CORRIGENDUM: "UNSTABLE SURFACE WAVES IN RUNNING WATER" [COMM. MATH. PHYS., 282, (2008) 733-796]

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It was pointed out by Michael Renardy that there were a few errors in Section 4 and Section 6 of [1]. First, in [1, Proposition 4.4] it was claimed that for a class of shear flows the wave speed of a neutral limiting mode must be the inflection value. As indicated in [2], however, two other possible values of the wave speed must be added, namely, the extremum value and the value at the bottom of the shear flow. We give its correct statement. Accordingly, we make changes to the statements of several assertions in Section 4 and Section 6 of [1]. Second, the proof of [1, Lemma 4.11] contains an error. We give a corrected proof, which works for more general shear flows. We note that these changes do not affect the main conclusions of [1]; see Remark 4.

Throughout, we follow the notation in [1] to study the Rayleigh equation

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi=0 \quad \text{in } (0,h) \tag{1}$$

with the boundary conditions

$$\begin{cases}(U(h)-c)^2\phi'(h)=(g+U'(h)(U(h)-c))\phi(h),\\ \phi(0)=0.\end{cases} \tag{2}$$

The main correction is to change [1, Proposition 4.4] to the following.

Proposition 4.4. *For $U\in\mathcal{K}^+$, a neutral limiting mode (ϕ_s,α_s,c_s) must be one of the following three cases:*

(1) *$c_s=U_s$ (inflection value) and ϕ_s solves (1)–(2) with $c=U_s$;*

(2) *$c_s=U(0)$ and $\phi_s>0$ in $(0,h)$ solves (1)–(2) with $c=U(0)$;*

(3) *$c_s=U_c$, an extremum value of U; if $y_c\in(0,h)$ is the largest such that $U(y_c)=U_c$, then $\phi_s(y)=0$ for $y\in[0,y_c]$ and $\phi_s>0$ in $y\in(y_c,h)$ solves (1) with the boundary conditions*

$$\begin{cases}(U(h)-c_s)^2\phi_s'(h)=(g+U'(h)(U(h)-c_s))\phi_s(h),\\ \phi_s(y_c)=0,\ \phi_s'(y_c)=0.\end{cases}$$

The case (2) can happen only when $U(y)>U(0)$ or $U(y)<U(0)$ for $y\in(0,h]$.

The proof of that $c_s=U_s, U(0)$ or U_c is sketched in [2] by completing the arguments in [1]. We will show properties of ϕ_s in cases (2) and (3).

Suppose $c_s\neq U_s$ or $U(0)$. Let $\tilde{y}\in(0,h]$ be the largest such that $U(\tilde{y})=c_s$. It is shown in [1] that ϕ_s is not identically zero and $\phi_s(y)=0$ for $y\in[0,\tilde{y}]$. We claim that ϕ_s vanishes nowhere in $(\tilde{y},h)$. Suppose otherwise that $\phi_s(\bar{y})=0$ for some $\bar{y}\in(\tilde{y},h)$. Then, $\phi_s\equiv0$ on $[\tilde{y},\bar{y}]$ by [1, Lemma 4.7]. Since the ordinary differential equation (1) is regular in $[\bar{y},h]$ furthermore $\phi_s\equiv0$ on $[\bar{y},h]$. A contradiction then proves the claim, and we may assume $\phi_s>0$ on $(\tilde{y},h)$. If $U'(\tilde{y})\neq0$, then from

$\phi_s(\tilde{y}) = \phi_s'(\tilde{y}) = 0$ and from the H^2-bound in [1, Lemma 4.6] of ϕ_s, we deduce that $\phi_s \equiv 0$ in a neighborhood of $\tilde{y}$ by the property of solutions of ODE (1) near $\tilde{y}$. Thus, c_s must be an extremum and ϕ_s has the stated properties. If $c_s = U(0)$ which is not an extremum value then $U - c_s \neq 0$ on $(0, h)$. Otherwise, $\phi_s \equiv 0$ on $(0, h)$ by the above argument. If $c_s = U(0)$ and $U(y) = U(0)$ only at $y = 0$ then ϕ_s cannot vanish on $(0, h]$ and hence we may assume $\phi_s > 0$ in $(0, h]$.

As pointed out in [2], the mistake in [1] was to conclude that $\phi_s = 0$ on $[0, \tilde{y}]$ would imply $\phi_s = 0$ in $[0, h]$ assuming $c_s \neq U_s$ only; the cases $c_s = U(0)$ and an extremum value were neglected.

Remark 1. As shown in [2] by means of examples and numerical computations, all three cases of neutral limiting modes with $c = U(0)$, an inflection value or an extremum value can occur. In case $c_s = U(0)$ or an extremum value, in view of Proposition 4.4 the wave number α_s is unique.

We must change the statement of [1, Theorem 4.2], accordingly. Let α_0 be the wave number of the neutral limiting mode with $c = U(0)$ and $\alpha_1, \cdots, \alpha_k$ be waves numbers of neutral limiting modes with c to be extremum of U.

Theorem 4.2. *For $U \in \mathcal{K}^+$, suppose that $-\alpha_{\max}^2 < 0$ is the lowest eigenvalue of the ordinary differential operator $-\frac{d^2}{dy^2} - K$ on the interval $(0, h)$ with the boundary conditions (4.5)–(4.6) in [1]. Let*

$$\bar{\alpha} = \begin{cases} 0 & \text{if } \min\{\alpha_0, \cdots, \alpha_k\} \geqslant \alpha_{\max}, \\ \max\{\alpha_i \mid \alpha_i < \alpha_{\max}\} & \text{if } \min\{\alpha_0, \cdots, \alpha_k\} < \alpha_{\max}. \end{cases}$$

Then, to each $\alpha \in (\bar{\alpha}, \alpha_{\max})$ there corresponds an unstable solution triple (ϕ, α, c) (with $Im\, c > 0$) of (1)–(2).

The proof is a minor modification of that in [1]. It is shown in [1] that for $U \in \mathcal{K}^+$, unstable modes bifurcate near a neutral mode with $c = U_s$ (inflection value) and $\alpha = \alpha_{\max}$ for α slightly less than $\alpha_{\max}$. The unstable mode then continues for smaller α. Since the continuation can only stop at a neutral limiting mode with $c = U(0)$ or an extremum value, there is an unstable mode for any $\alpha \in (\bar{\alpha}, \alpha_{\max})$. Of course, the interval $(\bar{\alpha}, \alpha_{\max})$ may not be the sharp interval for instability. To get a completed picture, one needs to study the local bifurcation of unstable modes near neutral modes with $c = U(0)$ or an extremum value.

In view of [1, Lemma 4.11] and the corrected Theorem 4.2, we make changes to the statement of [1, Corollary 4.12].

Corollary 4.12. *A monotone shear flow U with exactly one inflection point y_s and $-U''/(U - U(y_s)) > 0$ is unstable in the free surface setting, for wave numbers in an interval $(\bar{\alpha}, \alpha_{\max})$, where*

$$\bar{\alpha} = \begin{cases} 0 & \text{if } \alpha_0 \geqslant \alpha_{\max} \\ \alpha_0 & \text{if } \alpha_0 < \alpha_{\max} \end{cases},$$

and $\alpha_0, \alpha_{\max} > 0$ are as defined in Theorem 4.2.

In Section 6 of [1], we made the same mistake as in [1, Proposition 4.4] in characterizing neutral limiting modes for the class $\mathcal{F}$. Lemma 6.2 and Proposition 6.6 of [1] must be replaced by the following.

Lemma 6.2. *For $U \in \mathcal{F}$, we have the same characterization of neutral limiting modes as in Proposition* 4.4.

For the proof, we first derive an H^2-bound for the sequence of unstable modes approaching a neutral limiting mode. Then, the same arguments as in the proof of Proposition 4.4 apply. The H^2 bound can be obtained by a slight adjustment of the proof of [1, Lemma 4.6].

Changes are required for several assertions in [1, Section 6]. The conclusions of Lemma 6.3 and Theorem 6.4 in [1] should be dropped. In fact, it was claimed in [1, Theorem 6.4] that a shear flow without inflection is stable. But, it was shown in [2] by numerical computations that $U(y) = 1 - y^2$ for $y \in (-1,1)$, has unstable modes for wave number in $(\alpha_{\min}, +\infty)$, where $\alpha_{\min}$ is the wave number of the neutral mode with c the extremum value 1. The description in [1, Theorem 6.7] of the interval of unstable wave number must include the cases of the neutral modes with $c = U(0)$ and extremum values in addition to the case with c to be inflection values.

Another corrigendum is in the proof of [1, Lemma 4.11] due to the incorrect use of the Sturm-Liouville comparison theorem. We fix it in the following.

Proof of Lemma 4.11 in [1]. Let $K = -U''/(U - U_s)$ and μ_0 be the lowest eigenvalue of $-\frac{d^2}{dy^2} - K$ in $(0,h)$ with the Dirichlet boundary conditions at $y = 0$ and $y = h$. We consider three cases.

Case 1: $\mu_0 < 0$. By the argument in [1, Remark 4.10], the lowest eigenvalue of $-\frac{d^2}{dy^2} - K$ with boundary conditions (2) for $c = U_s$ is negative, which we denote by $-\alpha^2$. Then, a solution to (1)-(2) is found for $c = U_s$ and α.

Case 2: $\mu_0 > 0$. We modify the arguments in the proof of [1, Lemma 4.11]. Let ϕ_α be the solution of the Rayleigh equation

$$\phi_\alpha'' + (K - \alpha^2)\phi_\alpha = 0 \quad \text{in } (0,h)$$

with $\phi_\alpha(0) = 0$ and $\phi_\alpha'(0) = 1$. We claim that

$$\phi_\alpha > 0 \quad \text{on } (0,h] \quad \text{for any } \alpha \geqslant 0. \tag{3}$$

Suppose on the contrary that there exists $\alpha_0 \geqslant 0$ such that ϕ_{α_0} has a zero in $(0,h]$. Let $y_0 \in (0,h]$ be such that $\phi_{\alpha_0}(y) > 0$ for $y \in (0,y_0)$ and $\phi_{\alpha_0}(y_0) = 0$. Thus, $-\alpha_0^2$ is the lowest eigenvalue of $-\frac{d^2}{dy^2} - K$ on $(0,y_0)$ with the Dirichlet boundary conditions $\phi(0) = 0 = \phi(y_0)$. Since the lowest eigenvalue of $-\frac{d^2}{dy^2} - K$ in $(0,a)$ with the zero Dirichlet conditions is decreasing in $a \in (0,h]$, it follows that $\mu_0 \leqslant -\alpha_0^2 \leqslant 0$. A contradiction then proves (3). The rest of the proof is the same as that in [1].

Case 3: $\mu_0 = 0$. The same argument as in Case 2 shows that $\phi_\alpha > 0$ on $(0,h]$ for any $\alpha > 0$. Note that $\phi_0(h) = 0$. Since ϕ_0 is the zero eigenfunction of $-\frac{d^2}{dy^2} - K$

on $(0,h)$ with zero boundary conditions, $\lim_{\alpha\to 0+}\phi_\alpha(h)=0$. Hence,

$$\begin{aligned} f(\alpha) &= \frac{U(0)-U_s}{(U(h)-U_s)\phi_\alpha(h)} + \frac{\alpha^2}{(U(h)-U_s)\phi_\alpha(h)}\int_0^h (U-U_s)\phi_\alpha dy - \frac{g}{(U_s-U(h))^2} \\ &= \frac{1}{\phi_\alpha(h)}\left[\frac{U(0)-U_s}{U(h)-U_s} + \frac{\alpha^2}{(U(h)-U_s)}\int_0^h (U-U_s)\phi_\alpha dy\right] - \frac{g}{(U_s-U(h))^2} \end{aligned}$$

defined in [1, page 768] satisfies that $\lim_{\alpha\to 0+} f(\alpha) = -\infty$. The rest of the proof again follows by that in [1]. □

Remark 2. Indeed, [1, Lemma 4.11] holds true for any non-monotone U satisfying K is bounded and U_s is between $U(0)$ and $U(h)$.

Remark 3. If $U(0)$ is the strict absolute minimum or maximum and $c_s = U(0)$ then

$$-\alpha_s^2 = \inf_{\substack{\phi\in H^1(0,h)\\ \phi(0)=0}} \frac{\int_0^h \left(|\phi'(y)|^2 + \frac{U''(y)}{U(y)-U(0)}|\phi(y)|^2\right)dy - \left(\frac{g}{(U(h)-U(0))^2} + \frac{U'(h)}{U(h)-U(0)}\right)|\phi(h)|^2}{\int_0^h |\phi(y)|^2 dy}.$$

Choosing the test function $\phi = U - U(0)$, we have

$$-\alpha_s^2 \leqslant -\frac{g}{\int_0^h (U-U(0))^2 dy} < 0.$$

Thus, there exists unique neutral limiting mode with $c = U(0)$ for any $g > 0$. If $g = 0$, obviously, $\alpha_s = 0$ and $\phi_s = U - U(0)$. Then, for g small, $\alpha_s = O(\sqrt{g})$ and $\alpha_s \to +\infty$ as $g \to \infty$. Therefore, by Corollary 4.12, we obtain instability of monotone U with $\frac{U''}{U-U(y_s)} < 0$, for wave lengths as long as $O(1/\sqrt{g})$ when g is small.

Remark 4. The main results in [1] are not affected by the above changes. An important example used in [1] is

$$U(y) = A\sin(\beta(y-h/2)), \qquad y\in[0,h], \tag{4}$$

for any $h, \beta > 0$ satisfying $h\beta \leqslant \pi$ for arbitrary A and h. By the correct Corollary 4.2, such shear flow is unstable for wave numbers in the interval $(\bar\alpha, \alpha_{\max})$, instead of $(0, \alpha_{\max})$ as claimed in [1]. As shown in [1], there bifurcate small-amplitude periodic water waves with the shear background (4) and for any wave number α. These periodic waves are unstable when $\alpha \in (\bar\alpha, \alpha_{\max})$, supporting two conclusions in [1]: first, the bifurcation of periodic water waves is unrelated to stability of background flows; second, an arbitrarily small vorticity to the irrotational system of an arbitrary depth may induce instability of water waves. We refer the reader to Remarks 4.14 and 5.2 in [1] for details.

We also want to take this opportunity to correct a few typos: on page 756, equation (4.8a), the + before $g_r(U_s)$ should be −; on page 757, line -2, the + before $K(y)$ should be −.

References

[1] Vera Hur and Zhiwu Lin, *Unstable surface waves in running water*, Comm. Math. Phys., **282**, 733-796 (2008).

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