda - \mathcal{D}}\big(1 + f'(u_c)\big)u_{cx} = \frac{1}{\lambda - \mathcal{D}}\frac{d}{dx}\big((\mathcal{M}+1)u_c\big) \\ &= \frac{1}{c}\frac{\mathcal{D}}{\lambda - \mathcal{D}}(\mathcal{M}+1)u_c = \frac{1}{c}\big(\mathcal{E}^{\lambda,-} - 1\big)(\mathcal{M}+1)u_c, \end{aligned}$$

where we use the equation

$$\mathcal{L}_0 u_{cx} = \mathcal{M}u_{cx} + \left(1 - \frac{1}{c}\right)u_{cx} - \frac{1}{c}f'(u_c)u_{cx} = 0.$$

By Lemma 2.1, $\|w_\lambda\|_{L^2} \leqslant C$ (independent of λ), and

$$w_\lambda(x) \to -\frac{1}{c}(\mathcal{M}+1)u_c = \frac{1}{c^2}\big(u_c + f(u_c)\big) \tag{2.27}$$

strongly in L^2 when $\lambda \to 0+$. So by Lemma 2.8, we have $\|\tilde{v}_{\lambda_n}\|_{H^{\frac{m}{2}}} \leqslant C$. Then, as before, $\tilde{v}_{\lambda_n} \to \tilde{v}_0 \neq 0$ weakly in $H^{\frac{m}{2}}$. Since $\frac{k_{\lambda_n}}{\lambda_n}, \frac{1}{\|\tilde{v}_{\lambda_n}\|_{L_e^2}} \to 0$, we have $\mathcal{A}^0\tilde{v}_0 = 0$. So $\tilde{v}_0 = c_1 u_{cx}$ for some $c_1 \neq 0$. But since $(\tilde{v}_{\lambda_n}, u_{cx}) = 0$, we have $(\tilde{v}_0, u_{cx}) = 0$, a contradiction. This establishes the uniform bound for $\|v_\lambda\|_{L_e^2}$. The equation satisfied by v_λ is

$$\mathcal{A}^\lambda v_\lambda = \frac{k_\lambda}{\lambda_n}u_\lambda - c_\lambda\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}u_{cx} = \frac{k_\lambda}{\lambda_n}u_\lambda - c_\lambda w_\lambda.$$

Applying Lemma 2.8 to the above equation, we have $\|v_\lambda\|_{H^{\frac{m}{2}}} \leqslant C$ and thus $v_\lambda \to v_0$ weakly in $H^{\frac{m}{2}}$. By (2.27), v_0 satisfies

$$\mathcal{A}^0 v_0 = \mathcal{L}_0 v_0 = \frac{1}{c}(\mathcal{M}+1)u_c.$$

Taking ∂_c of (2.1), we have

$$\mathcal{L}_0\partial_c u_c = -\frac{1}{c}(\mathcal{M}+1)u_c. \tag{2.28}$$

Thus $\mathcal{L}_0(v_0 + \partial_c u_c) = 0$. Since $(v_0, u_{cx}) = \lim_{\lambda\to 0+}(v_\lambda, u_{cx}) = 0$, we have

$$v_0 = -\partial_c u_c + d_0 u_{cx}, \quad d_0 = (\partial_c u_c, u_{cx}) \,/\, \|u_{cx}\|_{L^2}^2.$$

Similar to the proof of $\|u_\lambda - u_0\|_{H^{\frac{m}{2}}} \to 0$, we have $\|v_\lambda - v_0\|_{H^{\frac{m}{2}}} \to 0$. We rewrite

$$u_\lambda = c_\lambda u_{cx} + \lambda v_\lambda = \bar{c}_\lambda u_{cx} + \lambda \bar{v}_\lambda,$$

where $\bar{c}_\lambda = c_\lambda + \lambda d_0$, $\bar{v}_\lambda = v_\lambda - d_0 u_{cx}$. Then $\bar{c}_\lambda \to 1$, $\bar{v}_\lambda \to -\partial_c u_c$, when $\lambda \to 0+$.

Now we compute $\lim_{\lambda\to 0+} \frac{k_\lambda}{\lambda^2}$. From (2.25), we have

$$\mathcal{A}^0 \frac{u_\lambda}{\lambda^2} + \frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}\left(\frac{\bar{c}_\lambda}{\lambda} u_{cx} + \bar{v}_\lambda\right) = \frac{k_\lambda}{\lambda^2} u_\lambda.$$

Taking the inner product of above with u_{cx}, we have

$$\frac{k_\lambda}{\lambda^2}(u_\lambda, u_{cx}) = \bar{c}_\lambda \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda^2} u_{cx}, u_{cx}\right) + \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} \bar{v}_\lambda, u_{cx}\right) = \bar{c}_\lambda I_1 + I_2.$$

For the first term, we have

$$\begin{aligned}
I_1 &= \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda^2} u_{cx}, u_{cx}\right) = \left(\frac{w_\lambda(x)}{\lambda}, u_{cx}\right) = \frac{1}{c}\left(\frac{\mathcal{D}}{(\lambda - \mathcal{D})\lambda}(\mathcal{M}+1)u_c, u_{cx}\right) \\
&= \frac{1}{c}\left(\frac{1}{\lambda - \mathcal{D}}(\mathcal{M}+1)u_c, u_{cx}\right) - \frac{1}{c\lambda}\big((\mathcal{M}+1)u_c, u_{cx}\big) \\
&= -\frac{1}{c^2}\big(\big(\mathcal{E}^{\lambda,-} - 1\big)(\mathcal{M}+1)u_c, u_c\big) - \frac{1}{c^2\lambda}\big(u_c + f(u_c), u_{cx}\big) \\
&= -\frac{1}{c^2}\big(\big(\mathcal{E}^{\lambda,-} - 1\big)(\mathcal{M}+1)u_c, u_c\big) \to \frac{1}{c^2}\big((\mathcal{M}+1)u_c, u_c\big), \quad \text{when } \lambda \to 0+.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
I_2 &= \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} \bar{v}_\lambda, u_{cx}\right) = \frac{1}{c}\left(\frac{1}{\lambda - \mathcal{D}}\big(1 + f'(u_c)\big)\bar{v}_\lambda, u_{cx}\right) \\
&= -\frac{1}{c^2}\left(\frac{\mathcal{D}}{\lambda - \mathcal{D}}\big(1 + f'(u_c)\big)\bar{v}_\lambda, u_c\right) = -\frac{1}{c^2}\big(\big(\mathcal{E}^{\lambda,-} - 1\big)\big(1 + f'(u_c)\big)\bar{v}_\lambda, u_c\big) \\
&\to -\frac{1}{c^2}\big(\big(1 + f'(u_c)\big)\partial_c u_c, u_c\big), \quad \text{when } \lambda \to 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{\lambda\to 0+} \frac{k_\lambda}{\lambda^2} &= \lim_{\lambda\to 0+} \frac{\bar{c}_\lambda I_1 + I_2}{(u_\lambda, u_{cx})} \\
&= \left[\frac{1}{c^2}\big((\mathcal{M}+1)u_c, u_c\big) - \frac{1}{c^2}\big(\big(1 + f'(u_c)\big)\partial_c u_c, u_c\big)\right] / \|u_{cx}\|_{L^2}^2 \\
&= -\frac{1}{c}\big((\mathcal{M}+1)\partial_c u_c, u_c\big) = -\frac{1}{c}\frac{dP}{dc},
\end{aligned}$$

since by (2.28)

$$(\mathcal{M}+1)u_c - \big(1+f'(u_c)\big)\partial_c u_c = -c(\mathcal{M}+1)\partial_c u_c. \qquad \square$$

3. Regularized Boussinesq type

Consider a solitary wave $u(x,t) = u_c(x-ct)$ $(c^2 > 1)$ of the regularized Boussinesq (RBou) type equation (1.3). Then u_c satisfies the equation

$$\mathcal{M}u_c + \left(1 - \frac{1}{c^2}\right)u_c - \frac{1}{c^2} f(u_c) = 0. \tag{3.1}$$

The linearized equation in the traveling frame $(x-ct, t)$ is

$$(\partial_t - c\partial_x)^2(u + \mathcal{M}u) - \partial_x^2\big(u + f'(u_c)u\big) = 0. \tag{3.2}$$

For a growing mode $e^{\lambda t}u(x)$ $(\operatorname{Re}\lambda > 0)$, $u(x)$ satisfies

$$(\lambda - c\partial_x)^2(u + \mathcal{M}u) - \partial_x^2\big(u + f'(u_c)u\big) = 0. \tag{3.3}$$

So we define the following dispersion operator $\mathcal{A}^\lambda : H^m \to L^2$ $(\lambda > 0)$

$$\mathcal{A}^\lambda u = \mathcal{M}u + u - \left(\frac{\partial_x}{\lambda - c\partial_x}\right)^2\big(u + f'(u_c)u\big)$$

and the existence of a purely growing mode is reduced to find $\lambda > 0$ such that $\mathcal{A}^\lambda$ has a nontrivial kernel. Since when $\lambda \to 0+$,

$$\frac{\partial_x}{\lambda - c\partial_x} = \frac{\mathcal{D}}{\lambda - \mathcal{D}} = \frac{1}{c}\big(\mathcal{E}^{\lambda,-} - 1\big) \to -\frac{1}{c} \quad \text{strongly in } L^2, \tag{3.4}$$

the zero limit of the operator$\mathcal{A}^\lambda$ is

$$\mathcal{L}_0 =: \mathcal{M} + \left(1 - \frac{1}{c^2}\right) - \frac{1}{c^2} f'(u_c). \tag{3.5}$$

The proof of Theorem 1 for RBou case is very similar to the BBM case, so we only give a sketch of the proof of the moving kernel formula.

Lemma 3.1. *Assume* $\ker \mathcal{L}^0 = \{u_{cx}\}$. *For* $\lambda > 0$ *small enough, let* $k_\lambda \in \mathbf{R}$ *to be the only eigenvalue of* $\mathcal{A}^\lambda$ *near zero. Then we have*

$$\lim_{\lambda \to 0+} \frac{k_\lambda}{\lambda^2} = -\frac{1}{c^2}\frac{dP}{dc} \,/\, \|u_{cx}\|_{L^2}^2,$$

where

$$P(c) = c\big((\mathcal{M}+1)u_c, u_c\big). \tag{3.6}$$

Proof. For $\lambda > 0$ small enough, let

$$u_\lambda \in H^m(\mathbf{R}), \qquad k_\lambda \in \mathbf{R}, \qquad \lim_{\lambda \to 0+} k_\lambda = 0,$$

such that $(\mathcal{A}^\lambda - k_\lambda)u_\lambda = 0$. We normalize u_λ by setting $\|u_\lambda\|_{L^2_e} = 1$. Then as in the BBM case, we have $\|u_\lambda\|_{H^{\frac{m}{2}}} \leqslant C$ and $u_\lambda \to u_{cx}$ in $H^{\frac{m}{2}}$ by a renormalization, under our assumption that $\ker \mathcal{L}_0 = \{u_{cx}\}$.

First, we show that $\lim_{\lambda \to 0+} \frac{k_\lambda}{\lambda} = 0$. As in the BBM case, we have

$$\frac{k_\lambda}{\lambda}(u_\lambda, u_{cx}) = \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} u_\lambda, u_{cx}\right) =: m(\lambda),$$

where

$$\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} = \frac{1}{c^2}\left(\frac{2}{\lambda - \mathcal{D}} - \frac{\lambda}{(\lambda - \mathcal{D})^2}\right)\bigl(1 + f'(u_c)\bigr).$$

We have

$$\begin{aligned} m(\lambda) &= \frac{1}{c^2}\left(\left[\frac{2}{\lambda - \mathcal{D}} - \frac{\lambda}{(\lambda - \mathcal{D})^2}\right]\bigl(1 + f'(u_c)\bigr)u_\lambda, u_{cx}\right) \\ &= \frac{1}{c^2}\left(\bigl(1 + f'(u_c)\bigr)u_\lambda, \left(\frac{2}{\lambda + \mathcal{D}} - \frac{\lambda}{(\lambda + \mathcal{D})^2}\right)u_{cx}\right) \\ &= \frac{1}{c^3}\left(\bigl(1 + f'(u_c)\bigr)u_\lambda, \left(\frac{2\mathcal{D}}{\lambda + \mathcal{D}} - \frac{\lambda\mathcal{D}}{(\lambda + \mathcal{D})^2}\right)u_c\right) \\ &= \frac{1}{c^3}\bigl(\bigl(1 + f'(u_c)\bigr)u_\lambda, \bigl(1 - \mathcal{E}^{\lambda,+}\bigr)\bigl(2 - \mathcal{E}^{\lambda,+}\bigr)u_c\bigr) \\ &\to \frac{2}{c^3}\bigl(\bigl(1 + f'(u_c)\bigr)u_{cx}, u_c\bigr) = 0, \end{aligned}$$

and thus

$$\lim_{\lambda \to 0+} \frac{k_\lambda}{\lambda} = \lim_{\lambda \to 0+} \frac{m(\lambda)}{(u_\lambda, u_{cx})} = 0.$$

Similarly to the BBM case, we can show that $u_\lambda = \bar{c}_\lambda u_{cx} + \lambda \bar{v}_\lambda$, with $\bar{c}_\lambda \to 1$, $\bar{v}_\lambda \to -\partial_c u_c$ in $H^{\frac{m}{2}}$ when $\lambda \to 0+$. In the proof, we use the facts that

$$\begin{aligned} w_\lambda(x) &= \frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} u_{cx} = \frac{1}{c^2}\left(\frac{2}{\lambda - \mathcal{D}} - \frac{\lambda}{(\lambda - \mathcal{D})^2}\right)\bigl(1 + f'(u_c)\bigr)u_{cx} \\ &= \left(\frac{2}{\lambda - \mathcal{D}} - \frac{\lambda}{(\lambda - \mathcal{D})^2}\right)(\mathcal{M} + 1)u_{cx} = \frac{1}{c}\left(\frac{2\mathcal{D}}{\lambda - \mathcal{D}} - \frac{\lambda\mathcal{D}}{(\lambda - \mathcal{D})^2}\right)(\mathcal{M} + 1)u_c \\ &= \frac{1}{c}\bigl(\mathcal{E}^{\lambda,-} - 1\bigr)\bigl(2 - \mathcal{E}^{\lambda,-}\bigr)(\mathcal{M} + 1)u_c \to -\frac{2}{c}(\mathcal{M} + 1)u_c, \quad \text{when } \lambda \to 0+. \end{aligned}$$

and

$$\mathcal{L}_0 \partial_c u_c = -\frac{2}{c^3}\big(u_c + f(u_c)\big) = -\frac{2}{c}(\mathcal{M}+1)u_c. \tag{3.7}$$

Next, we compute $\lim_{\lambda\to 0+} \frac{k_\lambda}{\lambda^2}$ by using

$$\frac{k_\lambda}{\lambda^2}(u_\lambda, u_{cx}) = \bar{c}_\lambda \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda^2} u_{cx}, u_{cx}\right) + \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} \bar{v}_\lambda, u_{cx}\right) = \bar{c}_\lambda I_1 + I_2.$$

For the first term, we have

$$\begin{aligned}
I_1 &= \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda^2} u_{cx}, u_{cx}\right) = \left(\frac{w_\lambda(x)}{\lambda}, u_{cx}\right) \\
&= \frac{1}{c}\left(\left[\frac{2\mathcal{D}}{(\lambda-\mathcal{D})\lambda} - \frac{\mathcal{D}}{(\lambda-\mathcal{D})^2}\right](\mathcal{M}+1)u_c, u_{cx}\right) \\
&= -\frac{1}{c^2}\left(\left[\frac{2\mathcal{D}^2}{(\lambda-\mathcal{D})\lambda} - \frac{\mathcal{D}^2}{(\lambda-\mathcal{D})^2}\right](\mathcal{M}+1)u_c, u_c\right) \\
&= -\frac{2}{c^2}\big(\big(\mathcal{E}^{\lambda,-}-1\big)(\mathcal{M}+1)u_c, u_c\big) + \frac{1}{c^2\lambda}\big(\mathcal{D}(\mathcal{M}+1)u_c, u_c\big) \\
&\quad + \frac{1}{c^2}\big(\big(\mathcal{E}^{\lambda,-}-1\big)^2(\mathcal{M}+1)u_c, u_c\big) \\
&\to \frac{3}{c^2}\big((\mathcal{M}+1)u_c, u_c\big), \quad \text{when } \lambda \to 0+,
\end{aligned}$$

since $\mathcal{E}^{\lambda,-} \to 0$ and

$$\big(\mathcal{D}(\mathcal{M}+1)u_c, u_c\big) = c\big(u_{cx}, (\mathcal{M}+1)u_c\big) = \frac{1}{c}\big(u_{cx}, u_c + f(u_c)\big) = 0.$$

For the second term, we have

$$\begin{aligned}
I_2 &= \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda} \bar{v}_\lambda, u_{cx}\right) = \frac{1}{c^2}\left(\left(\frac{2}{\lambda-\mathcal{D}} - \frac{\lambda}{(\lambda-\mathcal{D})^2}\right)\big(1+f'(u_c)\big)\bar{v}_\lambda, u_{cx}\right) \\
&= -\frac{1}{c^3}\left(\left(\frac{2\mathcal{D}}{\lambda-\mathcal{D}} - \frac{\lambda\mathcal{D}}{(\lambda-\mathcal{D})^2}\right)\big(1+f'(u_c)\big)\bar{v}_\lambda, u_c\right) \\
&= -\frac{1}{c^3}\big(\big(\mathcal{E}^{\lambda,-}-1\big)\big(2-\mathcal{E}^{\lambda,-}\big)\big(1+f'(u_c)\big)\bar{v}_\lambda, u_c\big) \\
&\to -\frac{2}{c^3}\big(\big(1+f'(u_c)\big)\partial_c u_c, u_c\big), \quad \text{when } \lambda \to 0+.
\end{aligned}$$

Thus

$$\begin{aligned}\lim_{\lambda\to 0+}\frac{k_\lambda}{\lambda^2} &= \lim_{\lambda\to 0+}\frac{\bar{c}_\lambda I_1 + I_2}{(u_\lambda, u_{cx})}\\ &= \left[\frac{3}{c^2}\big((\mathcal{M}+1)u_c, u_c\big) - \frac{2}{c^3}\big(\big(1+f'(u_c)\big)\partial_c u_c, u_c\big)\right] / \|u_{cx}\|_{L^2}^2\\ &= \left[-\frac{1}{c^2}\big((\mathcal{M}+1)u_c, u_c\big) - \frac{2}{c}\big((\mathcal{M}+1)\partial_c u_c, u_c\big)\right] / \|u_{cx}\|_{L^2}^2\\ &= -\frac{1}{c^2}\frac{dP}{dc} / \|u_{cx}\|_{L^2}^2,\end{aligned}$$

since by (3.7)

$$\big(1+f'(u_c)\big)\partial_c u_c = c^2(\mathcal{M}+1)\partial_c u_c + 2c(\mathcal{M}+1)u_c. \qquad \square$$

As a corollary of the above proof, we show Theorem 2 for the RBou case. We skip the proof of Theorem 2 for the BBM and KDV cases, since they are very similar. Theorem 2 (RBou) follows from the next lemma.

Lemma 3.2. *Assume* $\ker \mathcal{L}_0 = \{u_{cx}\}$. *If there is a sequence of purely growing modes* $e^{\lambda_n t}u_n(x)$ $(\lambda_n > 0)$ *for solitary waves* u_{c_n} *of* (1.3)*, with* $\lambda_n \to 0+$, $c_n \to c_0$*, then we must have* $P'(c_0) = 0$.

Proof. The proof is almost the same as that of Lemma 3.1, so we only sketch it. The only difference is that now the computations depend on the parameter c_n. Denote $\mathcal{E}_n^\pm = \frac{\lambda_n}{\lambda_n \pm c_n\partial_x}$, then by the same argument as in the proof of Lemma 2.1, we have $\text{s-lim}_{n\to\infty}\mathcal{E}_n^\pm = 0$. Then the operator

$$\mathcal{A}^{\lambda_n,c_n} = \mathcal{M} + 1 - \left(\frac{\partial_x}{\lambda_n - c_n\partial_x}\right)^2\big(1+f'(u_{c_n})\big)$$

converges to

$$\mathcal{L}_0 =: \mathcal{M} + \left(1 - \frac{1}{c_0^2}\right) - \frac{1}{c_0^2}f'(u_{c_0})$$

strongly in L^2. We have $\mathcal{A}^{\lambda_n,c_n}u_n = 0$ and we normalize u_n by $\|u_n\|_{L^2_{e_n}} = 1$, where $e_n = |f'(u_{c_n})|^2$. As before, it can be shown that $\|u_n\|_{H^{\frac{m}{2}}} \leqslant C$ (independent of n) and $u_n \to u_{c_0x}$ in $H^{\frac{m}{2}}$. Moreover, we have $u_n = \bar{c}_n u_{c_nx} + \lambda_n\bar{v}_n$, where $\bar{c}_n \to 1$ and $\bar{v}_n \to -\partial_c u_c|_{c_0}$ in $H^{\frac{m}{2}}$. From $\mathcal{A}^{\lambda_n,c_n}u_n = 0$, it follows that

$$0 = \bar{c}_n\left(\frac{\mathcal{A}^{\lambda_n,c_n} - \mathcal{A}^{0,c_n}}{\lambda_n^2}u_{c_nx}, u_{c_nx}\right) + \left(\frac{\mathcal{A}^{\lambda_n,c_n} - \mathcal{A}^{0,c_n}}{\lambda_n}\bar{v}_n, u_{c_nx}\right) = \bar{c}_n I_1 + I_2,$$

where

$$\mathcal{A}^{0,c_n} = \mathcal{M} + \left(1 - \frac{1}{c_n^2}\right) - \frac{1}{c_n^2} f'(u_{c_n}).$$

By the same computations as in the proof of Lemma 3.1,

$$\begin{aligned} I_1 &= -\frac{2}{c_n^2}\big((\mathcal{E}_n^- - 1)(\mathcal{M}+1)u_{c_n}, u_{c_n}\big) + \frac{1}{c_n^2}\big((\mathcal{E}_n^- - 1)^2(\mathcal{M}+1)u_{c_n}, u_{c_n}\big) \\ &\to \frac{3}{c_0^2}\big((\mathcal{M}+1)u_{c_0}, u_{c_0}\big), \quad \text{when } n \to \infty, \end{aligned}$$

and

$$I_2 \to -\frac{2}{c_0^3}\big(\big(1 + f'(u_{c_0})\big)\partial_c u_c|_{c_0}, u_{c_0}\big), \quad \text{when } n \to \infty.$$

Thus

$$0 = \lim_{n\to\infty}(\bar{c}_n I_1 + I_2) = -\frac{1}{c_0^2}\frac{dP}{dc}(c_0)$$

and the lemma is proved. □

4. KDV type

Consider a solitary wave $u(x,t) = u_c(x - ct)$ $(c > 0)$ of the KDV type equations (1.2). Then u_c satisfies the equation

$$\mathcal{M}u_c + cu_c - f(u_c) = 0. \tag{4.1}$$

The linearized equation is

$$(\partial_t - c\partial_x)u + \partial_x\big(f'(u_c)u - \mathcal{M}u\big) = 0 \tag{4.2}$$

and for a growing mode solution $e^{\lambda t}u(x)$ $(\operatorname{Re}\lambda > 0)$, $u(x)$ satisfies

$$(\lambda - c\partial_x)u + \partial_x\big(f'(u_c)u - \mathcal{M}u\big) = 0. \tag{4.3}$$

We define the following dispersion operator $\mathcal{A}^\lambda : H^m \to L^2$ $(\operatorname{Re}\lambda > 0)$

$$\mathcal{A}^\lambda u = cu + \frac{c\partial_x}{\lambda - c\partial_x}\big(f'(u_c)u - \mathcal{M}u\big)$$

and as before the existence of a purely growing mode is reduced to find $\lambda > 0$ such that $\mathcal{A}^\lambda$ has a nontrivial kernel. When $\lambda \to 0+$, $\mathcal{A}^\lambda$ converges to the zero-limit operator

$$\mathcal{L}_0 =: \mathcal{M} + c - f'(u_c). \tag{4.4}$$

The proof of Theorem 1 for KDV is similar to the BBM and RBou cases. So we only indicate some differences due to the different structure of the operator $\mathcal{A}^\lambda$. To prove the essential spectrum bound

$$\sigma_{\mathrm{ess}}(\mathcal{A}^\lambda) \subset \left\{z \,\middle|\, \operatorname{Re} z \geqslant \frac{1}{2}c\right\}, \tag{4.5}$$

we need to establish analogues of Lemmas 2.2 and 2.3. First, we note that, for any $u \in H^m(\mathbf{R})$,

$$\begin{aligned}\operatorname{Re}\left(-\frac{c\partial_x}{\lambda - c\partial_x}\mathcal{M}u, u\right) &= \operatorname{Re}\int \frac{-ick}{\lambda - ick}\alpha(k)\left|\hat{\phi}(k)\right|^2 dk \\ &= \int \frac{(ck)^2}{\lambda^2 + (ck)^2}\alpha(k)\left|\hat{\phi}(k)\right|^2 dk \geqslant 0.\end{aligned} \tag{4.6}$$

So by estimates as in the proof of Lemma 2.2, for any sequence

$$\{u_n\} \in H^m(\mathbf{R}), \quad \|u_n\|_2 = 1, \quad \operatorname{supp} u_n \subset \left\{x \,\middle|\, |x| \geqslant n\right\},$$

and any complex number z with $\operatorname{Re} z \leqslant \frac{1}{2}c$, we have

$$\operatorname{Re}\left((\mathcal{A}^\lambda - z)u_n, u_n\right) \geqslant \frac{1}{4}c,$$

when n is large enough. Since

$$\left[\mathcal{A}^\lambda, \chi_d\right] = \left(1 - \mathcal{E}^{\lambda,-}\right)[\mathcal{M}, \chi_d] + \left[\mathcal{E}^{\lambda,-}, \chi_d\right]\left(f'(u_c) - \mathcal{M}\right),$$

the conclusion of Lemma 2.3 still holds true by the same proof. Thus the essential spectrum bound (4.5) is obtained as before. The non-existence of growing modes for large λ is proved in the following lemma.

Lemma 4.1. *There exists $\Lambda > 0$, such that when $\lambda > \Lambda$, $\mathcal{A}^\lambda$ has no eigenvalues in $\{z \mid \operatorname{Re} z \leqslant 0\}$.*

Proof. Suppose otherwise, then there exists a sequence $\{\lambda_n\} \to +\infty$, $\{k_n\} \in \mathbb{C}$, and $\{u_n\} \in H^m(\mathbf{R})$, such that $\operatorname{Re} k_n \leqslant 0$ and $(\mathcal{A}^{\lambda_n} - k_n)u_n = 0$. Let $K > 0$ be such that $\alpha(k) \geqslant a|k|^m$ when $|k| \geqslant K$. For any $\delta, \varepsilon > 0$, and large n, we have $\delta\lambda_n \geqslant K$ and

$$\begin{aligned} 0 &\geqslant \operatorname{Re}\left(\mathcal{A}^{\lambda_n}u_n, u_n\right) \\ &\geqslant \int \frac{(ck)^2\alpha(k)}{\lambda_n^2 + (ck)^2}\left|\hat{u}_n(k)\right|^2 dk + c\|u_n\|_{L^2}^2 - \max\left|f'(u_c)\right|\|u_n\|_{L^2}\left\|\frac{c\partial_x}{\lambda_n + c\partial_x}u_n\right\|_{L^2} \\ &\geqslant \int \frac{(ck)^2\alpha(k)}{\lambda_n^2 + (ck)^2}\left|\hat{u}_n(k)\right|^2 dk + (c-\varepsilon)\|u_n\|_{L^2}^2 - \frac{\max|f'(u_c)|^2}{4\varepsilon}\int \frac{(ck)^2}{\lambda_n^2 + (ck)^2}\left|\hat{u}_n(k)\right|^2 dk \\ &\geqslant \int \frac{(ck)^2\alpha(k)}{\lambda_n^2 + (ck)^2}\left|\hat{u}_n(k)\right|^2 dk + (c-\varepsilon)\|u_n\|_{L^2}^2 - \frac{\max|f'(u_c)|^2}{4\varepsilon a(\delta\lambda_n)^m}\int_{|k|\geqslant\delta\lambda_n} \frac{(ck)^2\alpha(k)}{\lambda_n^2 + (ck)^2}\left|\hat{u}_n(k)\right|^2 dk \end{aligned}$$

$$- \frac{\max|f'(u_c)|^2 c^2 \delta^2}{4\varepsilon} \int\limits_{|k| \leqslant \delta\lambda_n} |\hat{u}_n(k)|^2 \, dk$$

$$\geqslant \left(1 - \frac{\max|f'(u_c)|^2}{4\varepsilon a(\delta\lambda_n)^m}\right) \int \frac{(ck)^2 \alpha(k)}{\lambda_n^2 + (ck)^2} |\hat{u}_n(k)|^2 \, dk + \left(c - \varepsilon - \frac{\max|f'(u_c)|^2 c^2 \delta^2}{4\varepsilon}\right) \|u_n\|_{L^2}^2$$

$$> 0, \quad \text{when } n \text{ is large enough,}$$

by choosing $\varepsilon, \delta > 0$ such that

$$c - \varepsilon - \frac{\max|f'(u_c)|c^2\delta^2}{4\varepsilon} > 0.$$

This is a contradiction and the lemma is proved. □

The eigenvalues of $\mathcal{A}^\lambda$ for small λ are also studied by the asymptotic perturbation theory. The required analogues of Lemmas 2.5 and 2.6 can be proved in the same way. The discrete eigenvalues of $\mathcal{A}^0 = \mathcal{L}_0$ are perturbed to get the eigenvalues of $\mathcal{A}^\lambda$ for small λ, in the sense of Proposition 2. The instability criterion in Theorem 1 can be proved in the same way, by deriving the following moving kernel formula: for $\lambda > 0$ small enough, let $k_\lambda \in \mathbf{R}$ to be the only eigenvalue of $\mathcal{A}^\lambda$ near zero, then

$$\lim_{\lambda \to 0+} \frac{k_\lambda}{\lambda^2} = -\frac{dP}{dc} \Big/ \|u_{cx}\|_{L^2}^2, \tag{4.7}$$

where

$$P(c) = \frac{1}{2}(u_c, u_c). \tag{4.8}$$

We sketch the proof of (4.7) below. First, similar to Lemma 2.8, we have the following a priori estimate:

For $\lambda > 0$ small enough, if $(\mathcal{A}^\lambda - z)u = v$, $z \in \mathbb{C}$ with $\operatorname{Re} z \leqslant \frac{1}{2}c$ and $v \in L^2$, then

$$\|u\|_{H^{\frac{m}{2}}} \leqslant C\big(\|u\|_{L_e^2} + \|v\|_{L^2}\big), \tag{4.9}$$

for a constant C independent of λ.

To prove (4.9), we note that for any $\varepsilon > 0$

$$\operatorname{Re}\big(\mathcal{A}^\lambda u, u\big) - \frac{1}{2}c\|u\|_{L^2}^2 \leqslant \|u\|_{L^2}\|v\|_{L^2} \leqslant \varepsilon\|u\|_{L^2}^2 + \frac{1}{4\varepsilon}\|v\|_{L^2}^2$$

and for any $\delta > 0$, when $\lambda \leqslant cK$,

$$\begin{aligned}\operatorname{Re}(\mathcal{A}^\lambda u,u) &\geqslant \int \frac{(ck)^2\alpha(k)}{\lambda^2+(ck)^2}\left|\hat{u}(k)\right|^2 dk + c\|u\|_{L^2}^2 - \|u\|_{L^2_e}\|u_n\|_{L^2}\\ &\geqslant \frac{a}{2}\int\limits_{|k|\geqslant K} |k|^m\left|\hat{u}(k)\right|^2 dk + c\|u\|_{L^2}^2 - \varepsilon\|u\|_{L^2}^2 - \frac{1}{4\varepsilon}\|u\|_{L^2_e}^2\\ &\geqslant \min\left\{\frac{a}{2},\frac{\delta}{K^m}\right\}\int |k|^m\left|\hat{u}(k)\right|^2 dk + (c-\delta-\varepsilon)\|u\|_{L^2}^2 - \frac{1}{4\varepsilon}\|u\|_{L^2_e}^2.\end{aligned}$$

Thus by choosing δ,ε to be small, we get the estimate (4.9).

To prove (4.7), we follow the same procedures as in the BBM and RBou cases. Let $u_\lambda \in H^m(\mathbf{R})$ be the solution of $(\mathcal{A}^\lambda - k_\lambda)u_\lambda = 0$ with $k_\lambda \in \mathbf{R}$ and $\lim_{\lambda\to 0+} k_\lambda = 0$. We normalize u_λ by setting $\|u_\lambda\|_{L^2_e} = 1$. Then by (4.9), we have $\|u_\lambda\|_{H^{\frac{m}{2}}} \leqslant C$ and as before, after a renormalization $u_\lambda \to u_0 = u_{cx}$ in $H^{\frac{m}{2}}$. We have $\lim_{\lambda\to 0+}\frac{k_\lambda}{\lambda} = 0$, since

$$\begin{aligned}\frac{k_\lambda}{\lambda}(u_\lambda,u_{cx}) &= \left(\frac{\mathcal{A}^\lambda-\mathcal{A}^0}{\lambda}u_\lambda,u_{cx}\right) = \left(\frac{1}{\lambda-\mathcal{D}}\big(f'(u_c)-\mathcal{M}\big)u_\lambda,u_{cx}\right)\\ &= \frac{1}{c}\left(\big(f'(u_c)-\mathcal{M}\big)u_\lambda,\frac{\mathcal{D}}{\lambda-\mathcal{D}}u_c\right)\\ &\to -\frac{1}{c}\big(\big(f'(u_c)-\mathcal{M}\big)u_{cx},u_c\big) = -(u_{cx},u_c) = 0.\end{aligned}$$

Similarly as before, we can show that $u_\lambda = \bar{c}_\lambda u_{cx} + \lambda\bar{v}_\lambda$, with $\bar{c}_\lambda \to 1$, $\bar{v}_\lambda \to -\partial_c u_c$ in $H^{\frac{m}{2}}$, when $\lambda \to 0+$. In the proof, we use the facts that

$$\begin{aligned}w_\lambda(x) &= \frac{\mathcal{A}^\lambda-\mathcal{A}^0}{\lambda}u_{cx} = \frac{1}{\lambda-\mathcal{D}}\big(f'(u_c)-\mathcal{M}\big)u_{cx}\\ &= \frac{\mathcal{D}}{\lambda-\mathcal{D}}u_c \to -u_c, \quad \text{when } \lambda\to 0+,\end{aligned}$$

and $\mathcal{L}_0\partial_c u_c = -u_c$. Now

$$\frac{k_\lambda}{\lambda^2}(u_\lambda,u_{cx}) = \bar{c}_\lambda\left(\frac{\mathcal{A}^\lambda-\mathcal{A}^0}{\lambda^2}u_{cx},u_{cx}\right) + \left(\frac{\mathcal{A}^\lambda-\mathcal{A}^0}{\lambda}\bar{v}_\lambda,u_{cx}\right) = \bar{c}_\lambda I_1 + I_2$$

and

$$\begin{aligned}I_1 &= \left(\frac{w_\lambda(x)}{\lambda},u_{cx}\right) = \left(\frac{1}{(\lambda-\mathcal{D})\lambda}\big(f'(u_c)-\mathcal{M}\big)u_{cx},u_{cx}\right)\\ &= -\frac{1}{c}\left(\frac{1}{\lambda-\mathcal{D}}\big(f'(u_c)-\mathcal{M}\big)u_{cx},u_c\right) + \frac{1}{c\lambda}\big(\big(f'(u_c)-\mathcal{M}\big)u_{cx},u_c\big)\\ &= -\frac{1}{c}\big(\big(\mathcal{E}^{\lambda,-}-1\big)u_c,u_c\big) \to \frac{1}{c}(u_c,u_c),\end{aligned}$$

$$I_2 = \left(\frac{\mathcal{A}^\lambda - \mathcal{A}^0}{\lambda}\bar{v}_\lambda, u_{cx}\right) = -\frac{1}{c}\left(\frac{\mathcal{D}}{\lambda - \mathcal{D}}\big(f'(u_c) - \mathcal{M}\big)\bar{v}_\lambda, u_c\right)$$
$$\to -\frac{1}{c}\big(\big(f'(u_c) - \mathcal{M}\big)\partial_c u_c, u_c\big) = -\frac{1}{c}(c\partial_c u_c + u_c, u_c),$$

so

$$\lim_{\lambda\to 0+}\frac{k_\lambda}{\lambda^2} = \lim_{\lambda\to 0+}\frac{\bar{c}_\lambda I_1 + I_2}{(u_\lambda, u_{cx})} = -(\partial_c u_c, u_c)/(u_{cx}, u_{cx}) = -\frac{dP}{dc}\ /\ \|u_{cx}\|_{L^2}^2.$$

5. Discussions

(a) About the spectral assumption for $\mathcal{L}_0$. When $\mathcal{M} = -\frac{d}{dx^2}$, the assumption (1.4) that $\ker(\mathcal{L}_0) = \{u_{cx}\}$ is true because the second order ODE $\mathcal{L}_0\psi = 0$ has two solutions which decay and grow at infinity respectively, and thus u_{cx} is the only decaying solution. Moreover, the solitary waves in such case can be shown to be positive and single-humped. Thus by the Sturm–Liouville theory for second order ODE operators, $n^-(\mathcal{L}_0) = 1$ since u_{cx} has exactly one zero. The proof of (1.4) for non-local dispersive operator $\mathcal{M}$ is much more delicate. In [1,3], (1.4) is proved for solitary waves of some KDV type equations, such as the intermediate long-wave equation [29] with

$$f(u) = u^2 \quad\text{and}\quad \alpha(k) = k\coth(kH) - H^{-1}.$$

The assumption (1.4) is related to the bifurcation of solitary waves, more specifically $\ker\mathcal{L}_0 = \{u_{cx}\}$ implies the non-existence of secondary bifurcations at c, that is, the solitary wave branch $u_c(x)$ is locally unique. Even in cases of multiple branches of solitary waves, (1.4) is still valid in each branch. We note that $\ker\mathcal{L}_0$ also monitors the changes of $n^-(\mathcal{L}_0)$ with respect to c. For example, when (1.4) is valid in a certain range of c, $n^-(\mathcal{L}_0)$ must remain unchanged in this range. Since otherwise, by continuation there is a crossing of eigenvalues through the origin at some c, which will increase the dimension of $\ker\mathcal{L}_0$.This observation has been used in some problems [3,34] to get $n^-(\mathcal{L}_0)$ for large waves from small waves for which $n^-(\mathcal{L}_0)$ is computable. At secondary bifurcation and turning points, the increase of $\ker(\mathcal{L}_0)$ signals the increase or decrease of $n^-(\mathcal{L}_0)$ when these transition points are crossed. One such example is the solitary waves for full water wave problem [34], for which the existence of infinitely many turning points leads $n^-(\mathcal{L}_0)$ to increase without bound by a result of Plotnikov [43].

The assumption (1.4) is required in all existing proof of nonlinear orbital stability [15,22,48].

(b) The sign-changing symbol. We assume $\alpha(k) \geqslant 0$ in our proof of Theorem 1. The proof can be easily modified to treat sign-changing symbols. Let $-\gamma = \inf\alpha(k) < 0$. Consider solitary wave solutions of KDV, BBM, and RBou type equations with

$$c > \gamma, \qquad 1 - \frac{1}{c} > \gamma \quad\text{and}\quad 1 - \frac{1}{c^2} > \gamma, \tag{5.1}$$

respectively. The condition (5.1) on c is to ensure that the essential spectrum of $\mathcal{L}_0$ lies in the positive axis, which seems to be necessary to get decaying solitary waves, such as in [7] and [5] for fifth order KDV and Benjamin equations with $\alpha(k) = -k^2 + \delta k^4$ and $-|k| + \delta k^2$, respectively. Denote $\tilde{\mathcal{M}}$ to be the multiplier operator with the nonnegative symbol $\tilde{\alpha}(k) = \alpha(k) + \gamma$.

The proof of Theorem 1 remains unchanged, by replacing $\mathcal{M}$ with $\tilde{\mathcal{M}} - \gamma$ and using the nonnegative symbol $\tilde{\alpha}(k)$ in all the estimates. The same proof still go through because of the condition (5.1). However, for sign-changing symbols the solitary waves might be highly oscillatory in some parameter range [5,7]. It is conceivable that such oscillatory waves are energy saddle with $n^-(\mathcal{L}_0) \geqslant 2$, whose stability cannot be studied by the traditional idea of proving energy minimizers. Theorem 1 gives a sufficient condition for instability in such cases.

(c) Comparisons with the Evans function method. In [41], Pego and Weinstein used the Evans function technique to obtain the instability criterion $dP/dc < 0$ for the case $\mathcal{M} = -\frac{d}{dx^2}$. In their paper, the eigenvalue problems (2.4), (3.3) and (4.3) are written as a first order system in x, depending on the parameter λ. The Evans function $D(\lambda)$ is a Wronskian-like function whose zeros in the right half-plane correspond to unstable eigenvalues, and it measures the intersection of subspaces of solutions exponentially decaying at $+\infty$ and $-\infty$. This method was first introduced by J.W. Evans in a series papers including [21] and was further studied in [6]. In [41], it was shown that $D(\lambda) > 0$ when $\lambda > 0$ is big enough, $D(0) = D'(0) = 0$ and

$$D''(0) = \operatorname{sgn} dP/dc. \tag{5.2}$$

If $dP/dc < 0$, then $D(\lambda) < 0$ and a continuation argument yields the vanishing of $D(\lambda)$ at some $\lambda > 0$, which establishes a growing mode. A similar formula as (5.2) was derived in [17], for problems that can be written in a multi-symplectic form. However, there are several restrictions of the Evans function technique: (1) Only differential operators, that is, those with polynomial symbols, can be treated by this method, in order for the eigenvalue problems to be written as first order ODE systems. (2) The solitary waves must have an exponential decay. Moreover, certain assumptions for eigenvalues of the asymptotic systems are required in constructing the Evans function [41, (0.6), (0.7)]. Such assumptions need to be checked case by case, and their relations to the properties of solitary waves are not very clear. By comparison, our approach applies to very general dispersive operators, in particular, non-local operators. We impose no additional assumptions on the solitary waves. For example, we can allow slowly decaying or highly oscillatory solitary waves. Our only assumption (1.4) is closely related to the bifurcation of solitary waves, and it appears to be rather natural in the stability theory. Moreover, the Evans function method can only be used for the one-dimensional problems, since otherwise the first order system cannot be written. Our approach has no such restriction and might be used in the multi-dimensional settings.

(d) Some future problems. We mention some open issues related to our study.

(i) When the instability conditions in Theorem 1 are not satisfied, the stability of the solitary waves was proved only for the KDV and BBM type equations [15,48] when $n^-(\mathcal{L}_0) = 1$. However, when $n^-(\mathcal{L}_0) > 1$ such solitary waves are energy saddles with an even Morse index, and their stability is a very subtle issue and not well understood even for the finite-dimensional Hamiltonian systems. For regularized Boussinesq type problems, the solitary waves are always highly indefinite energy saddles. Moreover, this issue of studying stability of energy saddles also arise in the full water wave problem [14,34] and Boussinesq systems [42]. A natural first step is to study the spectral stability of these solitary waves, for which Theorem 2 about transition points of stability might be useful as discussed in Section 1. Lastly, we note that for the critical case with $dP/dc = 0$, the nonlinear instability of solitary waves was recently proved [18] for the generalized KDV equation with $\mathcal{M} = -\frac{d}{dx^2}$. However, such an instability is not due to the linearized growing modes.

(ii) Can we get nonlinear orbital instability with an exponential rate from the linear instability, particularly in the L^2 norm? This problem is open, even in the KDV and BBM cases where the nonlinear instability in the energy norm has been proved [15,23]. This problem is also relevant to full water waves and some other problems for which the blow-up issue is concerned. The L^2 instability results would be useful to distinguish the large scale instability of basic waves from the local blow-up instability.

Acknowledgments

This work is supported partly by the NSF grants DMS-0505460 and DMS-0707397. The author thanks Yue Liu for helpful discussions and Lixin Yan for discussions on the proof of Lemma 2.3. He also thanks the referee for useful comments.

Appendix A

In this Appendix A, we describe a different approach from [15] and [46] to prove nonlinear instability for some dispersive wave models. In [15] and [46], the Liapunov functional method of [22] was extended to show nonlinear instability of solitary waves of KDV and BBM type equations, under the assumptions $dP/dc < 0$ and (1.5). For the KDV case, the Liapunov functional constructed in [15] is of the form

$$A(t) = \int Y\big(x - x(t)\big)u(x,t)\,dx, \tag{A.1}$$

where $Y(x) = \int_{-\infty}^{x} y(z)\,dz$ and $y(x)$ is an energy decreasing direction under the constraint of the constant momentum $Q(u)$. By using the fact that the solitary wave considered is an (constrained) energy saddle with negative index one, it can be shown [22] that $A'(t) \geqslant \delta > 0$ in the orbital neighborhood of the solitary wave. The nonlinear instability would follow immediately if $A(t)$ is bounded, as considered in the setting of [22]. However, $A(t)$ defined by (A.1) is not bounded because the function $Y(x)$ is not in L^2 if $\int y\,dx \neq 0$. To overcome this difficulty, in [15] it was shown that $A(t) \leqslant C(1 + t^{\eta})$ for some $\eta < 1$, then the nonlinear instability still follows. Such an estimate was obtained by showing that the maximum of the anti-derivative of $u(x,t)$ has a sublinear growth. The same approach is used in [38,46] and [20] for other dispersive wave equations, and the sublinear estimates are usually highly technical to prove. Below, we show that such an estimate can be avoided by using another approach, which was first introduced in [31] for a Schrödinger type problem.

The idea in [31] is to make a small correction to the (energy) decreasing direction $y(x)$ used in constructing the Liapunov functional $A(t)$. The new direction, still decreasing, has the additional property that its integral over $\mathbf{R}$ is zero. Then the new anti-derivative $Y(x) \in L^2$ and thus $A(t)$ is bounded which implies nonlinear instability. The correction is by the following lemma, which is a generalization of [31, Lemma 5.2].

Lemma A.1. *For any $r(x) \neq 0 \in L^2(\mathbf{R})$, $c \in \mathbf{R}$ and $m \geqslant 1$, there exists a sequence $\{y_n\}$ in $H^m(\mathbf{R})$ such that*

$$\big(1+|x|\big)y_n(x) \in L^1(\mathbf{R}), \qquad \int y_n(x)\,dx = c,$$

$y_n \to 0$ in $H^{\frac{m}{2}}(\mathbf{R})$ and $(y_n, r) = 0$.

Proof. We choose $\varphi(x) \in C_0^\infty(\mathbf{R})$ such that $\int \varphi(x)\,dx = c$. We claim that there exists $\psi(x) \in C_0^\infty(\mathbf{R})$ such that $(\psi_x, r) \neq 0$. Suppose otherwise, for any $\psi(x) \in C_0^\infty(\mathbf{R}^1)$, we have $(\psi_x, r) = 0$. Then $r_x = 0$ in the distribution sense and thus $r \equiv$ constant. But $r \in L^2$, so $r = 0$, which is a contradiction. Define

$$y_n = \frac{1}{n}\varphi\left(\frac{x}{n}\right) - a_n \psi_x(x),$$

with

$$a_n = \frac{\int \frac{1}{n}\varphi(\frac{x}{n}) r(x)\,dx}{(\psi_x, r)}.$$

Then $(y_n, r) = 0$ and

$$|a_n| \leqslant \frac{\|\frac{1}{n}\varphi(\frac{1}{n}x)\|_2 \|r\|_2}{|(\psi_x, r)|} = O\left(\frac{1}{\sqrt{n}}\right) \to 0,$$

when $n \to \infty$. Let $\varphi_n(x) = \varphi(\frac{x}{n})$, then

$$\left\|\frac{1}{n}\varphi\left(\frac{x}{n}\right)\right\|^2_{H^{\frac{m}{2}}} = \frac{1}{n}\|\varphi\|^2_{L^2} + \frac{1}{n^2}\big\||D|^{\frac{m}{2}}\varphi_n\big\|^2_{L^2} = \frac{1}{n}\|\varphi\|^2_{L^2} + \frac{1}{n^{m+1}}\big\||D|^{\frac{m}{2}}\varphi\big\|^2_{L^2}$$
$$\to 0, \quad \text{when } n \to \infty,$$

where in the above we use the scaling formula

$$|D|^{\frac{m}{2}}\varphi_n(x) = \frac{1}{n^{\frac{m}{2}}}\left(|D|^{\frac{m}{2}}\varphi\right)\left(\frac{x}{n}\right)$$

as in the proof of Lemma 2.3. Therefore, $y_n \to 0$ in $H^{\frac{m}{2}}(\mathbf{R})$, $(1+|x|)y_n \in L^1$ and

$$\int y_n(x)\,dx = \int \frac{1}{n}\varphi\left(\frac{x}{n}\right)dx = \int \varphi(x)\,dx = c.$$

The lemma is proved. □

We start with an (constrained) energy decreasing direction $y(x)$ with $(1+|x|)y(x) \in L^1$, that is,

$$(\mathcal{H}y, y) < 0 \quad \text{and} \quad \big(y, Q'(u_c)\big) = 0,$$

where $\mathcal{H}$ is the second order variation of the argumented energy functional, for which the solitary wave is a critical point. Let $H^{\frac{m}{2}}$ to be the energy space, that is, m is the leading power of the operator $\mathcal{H}$. Choosing $c = \int y\,dx$, $r = Q'(u_c)$ in the above lemma, we get a sequence $\{y_n\} \in H^m$ with the properties listed in the lemma. Defining $\tilde{y}_n = y - y_n$, then we have

$$\big(1+|x|\big)\tilde{y}_n(x) \in L^1, \qquad \big(\tilde{y}_n, P'(u_c)\big) = 0, \qquad \int \tilde{y}_n\,dx = 0$$

and $(\mathcal{H}\tilde{y}_n, \tilde{y}_n) < 0$ when n is big enough. Thus for large n, the function $\tilde{y}_n$ is a new (constrained) energy decreasing direction with zero integral. The Liapunov functional $A(t)$ is defined as in (A.1) by using this new direction $\tilde{y}_n$. By [15, p. 409], $Y(x) = \int_{-\infty}^{x} \tilde{y}_n(z)\,dz \in L^2$, thus $A(t)$ is bounded and the nonlinear instability results. Above approach has the following physical interpretation: if a solitary wave is not an energy minimizer under the constraint of constant momentum, neither is it even under the additional constraint of constant mass. This rather general idea could be useful in proving nonlinear instability of (constrained) energy saddles with Morse index one, for other similar dispersive wave problems.

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