

## Some recent results on instability of ideal plane flows

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ABSTRACT. We study linear and nonlinear instability of steady incompressible inviscid flows. First, some sufficient conditions of linear instability for a class of shear flows are given. Second, nonlinear instability in the  $\mathbf{L}^2$ -norm of velocity is proved under the assumption that the growth rate exceeds half of the Liapunov exponent of the steady flow, for any bounded domain. We also estimate the growth of the semigroup generated by the linearized operator, in the  $\mathbf{L}^p$  ( $p > 1$ ) space of the velocity.

### 1. Introduction

In this paper, we study the stability and instability of ideal plane flows. We present some new results for linear and nonlinear instability of ideal plane flows. Meanwhile, we give a brief survey of some recent results on this topic. We consider the two-dimensional incompressible inviscid flow satisfying the Euler equation

$$(1.1a) \quad \partial_t u + (u \cdot \nabla u) + \nabla p = 0,$$

$$(1.1b) \quad \nabla \cdot u = 0,$$

in a bounded domain  $\Omega$  of class  $\mathbf{C}^2$  with smooth boundary  $\partial\Omega$  composed of a finite number of connected components  $\Lambda_i$ . The boundary condition is

$$(1.1c) \quad u \cdot n = 0 \quad \text{on } \partial\Omega,$$

where  $n$  stands for the unit outer normal of  $\partial\Omega$ . The global existence of Euler equation in two dimension is quite well understood (see [15], [14]). However, many basic questions about the behavior of the solutions remain open. The stability and instability of steady ideal plane flows is a classical problem, dated back to 19th century ([17]). To rigorously prove instability for the full nonlinear problem, first we must find an eigenvalue of the linearized operator with positive real part, that is, a growing mode to the linearized equation. To prove nonlinear instability, we need to find a norm  $\|\cdot\|$  such that the nonlinear term behaves as  $O(\|\cdot\|^2)$ . For the Euler equation, this is not easy since the nonlinear term involves derivative. The linearized equation of (1.1) around a steady flow  $u_0$  is

$$(1.2) \quad \partial_t v + u_0 \cdot \nabla v + v \cdot \nabla u_0 + \nabla p = 0, \text{ in } \Omega$$

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with boundary conditions (1.1b) and (1.1c). The linearized operator is

$$L_0 v = -u_0 \cdot \nabla v - v \cdot \nabla u_0 - \nabla p$$

with  $v$  satisfying boundary conditions (1.1b) and (1.1c). It is a non self-adjoint and degenerate operator. Thus it is rather difficult to find a growing mode of (1.2) or a discrete unstable eigenvalue of  $L_0$ . Even for the simplest steady flows, shear flows of the form  $(U(y), 0)$  with  $y_1 \leq y \leq y_2$ , very few sufficient conditions for existence of growing modes are known although many efforts have been devoted to it. For shear flows, to find growing modes is equivalent to solve the following classical Rayleigh equation

$$(1.3) \quad (U - c) \left( \frac{d^2}{dy^2} - \alpha^2 \right) \phi - U'' \phi = 0$$

with  $\phi(y_1) = \phi(y_2) = 0$ . Here  $\alpha$  is the wave number (positive real) in the  $x$ -direction and  $c = c_r + ic_i$  is the complex wave speed. The flow is linearly unstable if some non-trivial solution to (1.3) with  $c_i > 0$  exists. A classical result of Lord Rayleigh ([17]) is the necessary condition for instability that the basic velocity profile should have an inflection point at some point  $y = y_s$ , that is,  $U''(y_s) = 0$ . However it is only a necessary condition for instability. Until now there are very few sufficient conditions for instability. In 1930s, Tollmien ([18]) gave some formal argument to find unstable modes near neutral modes. Besides that, there are only some special examples for which linear instability can be proved rigorously, such as  $U(y) = \sin my$  or  $\cos my$  ( $m$  integer) studied in [6]. In [10], we obtain some sufficient conditions of linear instability for several classes of shear flows. One class of shear flows studied in [10] is the class  $\mathcal{K}^+$  flows, with the steady velocity profile  $U(y)$  in  $\mathbf{C}^2[y_1, y_2]$  and the following function

$$K(y) := -U''(y) / (U(y) - U_s)$$

being positive and bounded in  $[y_1, y_2]$ , where  $U_s$  is the only inflection value of  $U$ .

**THEOREM 1.1.** ([10]) *For a shear flow  $(U(y), 0)$  in class  $\mathcal{K}^+$ . Let  $-\alpha_{\max}^2$  be the lowest eigenvalue of  $-\frac{d^2}{dy^2} - K(y)$ , which is assumed to be positive. For all  $\alpha \in (0, \alpha_{\max})$ , there is an unstable solution (with  $\text{Im } c > 0$ ) to (1.3). The instability condition is sharp in the sense that there are no growing modes if  $\alpha \geq \alpha_{\max}$  or  $-\frac{d^2}{dy^2} - K(y)$  is nonnegative.*

A particular example of class  $\mathcal{K}^+$  flow is  $U(y) = \sin my$  or  $\cos my$  ( $m$  is any real number) and thus the result in [6] is easily recovered by Theorem 1.1. In [10], results of the same type as Theorem 1.1 have been obtained for plane rotating flows in an annulus and a sufficient condition for instability of odd shear flows is also given. In Section 2 of this paper, we study linear instability of more general shear flows including those with  $U(y)$  being monotone or satisfying  $U''(y) = g(U(y))k(y)$  for some function  $k(y) > 0$ . For these flows, we obtain a sufficient condition of instability which in some cases is sharp by numerical evidences. We note that there are very few results of linear instability for more general flows in a general domain. In [11], we obtain the first general sufficient condition for linear instability of steady flows satisfying

$$\omega_0 = -\Delta\psi_0 = g(\psi_0),$$

where  $\omega_0$  and  $\psi_0$  are steady vorticity and stream functions. We prove the existence of a growing mode assuming that the following elliptic operator

$$(1.4) \quad A := -\Delta - g'(\psi_0)\phi + g'(\psi_0)\tilde{P}$$

has an odd number of negative eigenvalues and no kernel. Here  $\tilde{P}$  is the projection operator of  $\mathbf{L}^2(\Omega)$  into  $\ker\{u_0 \cdot \nabla\}$ . It is also showed in [11] that if  $g' > 0$  and  $A$  is positive, then the steady flow is linearly stable.

After obtaining a growing solution to the linearized equation, we need to show that linear instability implies nonlinear instability. This problem has two main difficulties. One difficulty is that the nonlinear term contains derivative. Another one is related to the structure of the essential spectrum of the linearized operator  $L_0$ . Define the space

$$L_{\text{sol}}^2(\Omega) = \{v = (v_1, v_2) \in L^2(\Omega)^2 \mid \nabla \cdot v = 0, v \cdot n = 0 \text{ on } \partial\Omega\}.$$

For a general domain  $\Omega$ , so far there is no rigorous characterization of  $\sigma_{L_{\text{sol}}^2}^{\text{ess}}(L_0)$ . When  $\Omega$  is a torus, it was showed ([7]) that

$$\sigma_{L_{\text{sol}}^2}^{\text{ess}}(L_0) = \{z \in \mathbf{C} \mid -\mu \leq \text{Re } z \leq \mu\}.$$

Here the number  $\mu$  is the Liapunov exponent of the flow generated by the steady velocity field  $u_0$ . Note that  $\mu > 0$  is equivalent to that  $\psi_0$  has a nondegenerate saddle point. Thus for general steady flows, the operator  $L_0$  has unstable essential spectrum. If the discrete spectrum lies deeply inside the essential spectrum, nonlinear instability is difficult to prove. In recent years, the nonlinear instability problem has been studied by several authors ([1], [5], [8], [19]). All these results essentially need to assume that the discrete eigenvalue is dominant, or lies outside the essential spectrum region. The best result is obtained by Bardos, Guo and Strauss in [1]. They prove nonlinear instability in the  $\mathbf{L}^2$ -norm of vorticity if the linear growth rate exceeds the Liapunov exponent of the steady flow, for a general bounded domain. In Section 3, we prove nonlinear instability in the  $\mathbf{L}^2$ -norm of velocity if the growth rate exceeds half of the Liapunov exponent, for steady flows in a bounded domain. After this result is obtained, we show in [12] the completed nonlinear instability result in the  $\mathbf{L}^p$ -norm of velocity (for any  $p > 1$ ) with no restriction on the growth rate. The idea of using a coupled approach introduced here plays an important role in the proof of the completed result in [12] and the recent proof of nonlinear instability of Vlasov-Poisson system ([13]). Using the same idea, we can estimate the growth of the semigroup  $e^{tL_0}$  in the  $\mathbf{L}^p$  space of velocity for  $p > 1$ , in any bounded domain. These estimates are new to our knowledge.

In Section 4, we make some comments related to our results.

## 2. Extended study of shear flow instability

In this section, we extend Theorem 1.1 to more general shear flows.

**DEFINITION 2.1.** A velocity profile  $U(y)$  is said to be in class  $\mathcal{F}$ , if for each number  $c$  in the range of  $U$  but not an inflection value,  $U''$  takes the same sign at all points where  $U(y) = c$ .

Some examples in class  $\mathcal{F}$  are a monotone flow, a symmetric flow with monotone half part and a flow satisfying  $U''(y) = g(U(y))k(y)$  for some positive function  $k(y)$  and any continuous function  $g$ . In particular, all flows in class  $\mathcal{K}^+$  are in class  $\mathcal{F}$ .

DEFINITION 2.2. The triple  $(c_s, \alpha_s, \phi_s)$  with  $c_s$  real and  $\alpha_s$  positive is said to be a neutral limiting mode, if it is the limit of a sequence of growing solutions  $(c_k, \alpha_k, \phi_k)$  (with  $\text{Im } c_k > 0$ ) of the Rayleigh equation (1.3). Formally  $(c_s, \alpha_s, \phi_s)$  ought to satisfy the Rayleigh equation

$$(2.1) \quad (U - c_s) \left( \frac{d^2}{dy^2} - \alpha_s^2 \right) \phi_s - U'' \phi_s = 0.$$

We call  $c_s$  the neutral limiting phase speed and  $\alpha_s$  the neutral limiting wave number.

Our first step is to study all possible neutral limiting modes. It is showed in [10] that for any shear flow  $(U(y), 0)$  in class  $\mathcal{F}$ , the neutral limiting phase speed must be an inflection value  $U_s$  of  $U$ . Furthermore, the following result holds.

THEOREM 2.3. *If the flow is in class  $\mathcal{F}$ , then for any neutral limiting mode  $(c_s, \alpha_s, \phi_s)$  with positive  $\alpha_s$ , we have  $c_s = U_s$  and the function  $\phi_s$  must solve*

$$(2.2) \quad -\frac{d^2}{dy^2} \phi_s + \frac{U''}{U - U_s} \phi_s = -\alpha_s^2 \phi_s$$

with  $\phi_s(y_1) = \phi_s(y_2) = 0$ . Thus  $-\alpha_s^2$  is a negative eigenvalue of the operator  $-\frac{d^2}{dy^2} + \frac{U''}{U - U_s}$  with Dirichlet boundary condition.

To prove Theorem 2.3, we need the following lemmas, some of which are proved in [10].

LEMMA 2.4. [10] *If  $U(y)$  is in class  $\mathcal{F}$  then the neutral limiting phase speed must be an inflection value of  $U$ .*

LEMMA 2.5. [10] *Let  $\phi$  be a solution of (1.3) with complex eigenvalue  $c = c_r + ic_i$  ( $c_i \neq 0$ ), then*

$$\int_{y_1}^{y_2} \left( |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{U''(U - q)}{|U - c|^2} |\phi|^2 \right) dy = 0$$

for every real number  $q$ .

LEMMA 2.6. *For a flow  $U(y)$  in class  $\mathcal{F}$ , if  $\{(c_k, \alpha_k, \phi_k)\}$  (with  $\text{Im } c_k > 0$  and  $\|\phi_k\|_2 = 1$ ) are a sequence of solutions to equation (1.3) as in Definition 2.2 of the neutral limiting mode  $(c_s, \alpha_s, \phi_s)$ . Then we have*

$$\|\phi_k\|_{\mathbf{H}^2} \leq C \text{ (independent of } k \text{)}.$$

PROOF. By Lemma 2.4,  $\{c_k\}$  tends to an inflection value  $U_s^i$  of  $U(y)$ . Let  $y^0, y^1, \dots, y^{n_i}$  be all the points satisfying  $U = U_s^i$  and

$$E_\varepsilon = \{y \in [y_1, y_2] \mid |y - y^j| < \varepsilon \text{ for some } j\}.$$

We choose  $\varepsilon$  small enough such that the intervals  $I_\varepsilon^j = (y^j - \varepsilon, y^j + \varepsilon)$  are disjoint and on each interval  $I_\varepsilon^j$

$$(2.3) \quad U''(y) = u_j(y) (U - U_s^i)^{m_j} \text{ with } u_j(y) \neq 0.$$

Let  $\mu_j = (-1)^{m_j} \text{sign } u_j$ , then  $\mu_j$  must be the same by the definition of class  $\mathcal{F}$  and we denote it by  $\mu$ . There exists  $\delta_0 > 0$  such that

$$|U(y) - c_k| \geq \delta_0, \text{ if } y \in E_\varepsilon^c \text{ and } k \text{ large enough.}$$

In the following, we use  $\phi, c, \alpha, U_s$  to replace  $\phi_k, c_k, \alpha_k, U_s^i$  for simplicity and denote  $c = c_r + ic_i$  ( $c_i > 0$ ). By Lemma 2.5

$$(2.4) \quad \int_{y_1}^{y_2} \left( |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{U''(U-q)}{|U-c_r|^2 + c_i^2} |\phi|^2 \right) dy = 0,$$

for any real  $q$ .

If  $\mu = -1$ , let  $q = U_s - 2(U_s - c_r)$ . Then by (2.4)

$$\begin{aligned} \int_{y_1}^{y_2} \left( |\phi'|^2 + \alpha^2 |\phi|^2 \right) dy &= \int_{y_1}^{y_2} \frac{U''(U-q)}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &= \int_{E_\varepsilon^c} \frac{U''(U-q)}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &\quad + \sum_j \int_{I_\varepsilon^j} \frac{|u_j(y)| |U-U_s|^{m_j-1} (U-U_s)(U-q)}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &< c_1 \int_{E_\varepsilon^c} |\phi|^2 dy \\ &\quad + \sum_j \int_{I_\varepsilon^j} \frac{|u_j(y)| |U-U_s|^{m_j-1} \left( (U-c_r)^2 - (U_s-c_r)^2 \right)}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &< c_1 \int_{E_\varepsilon^c} |\phi|^2 dy + \sum_j \int_{I_\varepsilon^j} |u_j(y)| |U-U_s|^{m_j-1} |\phi|^2 dy \\ &< C. \end{aligned}$$

If  $\mu = 1$ , let  $q = U_s$ , then

$$\begin{aligned} \int_{y_1}^{y_2} \left( |\phi'|^2 + \alpha^2 |\phi|^2 \right) dy &= \int_{E_\varepsilon^c} \frac{U''(U-U_s)}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &\quad - \sum_j \int_{I_\varepsilon^j} \frac{|u_j(y)| |U-U_s^i|^{m_j-1} (U-U_s)^2}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &< \int_{E_\varepsilon^c} \frac{U''(U-U_s)}{|U-c_r|^2 + c_i^2} |\phi|^2 dy \\ &< C. \end{aligned}$$

In the above we use  $C$  to denote some constant independent of  $\phi$ .

The estimate of  $\|\phi''\|_2$  follows from the estimate of  $\|\phi'\|_{\mathbf{H}^1}$ . The proof is the same as for the class  $\mathcal{K}^+$  case in [10]. So we skip the details.  $\square$

With the apriori estimate in Lemma 2.6, Theorem 2.3 follows by the same proof as in [10].

Our next step is to study the bifurcation near a neutral mode. Tollmien ([18]) and later C. C. Lin ([9]) formally showed that unstable modes can bifurcate from neutral modes under some conditions. In [10], the bifurcation of unstable modes from neutral modes is rigorously proved for shear flows in class  $\mathcal{K}^+$ . We note that for such flows, there is only one inflection value  $U_s$  and  $-\frac{U''}{U-U_s}$  is positive. In this case, the bifurcation of an unstable mode from a neutral mode can be established if and only if the perturbed wave number is slightly to the left of the neutral wave

number. This one-way bifurcation property of neutral modes plus Theorem 2.3 and the continuation property of unstable modes enable us to prove Theorem 1.1 ([10]). Here we use the same strategy for flows in class  $\mathcal{F}$ . Let  $U_s^1, U_s^2, \dots, U_s^n$  be all the inflection values of  $U(y)$ . We consider a generic flow in class  $\mathcal{F}$  for which  $-\frac{U''}{U-U_s^i}$  at each point taking  $U$ -value  $U_s^i$  is nonzero. We call these flows in class  $\mathcal{F}^+$ . Note that for each inflection value  $U_s^i$  the sign  $-\frac{U''}{U-U_s^i}$  is the same for all  $U_s^i$ -inflection points. We call a neutral mode positive if  $-\frac{U''}{U-U_s^i}$  is positive for its neutral phase speed  $U_s$ , and negative otherwise. We can also do the same perturbation analysis around neutral modes as in [10] for class  $\mathcal{F}^+$  flows. The conclusion is the following: for a positive (negative) neutral mode, an unstable mode bifurcates from a neutral mode if and only if the perturbed wave number is slightly to the left (right) of the neutral wave number.

For each neutral wave number  $\alpha_s$ , by Theorem 2.3,  $-\alpha_s^2$  is a negative eigenvalue of one of the operators  $-\frac{d^2}{dy^2} + \frac{U''}{U-U_s^i}$  and vice versa. We list all neutral wave numbers in the increasing order. From this sequence, we pick the smallest one from consecutive negative neutral wave numbers and the largest one from consecutive positive neutral wave numbers. In this way, we get a new sequence. If the smallest number in this sequence is a positive neutral wave number, we add zero into the sequence. We note that the largest number in this sequence must be a positive neutral wave number, since no unstable modes exist to its right. Denote the obtained sequence be  $\alpha_-^0 < \alpha_+^0 < \dots < \alpha_-^m < \alpha_+^m$ . Here  $\alpha_-^0$  (might be 0),  $\dots, \alpha_-^m$  are negative neutral wave numbers and  $\alpha_+^0, \dots, \alpha_+^m$  are positive neutral wave numbers. We have the following Theorem.

**THEOREM 2.7.** *For a shear flow  $(U(y), 0)$  in class  $\mathcal{F}^+$ , we define  $\alpha_-^0 < \alpha_+^0 < \dots < \alpha_-^m < \alpha_+^m$  as above, then there exist unstable solutions to the Rayleigh equation (1.3) for each  $\alpha$  in the intervals  $(\alpha_-^0, \alpha_+^0), \dots, (\alpha_-^m, \alpha_+^m)$ . For  $\alpha \geq \alpha_+^m$  there is no unstable solution.*

The proof of this theorem is similar to that of Theorem 1.1 in [10], so we skip it.

**REMARK 2.8.** If  $\alpha_-^0 > 0$ , the unstable wave numbers in Theorem 2.7 start from a positive number. This onset of instability away for zero wave number actually appeared in numerical computations. In [2], the instability of the monotone flow

$$U(y) = y + 5y^3 + f \tanh 4 \left( y - \frac{1}{2} \right)$$

is studied. From figure 3(c) in [2], we see that for  $f$  in the range  $1.6 \leq f \leq 1.64$ , the Rayleigh equation (1.3) has unstable solutions only for the wave numbers in an interval away from 0. This is different from class  $\mathcal{K}^+$  flows, in which case instability always starts from zero wave number. For this example, the unstable wave number interval given by Theorem 2.7 is sharp.

In general, it is not clear whether or not the unstable wave number intervals in Theorem 2.7 are complete. However, by the proof of Theorem 2.7, to study instability in the other intervals  $(0, \alpha_-^0), (\alpha_+^0, \alpha_-^1), \dots, (\alpha_+^{m-1}, \alpha_-^m)$  we only need to look at any one wave number in each interval. That is, if there exists an unstable solution to (1.3) for some  $\alpha_0$  in  $(\alpha_+^i, \alpha_-^{i+1})$  then (1.3) have unstable solutions for all wave numbers in  $(\alpha_+^i, \alpha_-^{i+1})$ . This enables us to locate the complete unstable wave

number range with very little numerical work. In [8] Grenier considered nonlinear instability of shear flows with perturbed velocity in  $\mathbf{L}^2((-\infty, +\infty) \times [y_1, y_2])$ . He proved that the shear flow is nonlinearly unstable in the space  $(-\infty, +\infty) \times [y_1, y_2]$  if there is an unstable solution to (1.3) for some wave number. So we have the following.

**CORROLARY 2.9.** *For shear flows in class  $\mathcal{F}^+$ , the sharp condition for nonlinear instability in  $(-\infty, +\infty) \times [y_1, y_2]$  is that one of the operators  $-\frac{d^2}{dy^2} + \frac{U''}{U-U_s^i}$  has a negative eigenvalue. Here  $\{U_s^i\}$  are all the inflection values of  $U$ .*

From the proof of Theorem 2.7 or Theorem 1.1 (see [10] for details), we have the following result.

**CORROLARY 2.10.** *For any  $\alpha \in (0, \alpha_+^m)$ , denote  $n^+(\alpha)$  to be the number of positive neutral wave numbers greater than  $\alpha$  and  $n^-(\alpha)$  to be the number of negative neutral wave numbers less than  $\alpha$ . Then the Rayleigh equation (1.3) has at most  $n^+(\alpha) + n^-(\alpha)$  unstable solutions at the wave number  $\alpha$ .*

Finally we note we might generalize Theorem 2.7 to shear flows in class  $\mathcal{F}$ . A higher order bifurcation analysis around neutral modes is required, for the case when  $m_j > 1$  as in the proof of Lemma 2.6.

### 3. Nonlinear instability and semigroup estimate

The vorticity form of the Euler equation (1.1) is

$$(3.1) \quad \partial_t \omega + u \cdot \nabla \omega = 0.$$

Let  $\psi$  be the stream function, then  $u = \nabla^\perp \psi = (\partial_2 \psi, -\partial_1 \psi)$  and  $\omega \equiv -\Delta \psi$ . Denote  $u_0, \omega_0, \psi_0$  to be the steady velocity, vorticity and stream functions.

**THEOREM 3.1.** *Consider a bounded domain  $\Omega$  and a steady flow  $u_0$  with  $\omega_0 \in \mathbf{C}^1(\Omega)$ . Suppose there exists an exponentially growing solution  $e^{\lambda t} v(x^1, x^2)$  ( $\text{Re } \lambda > 0$ ) to the linearized equation (1.2) with  $v = (v_1, v_2) \in \mathbf{H}^1(\Omega)^2$  and  $\omega = \text{curl } v \in \mathbf{L}^\infty(\Omega)$ . Let  $\mu$  be the Liapunov exponent of the steady velocity field  $u_0$ . If  $\text{Re } \lambda > \frac{\mu}{2}$ , then  $u_0$  is nonlinearly unstable in the following sense:*

*There exists positive constants  $C, \theta_0, \delta_0$  and a family of solutions  $\{v_\delta, 0 \leq \delta < \delta_0\}$  to the Euler equation (1.1) satisfying*

$$\|\omega_\delta(0) - \omega_0\|_2 \leq \delta$$

and

$$\sup_{0 < t \leq C \ln |\delta|} \|v_\delta(t) - u_0\|_2 \geq \theta_0.$$

For simplicity, we only consider a simply connected domain  $\Omega$  in which case we can assume  $\psi|_{\partial\Omega} = 0$ . Thus

$$u = \text{curl}^{-1} \omega = \nabla^\perp (-\Delta)^{-1} \omega.$$

The more complicated boundary conditions for  $\psi$  can be handled as in [11] and [12].

We first state the main idea in the proof. The main difficulty is due to the derivative in the nonlinear term. To overcome it, we use the following coupled approach. The linearized operator for the vorticity equation (3.1) is

$$(3.2) \quad M_0 \omega := -u_0 \cdot \nabla \omega - u \cdot \nabla \omega_0 = -u_0 \cdot \nabla \omega - \nabla^\perp (-\Delta)^{-1} \omega \cdot \nabla \omega_0.$$

Denoting by  $\omega(t)$  the evolution of the small perturbation of vorticity satisfying

$$\partial_t \omega + M_0 \omega = -\text{curl}^{-1} \omega \cdot \nabla \omega$$

then we have

$$(3.3) \quad \omega(t) = e^{tM_0} \omega(0) + \int_0^t e^{(t-s)M_0} (-v \cdot \nabla \omega)(s) ds$$

and

$$(3.4) \quad v(t) = \text{curl}^{-1} \omega(t) = \text{curl}^{-1} (e^{tM_0} \omega(0)) - \int_0^t \text{curl}^{-1} e^{(t-s)M_0} (\nabla \cdot (v\omega))(s) ds$$

To study nonlinear instability, we estimate the growth of  $\|v(t)\|_2$ . The idea is to consider the term  $\nabla \cdot (v\omega)$  in (3.4) as a function in  $\mathbf{H}^{-1}$  and then use the regularizing effect of  $\text{curl}^{-1}$  to get back to  $\mathbf{L}^2$ . In this way the nonlinear term essentially becomes  $v\omega$  and is easy to be handled by the bootstrap argument as in [1].

To prove Theorem 3.1, we need the following two lemmas.

LEMMA 3.2. *Let  $\text{Re } \lambda$  be the maximal growth rate for the linearized equation (1.2). Then for any  $\varepsilon > 0$ , there exists  $C_\varepsilon$  such that*

$$(3.5) \quad \|e^{tM_0}\|_{\mathbf{H}^{-1} \rightarrow \mathbf{H}^{-1}} \leq C_\varepsilon e^{t \max\{\text{Re } \lambda + \varepsilon, \mu\}}.$$

PROOF. It is equivalent to show that

$$\|e^{tM_0^*}\|_{\mathbf{H}^1 \rightarrow \mathbf{H}^1} \leq C_\varepsilon e^{t \max\{\text{Re } \lambda + \varepsilon, \mu\}}.$$

Here

$$M_0^* = u_0 \cdot \nabla - (-\Delta)^{-1} (\nabla^\perp \omega_0 \cdot \nabla)$$

is the duality operator of  $M_0$ . We have

$$\|e^{tM_0^*}\|_{\mathbf{L}^2 \rightarrow \mathbf{L}^2} \leq C'_\varepsilon e^{t(\text{Re } \lambda + \frac{\varepsilon}{2})}$$

for some constant  $C'_\varepsilon$ , since  $M_0^*$  is compact perturbation of the operator  $u_0 \cdot \nabla$  which generates a isometry group and  $\text{Re } \lambda$  is the maximal real part of the discrete eigenvalues of  $M_0$  and  $M_0^*$ . Denoting  $\omega(t) = e^{tM_0^*} \omega(0)$ , we have

$$(3.6) \quad \begin{aligned} \omega(t, x) &= e^{tu_0 \cdot \nabla} \omega(0) - \int_0^t e^{(t-s)u_0 \cdot \nabla} (-\Delta)^{-1} (\nabla^\perp \omega_0 \cdot \nabla \omega(s)) ds \\ &= \omega(0, \mathbf{X}_0(t; x)) - \int_0^t \left( (-\Delta)^{-1} (\nabla^\perp \omega_0 \cdot \nabla \omega) \right) (s; \mathbf{X}_0(t-s; x)) ds, \end{aligned}$$

where  $\mathbf{X}_0(t; x)$  is the flow induced by the steady velocity field  $u_0$ . So

$$\begin{aligned} \partial \omega(t, x) &= \partial \omega(0, \mathbf{X}_0(t; x)) \frac{\partial \mathbf{X}_0}{\partial x}(t) \\ &\quad - \int_0^t \left( \partial (-\Delta)^{-1} (\nabla^\perp \omega_0 \cdot \nabla \omega) \right) (s; \mathbf{X}_0(t-s; x)) \frac{\partial \mathbf{X}_0}{\partial x}(t-s) ds. \end{aligned}$$

By the definition of the Liapunov exponent, there exists constant  $C''_\varepsilon$  such that

$$\left| \frac{\partial \mathbf{X}_0}{\partial x} \right| (t) \leq C''_\varepsilon e^{t(\mu + \frac{\varepsilon}{2})}.$$



Thus

$$\begin{aligned} \|\partial\omega(t)\|_2 &\leq \|\partial\omega(0)\|_2 C_\varepsilon'' e^{t(\mu+\frac{\varepsilon}{2})} + \int_0^t C \|\omega(s)\|_2 C_\varepsilon'' e^{(t-s)(\mu+\frac{\varepsilon}{2})} ds \\ &\leq \|\partial\omega(0)\|_2 C_\varepsilon'' e^{t(\mu+\frac{\varepsilon}{2})} + \int_0^t C C_\varepsilon' e^{s(\operatorname{Re}\lambda+\frac{\varepsilon}{2})} \|\omega(0)\|_2 C_\varepsilon'' e^{(t-s)(\mu+\frac{\varepsilon}{2})} ds \\ &\leq C_\varepsilon e^{t \max\{\operatorname{Re}\lambda+\varepsilon, \mu\}} \end{aligned}$$

and (3.5) is proved. Here we use the boundness of

$$(-\Delta)^{-1} : \mathbf{H}^{-1} \rightarrow \mathbf{H}^1.$$

and the fact that the Jacobian of the mapping  $x \rightarrow \mathbf{X}_0(t; x)$  is 1.  $\square$

The following bootstrap result is required.

**LEMMA 3.3.** *Let  $c_1 = 2 \|v_g\|_2$ . If  $\|v(t)\|_2 \leq c_1 \delta e^{\operatorname{Re}\lambda t}$  for  $0 \leq t \leq T$ , then there exists some constant  $C_1$  such that*

$$\|\omega(t)\|_4, \|v(t)\|_4 \leq C_1 \delta e^{\operatorname{Re}\lambda t}$$

for  $0 \leq t \leq T$ .

The proof basically follows from the same argument in [1], so we skip it.

**PROOF OF THEOREM 3.1.** We take the growing mode  $v_g$  with the largest growth rate  $\operatorname{Re}\lambda$ . We consider the general case when  $\lambda$  is complex and take the initial data to be  $v_\delta(0) = u_0 + \delta \operatorname{Im} v_g$ . There exists a constant  $c_0$  such that

$$\|e^{tL_0} \operatorname{Im} v_g\|_2 \geq c_0 e^{\operatorname{Re}\lambda t}.$$

By normalization, we can assume  $\|\operatorname{Im} \omega_g\|_2 = 1$  then  $\|\omega_\delta(0) - \omega_0\|_2 = \delta$ . Since  $\omega_\delta(0) \in \mathbf{L}^\infty$ , there exists a unique global weak solution to the Euler equation by Yudovich's theory (see [14]). Let  $\omega(t) = \omega_\delta(t) - \omega_0$  and  $v(t) = v_\delta(t) - u_0$ . Denote  $T$  to be the maximal time such that

$$\|v(t)\|_2 \leq c_1 \delta e^{\operatorname{Re}\lambda t}$$

Since  $c_1 = 2 \|v_g\|_2$ , we have  $\|v(0)\|_2 < c_1 \delta$  and thus  $T > 0$ . By (3.4), we have

$$v(t) = e^{tL_0} v(0) - \int_0^t \nabla^\perp (-\Delta)^{-1} e^{(t-s)M_0} (\nabla \cdot (v\omega))(s) ds = v_1 + v_2.$$

For  $0 \leq t \leq T$ , we have

$$\begin{aligned} \|v_2\|_2 &\leq \int_0^t C \left\| e^{(t-s)M_0} \right\|_{\mathbf{H}^{-1} \rightarrow \mathbf{H}^{-1}} \|\nabla \cdot (v\omega)(s)\|_{\mathbf{H}^{-1}} ds \\ &\leq \int_0^t C C_\varepsilon e^{(t-s) \max\{\operatorname{Re}\lambda+\varepsilon, \mu\}} \|v\omega(s)\|_2 ds \\ &\leq \int_0^t C C_\varepsilon e^{(t-s) \max\{\operatorname{Re}\lambda+\varepsilon, \mu\}} \|v(s)\|_4 \|\omega(s)\|_4 ds \\ &\leq \int_0^t C C_\varepsilon e^{(t-s) \max\{\operatorname{Re}\lambda+\varepsilon, \mu\}} (C_1 \delta e^{\operatorname{Re}\lambda s})^2 ds \quad (\text{by Lemma 3.3}) \\ &= C_2 (\delta e^{\operatorname{Re}\lambda t})^2 \end{aligned}$$

if  $\varepsilon$  is small, under our assumption that  $\operatorname{Re} \lambda > \frac{\mu}{2}$ . Let  $T^*$  be such that  $\delta e^{T^* \operatorname{Re} \lambda} = \theta$ , where

$$\theta = \frac{\min\{c_0, c_1\}}{4C_2}$$

We show that  $T^* \leq T$ . Suppose otherwise, we have  $T^* > T$ . Then at time  $T$ , we have

$$\begin{aligned} \|v(T)\|_2 &\leq \|v_1(T)\|_2 + \|v_2(T)\|_2 \\ &\leq \|e^{TL_0} \delta v_g\|_2 + C_2 (\delta e^{\operatorname{Re} \lambda T})^2 \\ &\leq (\|v_g\|_2 + C_2 \theta) \delta e^{T \operatorname{Re} \lambda} \\ &< \frac{3}{4} c_1 \delta e^{T \operatorname{Re} \lambda} \end{aligned}$$

which is a contradiction to the definition of  $T$ . So at time  $T^*$ , we must have

$$\begin{aligned} \|v(T^*)\|_2 &\geq \|v_L(T^*)\|_2 - \|v_N(T^*)\|_2 \\ &\geq c_0 \delta e^{\operatorname{Re} \lambda T^*} - C_2 (\delta e^{\operatorname{Re} \lambda T^*})^2 \\ &= c_0 \theta - C_2 \theta^2 \\ &\geq \frac{3}{4} \theta. \end{aligned}$$

We let  $\theta_0 = \frac{3}{4} \theta$  and the proof of Theorem 3.1 is finished.  $\square$

This coupled approach can be used to study the growth property of the semi-group  $e^{tL_0}$ . We consider a simply connected domain first. Define the space

$$H_p = \left\{ u \in \mathbf{L}^p(\Omega)^2 \mid \nabla \cdot u = 0, u \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

**THEOREM 3.4.** *Consider a simply connected bounded domain  $\Omega$ . Let  $\operatorname{Re} \lambda$  be the maximal growth rate for the linearized equation (1.2). Then for any  $\varepsilon > 0$  and  $p > 1$  there exists  $C_{\varepsilon, p}$  such that*

$$\|e^{tL_0}\|_p \leq C_{\varepsilon, p} e^{t \max\{\operatorname{Re} \lambda + \varepsilon, \mu\}}.$$

Here  $e^{tL_0} : H_p \rightarrow H_p$ .

**PROOF.** It is proved in [12] that there exists constant  $C$  such that

$$(3.7) \quad \|v\|_p \leq C \sup_{\substack{\psi \in \mathbf{W}_0^{1, p'} \\ \|\psi\|_{\mathbf{W}^{1, p'}} = 1}} \int \int_{\Omega} \psi \operatorname{curl} v dx^1 dx^2.$$

So we have

$$\begin{aligned}
\|e^{tL_0}v(0)\|_p &\leq C \sup_{\substack{\psi \in \mathbf{W}_0^{1,p'} \\ \|\psi\|_{\mathbf{W}^{1,p'}}=1}} \int \int_{\Omega} \psi \operatorname{curl} (e^{tL_0}v(0)) \, dx^1 dx^2 \\
&= C \sup_{\substack{\psi \in \mathbf{W}_0^{1,p'} \\ \|\psi\|_{\mathbf{W}^{1,p'}}=1}} \int \int_{\Omega} \psi e^{tM_0} (\partial_2 v_1(0) - \partial_1 v_2(0)) \, dx^1 dx^2 \\
&= C \sup_{\substack{\psi \in \mathbf{W}_0^{1,p'} \\ \|\psi\|_{\mathbf{W}^{1,p'}}=1}} \int \int_{\Omega} (e^{tM_0^*} \psi) (\partial_2 v_1(0) - \partial_1 v_2(0)) \, dx^1 dx^2 \\
&= C \sup_{\substack{\psi \in \mathbf{W}_0^{1,p'} \\ \|\psi\|_{\mathbf{W}^{1,p'}}=1}} \int \int_{\Omega} \partial (e^{tM_0^*} \psi) \cdot (-v_2(0), v_1(0)) \, dx^1 dx^2 + \oint_{\partial\Omega} e^{tM_0^*} \psi v \cdot dl \\
&= C \sup_{\substack{\psi \in \mathbf{W}_0^{1,p'} \\ \|\psi\|_{\mathbf{W}^{1,p'}}=1}} \int \int_{\Omega} \partial (e^{tM_0^*} \psi) \cdot (-v_2(0), v_1(0)) \, dx^1 dx^2 \\
&\leq C \left\| e^{tM_0^*} \psi \right\|_{\mathbf{W}^{1,p'}} \|v(0)\|_p,
\end{aligned}$$

where we use  $e^{tM_0^*} \psi|_{\partial\Omega} = 0$  which can be easily seen from the formula (3.6). By the same proof of Lemma 3.2, there exists constant  $C_{\varepsilon,p'}$  such that

$$\|e^{tM_0}\|_{\mathbf{W}^{1,p'} \rightarrow \mathbf{W}^{1,p'}} \leq C_{\varepsilon,p'} e^{t \max\{\operatorname{Re} \lambda + \varepsilon, \mu\}}.$$

Combining above, the conclusions follows.  $\square$

For a non-simply connected domain  $\Omega$  with  $\partial\Omega = \cup_i \Lambda_i$ , we consider the semi-group  $e^{tL_0}$  defined on the space

$$H_p^0 = \left\{ u \in \mathbf{L}^p(\Omega)^2 \mid \nabla \cdot u = 0, u \cdot n = 0 \text{ on } \partial\Omega \text{ and } \oint_{\Lambda_i} u \cdot dl = 0 \text{ for each } \Lambda_i \right\}.$$

We note that all possible growing modes must be in the space  $H_p^0$  (see [11]).

**THEOREM 3.5.** *Consider a non-simply connected bounded domain  $\Omega$ . Let  $\operatorname{Re} \lambda$  be the maximal growth rate for the linearized equation (1.2). Then for any  $\varepsilon > 0$  and  $p > 1$  there exists  $C_{\varepsilon,p}$  such that*

$$\|e^{tL_0}\|_p \leq C_{\varepsilon,p} e^{t \max\{\operatorname{Re} \lambda + \varepsilon, \mu\}}.$$

Here  $e^{tL_0} : H_p^0 \rightarrow H_p^0$ .

For the proof, we use the similar duality estimate for  $\mathbf{L}^p$ -norm of a function in  $H_p^0$  (see [12]) and follow the same line in the proof of Theorem 3.4.

In the formal level, Theorems 3.4 and 3.5 imply that the unstable essential spectrum of  $L_0$  in the space  $H_p$  ( $H_p^0$  for non-simply connected domain) lies in the strip

$$\{z \in \mathbf{C} \mid 0 \leq \operatorname{Re} z \leq \mu\}.$$

It is rigorously proved in [7] that for the case of a torus and  $p = 2$ , the unstable essential spectrum of  $L_0$  is the whole strip.

#### 4. Some comments

The choice of norm is very important in the stability or instability study of ideal plane flows. In [11], we construct a steady flow which is nonlinearly and linearly stable in the  $\mathbf{L}^2$ -norm of vorticity but linearly unstable in the  $\mathbf{L}^2$ -norm of velocity. For most cases, we are interested in the stability or instability in the large scale sense. We believe that the constrain of small  $\mathbf{L}^p$ -norm of initial vorticity perturbation is necessary to control the influence of very small scales. With this constrain, we show ([12]) that nonlinear instability in  $\mathbf{L}^p$  norm of velocity is indeed in the large scale sense.

In the following, we comment on some future problems related to our results.

For steady flows satisfying  $-\Delta\psi_0 = g(\psi_0)$  with  $g' > 0$ , if the operator  $A$  (defined in (1.4)) has no kernel, we conjecture that  $A > 0$  is the sharp condition for nonlinear stability. To prove it, first we need to close the gap between the linear and nonlinear stability conditions in [11]. Second, if  $A$  has a negative eigenvalue we need to find a growing mode. Currently, this is only proved for shear flows and rotating flows ([10]). We also note that if  $g'$  changes sign, so far there are no methods to study nonlinear stability. In this case, the usual energy-Casimir method is not applicable.

It is very interesting to utilize singular neutral modes in the study of shear flow instability, as what we did for regular neutral modes in Section 2 and [10]. This issue is important to study instability of shear flows in a stratified fluid and in a beta plane([3], [16]).

#### References

- [1] C. Bardos, Y. Guo and W. Strauss, *Stable and unstable ideal plane flows*, Chinese Annals Math., **23B**, 149-164 (2002).
- [2] N. J. Balmforth and P. J. Morrison, A necessary and sufficient instability condition for inviscid shear flow, *Studies in Appl. Math.*, **102**: 309-344 (1999).
- [3] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*, Cambridge Monographs on Mechanics and Appl. Math., 1981.
- [4] P. G. Drazin and L. N. Howard, *Hydrodynamic stability of parallel fluid of an inviscid fluid*, Adv. Appl. Math., **9**, 1-89(1966).
- [5] S. Friedlander, W. Strauss and M. M. Vishik, *Nonlinear instability in an ideal fluid*, Ann. I. H. P. (Anal Nonlin.), **14**(2), 187-209 (1997).
- [6] S. Friedlander, L. N. Howard, *Instability in parallel flow revisited*, Stud. Appl. Math., **101**, 1-21(1998).
- [7] Y. Latushkin and Roman Shvydkoy, *The essential spectrum of the linearized 2D Euler operator is a vertical band*, Contemp. Math., **327**, 299-304 (2003).
- [8] E. Grenier, *On the nonlinear instability of Euler and Prandtl equations*, Commun. Pure. Appl. Math., **53**, 1067-1091(2000).
- [9] C. C. Lin, *The Theory of Hydrodynamic Stability*, Cambridge Univ. Press, 1955.
- [10] Zhiwu Lin, *Instability of some ideal plane flows*, SIAM J. Math. Anal., **35**, no. 2, 318-356 (2003).
- [11] Zhiwu Lin, *Some stability and instability criteria for ideal plane flows*, Comm. Math. Phys., **246**, no. 1, 87-112(2004).
- [12] Zhiwu Lin, *Nonlinear instability of ideal plane flows*, Int. Math. Res. Not., **41**, 2147-2178(2004).
- [13] Zhiwu Lin, *Nonlinear instability of periodic waves for Vlasov-Poisson system*, to appear in Commun. Pure. Appl. Math.
- [14] Andrew J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge University Press, 2002.

- [15] C. Marchioro, M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer-Verlag 1994.
- [16] Joseph Pedlosky, *Geophysical fluid dynamics*, 2nd edition, Springer 1987.
- [17] Lord Rayleigh, *On the stability or instability of certain fluid motions*, Proc. London Math. Soc., **9**, 57-70(1880).
- [18] W. Tollmien, *Ein Allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen*, Nachr. Ges. Wiss. Göttingen Math. Phys. **50**, 79-114(1935).
- [19] M. M. Vishik and S. Friedlander, *Nonlinear instability in two dimensional ideal plane fluids: The case of a dominant eigenvalue*, to appear in Comm. Math. Phys.
- [20] G. Wolansky and M. Ghil, *Nonlinear stability for saddle solutions of ideal flows and symmetry breaking*, Comm. Math.Phys., **193** 713-736(1998).

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