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Some Stability and Instability Criteria for Ideal Plane Flows

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Abstract: We investigate stability and instability of steady ideal plane flows for an arbitrary bounded domain. First, we obtain some general criteria for linear and nonlinear stability. Second, we find a sufficient condition for the existence of a growing mode to the linearized equation. Third, we construct a steady flow which is nonlinearly and linearly stable in the L^2 norm of vorticity but linearly unstable in the L^2 norm of velocity.

1. Introduction

We consider an incompressible inviscid flow satisfying the Euler equation

$$\partial_t u + (u \cdot \nabla u) + \nabla p = 0, \tag{1a}$$

$$\nabla \cdot u = 0, \tag{1b}$$

in a bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$ composed of a finite number of connected components Λ_i. The boundary condition is

$$u \cdot n = 0 \quad \text{on } \partial\Omega,$$

where n stands for the unit outer normal of $\partial\Omega$. The vorticity form of (1) is given by

$$\partial_t \omega - \psi_y \partial_x \omega + \psi_x \partial_y \omega = 0, \tag{2}$$

where ψ is the stream function, and $\omega \equiv -\Delta\psi = -\left(\partial_x^2 + \partial_y^2\right)\psi$ is the vorticity. The boundary conditions associated with (2) are given by

$$\psi|_{\Lambda_i} = \Psi_i, \tag{3a}$$

and

$$\oint_{\Lambda_i} \frac{\partial \psi}{\partial n} = A_i, \tag{3b}$$

with Ψ_i depending on time only, and A_i being some constants.

A steady flow satisfying (2), (3) has a stream function ψ_0 satisfying

$$-\psi_{0_y} \partial_x \omega_0 + \psi_{0_x} \partial_y \omega_0 = 0, \tag{4}$$

where $\omega_0 \equiv -\Delta \psi_0$ is the associated vorticity. Consider ψ_0 satisfying the following elliptic equation:

$$-\Delta \psi = g(\psi) \tag{5}$$

with boundary conditions (3) and g being some differentiable function. Then $\omega_0 \equiv -\Delta \psi_0 = g(\psi_0)$ is a steady solution of (2). In this paper we study stability and instability of these steady solutions. If $g' > 0$, Wolansky and Gill [17] derived some linear and nonlinear stability criteria, using the energy-Casimir method and a supporting functional method. However their conditions involve some unspecified finite dimensional function spaces and are not easy to check. Our first theorem is a refinement of their results. We state the theorem only for a simply connected domain. For this case, the boundary conditions (3) can be simplified to be

$$\psi = 0 \ \text{ on } \partial\Omega.$$

First we introduce some notations. We call a real number ρ a critical value of ψ_0 if ψ_0 takes the value ρ at a critical point. The set of all critical values of ψ_0 has zero measure by Sard's Theorem. For any ρ which is not a critical value, the level set $\{\psi_0 = \rho\}$ consists of a finite number of disjoint closed curves, which we denote by $\Gamma_1(\rho), \Gamma_2(\rho), \cdots, \Gamma_{n_\rho}(\rho)$. Let $\mathbf{X} = \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ and $\mathbf{Y} = \mathbf{H}_0^1(\Omega)$, with

$$\|\psi\|_{\mathbf{X}}^2 = \int\int_\Omega |\Delta \psi|^2 \, dxdy, \quad \|\psi\|_{\mathbf{Y}}^2 = \int\int_\Omega |\nabla \psi|^2 \, dxdy.$$

Note that $\|\psi\|_{\mathbf{X}}^2$, $\|\psi\|_{\mathbf{Y}}^2$ are the enstrophy and energy of the flow with the stream function ψ. The linearized equation of (2) around the steady state (ψ_0, ω_0) is

$$\partial_t \tilde{\omega} - \psi_{0_y} \partial_x \tilde{\omega} + \psi_{0_x} \partial_y \tilde{\omega} = \tilde{\psi}_y \partial_x \omega_0 - \tilde{\psi}_x \partial_y \omega_0, \tag{6}$$

with

$$\tilde{\omega} = -\Delta \tilde{\psi}$$

and

$$\tilde{\psi} = 0 \ \text{ on } \partial\Omega.$$

We have the following result on nonlinear and linear stability.

Theorem 1.1. *Suppose* $\omega_0 = g(\psi_0)$ $\left(g \in \mathbf{C}^1\right)$ *is a steady flow with* $g' > 0$.

(i)*We define the functional* $a(\phi)$ *for* $\phi \in \mathbf{Y}$ *by*

$$a(\phi) = \int\int_\Omega |\nabla\phi|^2 \, dxdy - \int\int_\Omega g'(\psi_0)\, \phi^2 dxdy$$
$$+ \int_{\min \psi_0}^{\max \psi_0} g'(\rho) \frac{\left|\sum_{i=1}^{n_\rho} \oint_{\Gamma_i(\rho)} \frac{\phi}{|\nabla\psi_0|}\right|^2}{\sum_{i=1}^{n_\rho} \oint_{\Gamma_i(\rho)} \frac{1}{|\nabla\psi_0|}} d\rho \ .$$

If

$$\inf_{\|\phi\|_2 = 1} a(\phi) > 0, \tag{7}$$

then the flow is nonlinearly stable in the following sense: for any $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\|\psi(.,0) - \psi_0\|_{\mathbf{X}} < \delta \Rightarrow \sup_{t>0} \|\psi(.,t) - \psi_0\|_{\mathbf{X}} < \varepsilon,$$

where $\psi(.,t)$ *is the solution to (2) with the initial state* $\psi(.,0) \in \mathbf{X}$. *Condition (7) is equivalent to the operator B (defined by (14)) being positive.*

(ii)*We define the functional* $b(\phi)$ *for* $\phi \in \mathbf{Y}$ *by*

$$b(\phi) \ = \int\int_\Omega |\nabla\phi|^2 \, dxdy - \int\int_\Omega g'(\psi_0)\, \phi^2 dxdy$$
$$+ \int_{\min \psi_0}^{\max \psi_0} g'(\rho) \sum_{i=1}^{n_\rho} \frac{\left|\oint_{\Gamma_i(\rho)} \frac{\phi}{|\nabla\psi_0|}\right|^2}{\oint_{\Gamma_i(\rho)} \frac{1}{|\nabla\psi_0|}} d\rho \ .$$

If

$$\inf_{\|\phi\|_2 = 1} b(\phi) > 0, \tag{8}$$

then the flow is linearly stable in the following sense : for any $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\|\psi(.,0) - \psi_0\|_{\mathbf{X}} < \delta \Rightarrow \sup_{t>0} \|\psi(.,t) - \psi_0\|_{\mathbf{X}} < \varepsilon,$$

where $\psi(.,t)$ *is the solution to the linearized Euler equation (6) with the initial state* $\psi(.,0) \in \mathbf{X}$.

Note that we have $b(\phi) \geq a(\phi)$ for any $\phi \in \mathbf{Y}$ and the equality only holds in the case when each level set $\{\psi_0 = \rho\}$ consists of only a single curve.

Theorem 1.1(ii) also gives a criterion for spectral stability. That is, if $b(\phi)$ is positive definite then there is no exponentially growing solution to the linearized Euler equation (6). So the linearized operator has no unstable discrete eigenvalue. For shear flows and rotating flows, it was proved in [12] that this criterion is also necessary, namely if $b(\phi)$ is negative then we can find a growing mode. Now we give a new criterion for the existence of a growing mode, for steady flows satisfying (5) on a bounded domain and without the assumption that $g' > 0$. We state the result only for the simply connected case. For the non-simply connected case, see Sect. 5 for some modifications. Define the operator A_0: $\mathbf{X} \to \mathbf{L}^2(\Omega)$ as follows:

$$A_0\phi := -\Delta\phi - g'(\psi_0)\,\phi + g'(\psi_0)\,\tilde{P}\phi, \tag{9}$$

where $\tilde{P}$ is the orthogonal projection operator of $\mathbf{L}^2(\Omega)$ onto $\mathbf{S} = \ker L$ and $L = -\partial_y\psi_0\partial_x + \partial_x\psi_0\partial_y$ is defined on $\mathbf{H}_0^1(\Omega)$. We prove later that

$$b(\phi) = (A_0\phi, \phi). \tag{10}$$

So the fact that $b(\phi)$ is negative for some ϕ is equivalent to the existence of a negative eigenvalue of A_0, and condition (8) is equivalent to the operator A_0 being positive.

Theorem 1.2. *If A_0 has an odd number of negative eigenvalues and no kernel, then there exists a purely growing mode $e^{\lambda t}\omega$ (with $\lambda > 0$ and $\omega \in \mathbf{X}$) to the linearized Euler equation (6).*

It is hard to prove the existence of unstable discrete eigenvalues for the linearized Euler operator since it is degenerate and non-elliptic. Even for the simplest shear flow case, little has been known about sufficient conditions for the existence of unstable discrete eigenvalues ([2, 13]). We can modify the proof of Theorem 1.2 to get an instability criterion for the case when A_0 has a nontrivial kernel. For the case when A_0 has an even number of negative eigenvalues and no kernel, we cannot expect to find purely growing modes. Some new methods to find non-purely growing modes were developed in [13] for shear flows and rotating flows.

Now we sketch the main idea for the proof of Theorem 1.2. For a growing mode $\left(e^{\lambda t}\omega, e^{\lambda t}\psi\right)$ (with $\lambda > 0$) to the linearized Euler equation (6), (ω, ϕ) satisfies the following equations

$$\lambda\omega - \psi_{0_y}\partial_x\omega + \psi_{0_x}\partial_y\omega = \psi_y\partial_x\omega_0 - \psi_x\partial_y\omega_0, \tag{11}$$

$$\omega = -\Delta\psi, \tag{12}$$

$$\psi = 0 \text{ on } \partial\Omega.$$

Using the strategy in [12], we represent ω in terms of ψ by integrating (11) along the fluid trajectory, then plug it into the Poisson equation (12). The resulting equation can be written as $A_\lambda\psi = 0$. The operator A_λ is the minus Laplacian plus a bounded operator. For the existence of a purely growing mode, it suffices to show that for some $\lambda_0 > 0$, A_{λ_0} has a nontrivial kernel. The main difference from the case in [12] is that here A_λ is not self-adjoint. This makes the analysis more difficult. We use the infinite determinant method developed in [13]. We study the infinite determinant $d(\lambda)$ of $Id - \exp(-A_\lambda)$ as $\lambda \to 0^+$ and $\lambda \to \infty$. It turns out that $d(\lambda)$ is nonnegative when λ is sufficiently large. It can be shown that $d(\lambda)$ is negative as $\lambda \to 0^+$ under the conditions of Theorem 1.2. These two facts imply that for some $\lambda_0 > 0$ $d(\lambda_0) = 0$, which implies that A_{λ_0} has a nontrivial kernel.

The stability result in Theorem 1.1 is proved in the vorticity norm. This norm was also used in [1] to prove nonlinear instability from the existence of an unstable discrete eigenvalue of the linearized Euler operator. However, the stability problem in the L^2 norm of velocity (energy norm) is quite different. So far there is no general method to prove nonlinear stability and instability in the energy norm. In the last part of this paper, we construct a steady flow which is nonlinearly and linearly stable in the enstrophy norm $||\cdot||_{\mathbf{X}}$ but linearly unstable in the energy norm $||\cdot||_{\mathbf{Y}}$. This example illustrates the importance of the norm to adopt when studying stability of incompressible inviscid flows.

2. Stability Criteria

In this section, we prove Theorem 1.1. We need the following result from Wolansky and Gill [17]. We state it in the following form.

Lemma 2.1 ([17]). *The steady flow as in Theorem 1.1 is nonlinearly stable in* $\mathbf{X}$ *if for some integer* m*, the following functional* $e_m(\phi)$ *is positive definite in* $\mathbf{Y}$*:*

$$e_m(\phi) := \int\int_\Omega |\nabla\phi|^2\, dxdy - \int\int_\Omega g'(\psi_0)\,\phi^2 dxdy + \sum_{i=1}^m < \xi_i^0, \phi >^2,$$

where $< .,. >$ *is the* g'*weighted* $\mathbf{L}^2$ *inner product, and* $\{\xi_1^0, \cdots, \xi_m^0\}$ *is a* g' *weighted orthogonal basis of some* m*-dimensional subspace* W_m *of*

$$W = \{\psi \in \mathbf{Y}\,|\psi = h(\psi_0),\ h \text{ being measurable on the range of } \psi_0\}.$$

Denote $P(P_m)$ the orthogonal projection operators of $\mathbf{L}^2(\Omega)$ onto $W(W_m)$ and define the operators

$$B(B_m) : \mathbf{X} \to \mathbf{L}^2(\Omega), \tag{13}$$

$$B\phi\,(B_m\phi) = -\Delta\phi - g'(\psi_0)\,\phi + g'(\psi_0)\,P\phi(P_m\phi). \tag{14}$$

Then we readily see that $e_m(\phi) = (B_m\phi, \phi)$. Thus to show that $e_m(\phi)$ is positive definite, it is equivalent to show that B_m is positive.

Lemma 2.2. *If the operator* B *is positive, then for some integer* m *there exists a* m*-dimensional subspace* $W_m \subset W$ *such that the operator* B_m *is positive.*

Proof. Let $\zeta_1, \zeta_2, \cdots$ be a complete orthogonal basis of $\mathbf{L}^2(\Omega)$, and denote W_n the space spanned by

$$P\zeta_1, P\zeta_2, \cdots, P\zeta_n.$$

Let P_n be the corresponding orthogonal projection. Then it is readily seen that $P_n \to P$ strongly in the sense that for any $\phi \in \mathbf{X}$, $P_n\phi \to P\phi$ strongly in $\mathbf{L}^2(\Omega)$. If the conclusion of the lemma is not true, then for each $n \in \mathbf{N}$ we can find $\lambda_n \le 0$ and $\|\phi_n\|_2 = 1$ such that

$$B_n\phi_n = \lambda_n\phi_n. \tag{15}$$

Let $\lambda_n \to \lambda_0 \le 0$. Then it is obvious that $\|\phi_n\|_{\mathbf{H}^2(\Omega)} \le C$ (independent of n). So after taking some subsequence, we have $\phi_n \to \phi_0$ strongly in $\mathbf{L}^2(\Omega)$ with $\|\phi_0\|_2 = 1$. Since $B_n \to B$ strongly, we have $B_n\phi_n \to B\phi_0$ weakly. So taking the limit in (15), we have $B\phi_0 = \lambda_0\phi_0$ which is a contradiction to the positivity of B. □

Lemma 2.3. *For any regular point $(x, y) \in \Omega$ of the function ψ_0, let $\rho = \psi_0(x, y)$ and $(x, y) \in \Gamma_i(\rho)$, where $\{\psi_0 = \rho\} = \cup_{i=1}^{n_\rho} \Gamma_i(\rho)$. For any $\phi \in \mathbf{Y}$, we define two functions $\phi_1, \phi_2 \in \mathbf{L}^2(\Omega)$ in the following way:*

$$\phi_1(x, y) = \frac{\sum_{i=1}^{n_\rho} \oint_{\Gamma_i(\rho)} \frac{\phi}{|\nabla \psi_0|}}{\sum_{i=1}^{n_\rho} \oint_{\Gamma_i(\rho)} \frac{1}{|\nabla \psi_0|}} \tag{16}$$

and

$$\phi_2(x, y) = \frac{\oint_{\Gamma_i(\rho)} \frac{\phi}{|\nabla \psi_0|}}{\oint_{\Gamma_i(\rho)} \frac{1}{|\nabla \psi_0|}}. \tag{17}$$

Then we have $P\phi = \phi_1$ and $\tilde{P}\phi = \phi_2$ in the $\mathbf{L}^2$ sense. Here $\tilde{P}$ is the projection operator of $\mathbf{L}^2(\Omega)$ onto

$$\mathbf{S} = \ker\left(-\partial_y \psi_0 \partial_x + \partial_x \psi_0 \partial_y\right).$$

Proof. To show $P\phi = \phi_1$, we take any $\xi = h(\psi_0) \in W$. Then

$$\begin{aligned}
(\phi - \phi_1, \xi) &= \int\int_\Omega (\phi - \phi_1) h(\psi_0)\, dxdy \\
&= \int_{\min \psi_0}^{\max \psi_0} h(\rho) \left(\int_{\{\psi_0 = \rho\}} \frac{\phi - \phi_1}{|\nabla \psi_0|} \right) d\rho \quad \text{(by the co-area formula)} \\
&= \int_{\min \psi_0}^{\max \psi_0} h(\rho) \left(\sum_{i=1}^{n_\rho} \oint_{\Gamma_i(\rho)} \frac{\phi}{|\nabla \psi_0|} - \phi_1(\rho)|_{\psi_0 = \rho} \sum_{i=1}^{n_\rho} \oint_{\Gamma_i(\rho)} \frac{1}{|\nabla \psi_0|} \right) d\rho \\
&= 0.
\end{aligned}$$

So $\phi - \phi_1 \in W^\perp$. Since clearly $\phi_1 \in W$, we have $P\phi = \phi_1$.

To show that $\tilde{P}\phi = \phi_2$, we take any $\eta \in \mathbf{S}$. Then

$$\begin{aligned}
(\phi - \phi_2, \eta) &= \int\int_\Omega (\phi - \phi_2)\, \eta dxdy \\
&= \int_{\min \psi_0}^{\max \psi_0} \sum_{i=1}^{n_\rho} \left(\oint_{\Gamma_i(\rho)} \frac{(\phi - \phi_2)\, \eta}{|\nabla \psi_0|} \right) d\rho \\
&= \int_{\min \psi_0}^{\max \psi_0} \sum_{i=1}^{n_\rho} \eta|_{\Gamma_i(\rho)} \left(\oint_{\Gamma_i(\rho)} \frac{\phi}{|\nabla \psi_0|} - \phi_2|_{\Gamma_i(\rho)} \oint_{\Gamma_i(\rho)} \frac{1}{|\nabla \psi_0|} \right) d\rho \\
&\quad \text{(since } \eta, \phi_2 \text{ take constant values on each } \Gamma_i(\rho)\text{)} \\
&= 0.
\end{aligned}$$

So $\phi - \phi_2 \in \mathbf{S}^\perp$. Since $\phi_2 \in \mathbf{S}$, we have $\tilde{P}\phi = \phi_2$. □

Now Theorem 1.1(i) follows from the above three lemmas. By Lemmas 2.2 and 2.3, if the condition (7) is satisfied, then there exists some integer m such that B_m is positive. Then by Lemma 2.1, the steady flow is nonlinearly stable. Theorem 1.1(ii) can be proved in the same way.

Corollary 2.1. *Under assumption (8), there is no unstable discrete eigenvalue for the linearized Euler operator.*

Proof. This is a consequence of Theorem 1.1(ii). Here we give a direct proof of it. First we notice that if (ω, ψ) is a solution to (6), then for any $\xi \in \mathbf{S} = \ker\left(-\partial_y \psi_0 \partial_x + \partial_x \psi_0 \partial_y\right)$, the following two functionals

$$E(\omega, \psi) = \int\int_\Omega \left(\frac{|\omega|^2}{g'(\psi_0)} - |\nabla\psi|^2\right) dxdy$$
$$M_\xi(\omega) = \int\int_\Omega \omega\xi\, dxdy$$

are conserved. This can be checked by a straightforward computation.

If there exists a growing mode $\left(e^{\lambda t}\omega(x, y), e^{\lambda t}\psi(x, y)\right)$ to (6) with $\operatorname{Re}\lambda > 0$, then

$$E\left(e^{\lambda t}\omega, e^{\lambda t}\psi\right) = e^{2\operatorname{Re}\lambda t} E(\omega, \psi), M_g\left(e^{\lambda t}\omega\right) = e^{\lambda t} M_g(\omega)$$

are independent of t. Thus it follows that

$$E(\omega, \psi) = M_\xi(\omega) = 0$$

for any $\xi \in \mathbf{S}$.

Noticing that

$$\int\int_\Omega |\nabla\psi|^2\, dxdy = \int\int_\Omega \psi\omega^* dxdy,$$

where ω^* is the complex conjugate of ω, we have

$$\begin{aligned}
0 = E(\omega, \psi) &= \int\int_\Omega \left(\frac{|\omega|^2}{g'(\psi_0)} - 2\psi\omega^* + |\nabla\psi|^2\right) dxdy \\
&= \int\int_\Omega \left\{\left|\frac{\omega}{\sqrt{g'(\psi_0)}} - \psi\sqrt{g'(\psi_0)}\right|^2 - g'(\psi_0)|\psi|^2 + |\nabla\psi|^2\right\} dxdy \\
&= \int\int_\Omega \left|\frac{\omega}{\sqrt{g'(\psi_0)}} - \left(1 - \tilde{P}\right)\psi\sqrt{g'(\psi_0)}\right|^2 - g'(\psi_0)|\psi|^2 + |\nabla\psi|^2 \\
&\quad + g'(\psi_0)\left|\tilde{P}\psi\right|^2 \; dxdy \\
&\geq \int\int_\Omega |\nabla\psi|^2 - g'(\psi_0)|\psi|^2 + g'(\psi_0)\left|\tilde{P}\psi\right|^2 dxdy.
\end{aligned}$$

So if the last quadratic form in the above is positive, we get a contradiction. This proves the conclusion. Here for the equality in the third line above, we use the fact that

$$\frac{\omega}{\sqrt{g'(\psi_0)}} \in \mathbf{S}^\perp.$$

□

Remark 2.1. The two conditions (7) and (8) are the same if and only if $\{\psi_0 = \rho\}$ consists of only one closed curve for any ρ. We shall prove that in this case the linearized Euler operator has no growing modes.

We consider the steady flow with the stream function ψ_0 defined on a simply connected domain. Here ψ_0 satisfies the following elliptic equation:

$$-\Delta\psi_0 = g(\psi_0), \text{ in } \Omega$$
$$\psi_0 = 0, \text{ on } \partial\Omega.$$

Lemma 2.4. *If $\{\psi_0 = \rho\}$ consists of only one closed curve for any*

$$\rho \in (\min\psi_0, \max\psi_0),$$

then there are no exponentially growing modes to the linearized Euler equation (6).

Proof. First we define the generalized polar coordinates (r, θ) in the following way. Let l be the arc length variable on the stream line $\{\psi_0 = \rho\}$ and define

$$r = \psi_0(x, y), \quad \frac{2\pi}{v(r)} = \oint_{\{\psi_0 = r\}} \frac{1}{|\nabla\psi_0|}, \quad \theta = v(r)\int_0^l \frac{dl'}{|\nabla\psi_0|}.$$

Then $\min\psi_0 \le r \le \max\psi_0$, $0 \le \theta \le 2\pi$. If $e^{\lambda t}\psi$ $(\operatorname{Re}\lambda > 0)$ is a growing mode to (6), then ψ satisfies (see Lemma 3.1)

$$-\Delta\psi - g'(\psi_0)\psi + g'(\psi_0)\lambda\int_{-\infty}^0 e^{\lambda s}\psi(X(s; x, y), Y(s; x, y))\,ds = 0. \qquad (18)$$

Here $(X(s; x, y), Y(s; x, y))$ is the solution of the characteristic equation as defined in (22). In the polar coordinates (r, θ), the characteristic equation becomes

$$\begin{cases} \dot{r} = 0, \\ \dot{\theta} = -v(r). \end{cases}$$

Let

$$\psi(r, \theta) = \sum_{-\infty}^{+\infty} \psi_k(r)\, e^{ik\theta},$$

then (18) becomes

$$-\Delta\psi - g'(\psi_0)\psi + g'(\psi_0)\sum_{-\infty}^{+\infty} \frac{\lambda}{\lambda - ikv(r)}\psi_k(r)\, e^{ik\theta} = 0. \qquad (19)$$

Taking the inner product of (19) with ψ^*, we have

$$\int\int_\Omega |\nabla\psi|^2\,dxdy + \int_{\min\psi_0}^{\max\psi_0} \frac{1}{v(r)} g'(r) \sum_{-\infty}^{+\infty} \frac{ikv(r)}{a + bi - ikv(r)} |\psi_k(r)|^2\,dr = 0, \qquad (20)$$

where $\lambda = a + bi$ $(a > 0)$. Taking the imaginary part of (20), we get

$$\int_{\min\psi_0}^{\max\psi_0} \frac{1}{v(r)} g'(r) \sum_{-\infty}^{+\infty} \frac{akv(r)}{a^2 + (b - kv(r))^2}\,dr = 0.$$

So $a = 0$ which is a contradiction. Thus there are no growing modes to the linearized Euler equation (6).

3. Instability Criterion

We divide the proof of Theorem 1.2 into several steps.

3.1. Dispersion operators. In this part, we introduce dispersion operators A_λ and study their basic properties.

Definition 3.1. *The dispersion operators are a family of operators $A_\lambda\ \left(\lambda \in \mathbf{R}^+\right) : \mathbf{X} \to L^2(\Omega)$. Here*

$$A_\lambda \psi := -\Delta\psi - g'(\psi_0)\,\psi + g'(\psi_0)\,\lambda \int_{-\infty}^{0} e^{\lambda s} \psi\left(X(s;x,y), Y(s;x,y)\right) ds, \tag{21}$$

where $\left(X(s;x,y), Y(s;x,y)\right)$ is the solution to the characteristic equation

$$\begin{cases} \dot{X}(s) = -\partial_y \psi_0(X(s), Y(s)) \\ \dot{Y}(s) = \partial_x \psi_0(X(s), Y(s)), \end{cases} \tag{22}$$

with the initial value $X(0) = x$, $Y(0) = y$.

Remark 3.1. A_λ is well-defined. Denoting

$$K_\lambda \psi := -g'(\psi_0)\,\psi + g'(\psi_0)\,\lambda \int_{-\infty}^{0} e^{\lambda s} \psi\left(X(s;x,y), Y(s;x,y)\right) ds, \tag{23}$$

then we have

$$\|K_\lambda \psi\|_2 \le 2 \left\|g'(\psi_0)\right\|_\infty \|\psi\|_2 . \tag{24}$$

Indeed, for any function $\phi \in L^2(\Omega)$, we have

$$\begin{aligned}
&\left|\int\int_\Omega \int_{-\infty}^{0} \lambda e^{\lambda s} g'(\psi_0)\,\phi(x,y) \psi\left(X(s;x,y), Y(s;x,y)\right) ds dx dy\right| \\
&\le \left(\int\int_\Omega \int_{-\infty}^{0} \lambda e^{\lambda s} |\psi|^2 \left|g'(\psi_0)\right| \left(X(s;x,y), Y(s;x,y)\right) ds dx dy\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int\int_\Omega \int_{-\infty}^{0} \lambda e^{\lambda s} |\phi|^2 \left|g'(\psi_0)\right| ds dx dy\right)^{\frac{1}{2}} \\
&\le \left(\int_{-\infty}^{0} \lambda e^{\lambda s} \left\|g'(\psi_0)\right\|_\infty \int\int_\Omega |\psi|^2 dx ds\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{-\infty}^{0} \lambda e^{\lambda s} \left\|g'(\psi_0)\right\|_\infty \int\int_\Omega |\phi|^2 dx ds\right)^{\frac{1}{2}} \\
&= \left\|g'(\psi_0)\right\|_\infty \|\phi\|_2 \|\psi\|_2 \ .
\end{aligned}$$

Thus (24) follows. Here we used the fact that the Jacobian of the mapping

$$(x,y) \to \left(X(s;x,y), Y(s;x,y)\right)$$

is one.

The following lemma indicates the reason why we introduce A_λ.

Lemma 3.1. *Let $\lambda > 0$, then there exists a nontrivial solution*

$$\left(e^{\lambda t}\omega(x, y), e^{\lambda t}\psi(x, y)\right)$$

to (6) with $\omega \in C^1$ and $\psi \in \mathbf{X}$, if and only if there exists some $\psi \in \mathbf{X}$ such that $A_\lambda \psi = 0$. In this case

$$\omega = g'(\psi_0)\psi - g'(\psi_0)\lambda \int_{-\infty}^{0} e^{\lambda s}\psi(X(s; x, y), Y(s; x, y))\,ds. \tag{25}$$

Proof. If $\left(e^{\lambda t}\omega(x, y), e^{\lambda t}\psi(x, y)\right)$ is a solution to (6), then (ω, ψ) satisfies (11). We can rewrite (11) at $(X(s), Y(s))$ as

$$\begin{aligned}\frac{d}{ds}(e^{\lambda s}\omega((X(s), Y(s)))) &= e^{\lambda s}\left(\psi_y \partial_x \omega_0 - \psi_x \partial_y \omega_0\right)(X(s), Y(s)) \\ &= e^{\lambda s} g'(\psi_0)\left(-\partial_y \psi_0 \psi_x + \partial_x \psi_0 \psi_y\right)(X(s), Y(s)) \\ &= e^{\lambda s} g'(\psi_0)\frac{d\psi}{ds}(X(s), Y(s)).\end{aligned}$$

Integrating above from $-\infty$ to 0, we get

$$\begin{aligned}\omega(x, y) &= g'(\psi_0)\textstyle\int_{-\infty}^{0} e^{\lambda s}\frac{d\psi}{ds}(X(s; x, y), Y(s; x, y))\,ds \\ &= g'(\psi_0)\psi - g'(\psi_0)\lambda \textstyle\int_{-\infty}^{0} e^{\lambda s}\psi(X(s; x, y), Y(s; x, y))\,ds.\end{aligned}$$

Plugging the above equality into the Poisson equation, we get

$$-\Delta\psi = g'(\psi_0)\psi - g'(\psi_0)\lambda \int_{-\infty}^{0} e^{\lambda s}\psi(X(s; x, y), Y(s; x, y))\,ds,$$

which is exactly $A_\lambda \psi = 0$.

Conversely, if $\omega \in C^1$ and satisfies (25), we can show that it satisfies (11) by the same argument as in [12]. □

In the following, we show that (ω, ψ) is a weak solution to (11). Moreover ω is differentiable almost everywhere.

Lemma 3.2. *Given $\psi \in \mathbf{Y}$ satisfying $A_\lambda \psi = 0$ and $\omega(x, y)$ defined by (25), then (ψ, ω) is a weak solution of (11). Moreover ψ is differentiable besides the critical set, which is the set of all points (x, y) such that $\psi_0(x, y)$ is equal to the critical value at a saddle point. So according to Lemma 3.1 (ψ, ω) satisfies (11) almost everywhere in the classical sense.*

Proof. To show that (ψ, ω) is a weak solution of (11), we take any $\phi \in C_0^1(\Omega)$, then

$$\begin{aligned}&\int\int_\Omega \left(\psi_{0_y}\partial_x\phi - \psi_{0_x}\partial_y\phi\right)\omega\,dxdy \\ &= -\int\int_\Omega \left(\psi_{0_y}\partial_x\phi - \psi_{0_x}\partial_y\phi\right) g'(\psi_0)\int_{-\infty}^{0}\lambda e^{\lambda s}\psi(X(s), Y(s))ds\;dxdy \\ &\quad + \int\int_\Omega \left(\psi_{0_y}\partial_x\phi - \psi_{0_x}\partial_y\phi\right) g'(\psi_0)\psi(x, y)\,dxdy \\ &= I + II.\end{aligned}$$

For the first term, we have

$$
\begin{aligned}
I &= -\int_{-\infty}^{0} \lambda e^{\lambda s} \int\int_{\Omega} g'(\psi_0)\left(\psi_{0_y}\partial_x\phi - \psi_{0_x}\partial_y\phi\right)(X(-s), Y(-s))\,\psi(x,y)\,dxdyds \\
&= \int\int_{\Omega} g'(\psi_0)\int_{-\infty}^{0}\lambda e^{\lambda s}\left(-\frac{d}{ds}\phi(X(-s),Y(-s))\right)ds\psi(x,y)\,dxdy \\
&= \int\int_{\Omega} g'(\psi_0)\left(-\lambda\phi(x,y) + \int_{-\infty}^{0}\lambda^2 e^{\lambda s}\phi(X(-s),Y(-s))\,ds\right)\psi(x,y)\,dxdy \\
&= -\int\int_{\Omega} g'(\psi_0)\,\lambda\phi(x,y)\,\psi(x,y)\,dxdy \\
&\quad + \int_{-\infty}^{0}\lambda^2 e^{\lambda s}\int\int_{\Omega} g'(\psi_0)\,\phi(X(-s),Y(-s))\,\psi(x,y)\,dxdy \\
&= \lambda\int\int_{\Omega}\left(-g'(\psi_0)\,\psi + g'(\psi_0)\,\lambda\int_{-\infty}^{0} e^{\lambda s}\psi(X(s;x,y),Y(s;x,y))\,ds\right) \\
&\quad \times \phi(x,y)\,dxdy \\
&= -\lambda\int\int_{\Omega}\omega\phi\,dxdy.
\end{aligned}
$$

Here in the first and fourth equality we change the variable

$$
(x, y) \to (X(s;x,y), Y(s;x,y)).
$$

By integration by parts

$$
\begin{aligned}
II &= \int\int_{\Omega}\left(\psi_{0_y}\partial_x\phi - \psi_{0_x}\partial_y\phi\right) g'(\psi_0)\,\psi(x,y)\,dxdy \\
&= \int\int_{\Omega}\phi\left(-\psi_{0_y}\partial_x + \psi_{0_x}\partial_y\right)\left(g'(\psi_0)\,\psi(x,y)\right)dxdy \\
&= \int\int_{\Omega} g'(\psi_0)\left(-\psi_{0_y}\partial_x\psi + \psi_{0_x}\partial_y\psi\right)\phi\,dxdy \\
&= \int\int_{\Omega}\left(\psi_y\partial_x\omega_0 - \psi_x\partial_y\omega_0\right)\phi\,dxdy.
\end{aligned}
$$

So

$$
\begin{aligned}
\int\int_{\Omega}\left(\psi_{0_y}\partial_x\phi - \psi_{0_x}\partial_y\phi\right)\omega\,dxdy &= I + II \\
&= \int\int_{\Omega}\left(-\lambda\omega + \psi_y\partial_x\omega_0 - \psi_x\partial_y\omega_0\right)\phi\,dxdy
\end{aligned}
$$

which means that (ψ, ω) is a weak solution of (11).

Taking the derivative ∂_x on the right hand side of (25), we get the expression

$$
\begin{aligned}
&\partial_x\left(g'(\psi_0)\,\psi(x,y)\right) - \partial_x\left(g'(\psi_0)\right)\int_{-\infty}^{0}\lambda e^{\lambda s}\psi(X(s),Y(s))\,ds \; - g'(\psi_0) \\
&\times\int_{-\infty}^{0}\lambda e^{\lambda s}\left(\partial_x\psi(X(s),Y(s))\,\frac{\partial X(s;x,y)}{\partial x} + \partial_y\psi(X(s),Y(s))\,\frac{\partial Y(s;x,y)}{\partial x}\right)ds.
\end{aligned}
$$

If (x, y) is not in the critical set, then the fluid particle with the initial position (x, y) has a periodic trajectory. So $\frac{\partial X(s;x,y)}{\partial x}$ and $\frac{\partial Y(s;x,y)}{\partial x}$ can only have linear growth and the third term above is finite. Since the other terms are also finite, we prove that ω defined by (25) is differentiable if (x, y) is not in the critical set. □

Note that in [1], it was shown that if the growth rate $\operatorname{Re}\lambda$ is greater than the Liapunov exponent of the steady flow, then the growing mode $\omega \in \mathbf{H}^1(\Omega)$.

Next we study some properties of A_λ.

Lemma 3.3. *A_λ is a densely defined closed operator and for any ξ in its resolvent set $\rho(A_\lambda)$, $(\xi - A_\lambda)^{-1}$ is a trace class operator. The eigenvalues of A_λ appear in complex conjugate pairs and they are all discrete with finite multiplicity.*

Proof. Denote

$$A = -\Delta$$

with $D(A) = \mathbf{X}$. Then clearly $(\xi - A)^{-1}$ is a trace class operator for any $\xi \in \rho(A)$. We have

$$\left\| (A + l)^{-1} \right\| \leq \frac{1}{l}$$

for any $l > 0$. By Remark 3.1, $A_\lambda - A = K_\lambda$ are uniformly bounded operators with $\|K_\lambda\| \leq 2 \left\| g'(\psi_0) \right\|_\infty$. We have

$$A_\lambda + l = A + l + K_\lambda = \left(1 + K_\lambda (A + l)^{-1}\right)(A + l).$$

So if $2 \left\| g'(\psi_0) \right\|_\infty < l$, then $-l \in \rho(A_\lambda)$ and

$$(A_\lambda + l)^{-1} = (A + l)^{-1} \left(1 + K_\lambda (A + l)^{-1}\right)^{-1}.$$

This is the multiplication of a bounded operator with a trace class operator, so it is also in trace class. For any $\xi \in \rho(A_\lambda)$, from formula

$$(\xi - A_\lambda)^{-1} = (-l - A_\lambda)^{-1} + (\xi + l)(\xi - A_\lambda)^{-1}(-l - A_\lambda)^{-1},$$

we can see that $(\xi - A_\lambda)^{-1}$ is in trace class.

Now the conclusions about the eigenvalues of A_λ follow from the trace class property just proved and the fact that A_λ commutes with complex conjugation. □

Lemma 3.4. *There exists $\Lambda_0 > 0$ such that if $\lambda > \Lambda_0$ then A_λ has no negative eigenvalues.*

Proof. First we show that for all $\psi \in H_0^1(\Omega)$,

$$\|K_\lambda \psi\|_2 \leq \frac{|\omega_0|_{C^1}}{\lambda} \|\nabla \psi\|_2.$$

In fact, for any $\phi \in L^2(\Omega)$,

$$
\begin{aligned}
|(K_\lambda \psi, \phi)| &\le \int\int_\Omega \int_{-\infty}^0 e^{\lambda s} \left|\psi_y \partial_x \omega_0 - \psi_x \partial_y \omega_0\right| (X(s), Y(s)) \,.\, |\phi| \, dxdyds \\
&\le |\omega_0|_{C^1} \int_{-\infty}^0 e^{\lambda s} \int\int_\Omega |\nabla \psi| (X(s), Y(s)) \, |\phi| \, dxdyds \\
&\le |\omega_0|_{C^1} \int_{-\infty}^0 e^{\lambda s} \left(\int\int_\Omega |\nabla \psi|^2 (X(s), Y(s)) \, dxdy \right)^{\frac{1}{2}} \\
&\quad \left(\int\int_\Omega |\phi|^2 \, dxdy \right)^{\frac{1}{2}} ds = |\omega_0|_{C^1} \frac{1}{\lambda} \|\nabla \psi\|_2 \|\phi\|_2 \,.
\end{aligned}
$$

Suppose there exists some negative eigenvalue for A_λ, that is

$$
-\Delta \psi + K_\lambda \psi = k\psi, \tag{26}
$$

for some $k < 0$, $0 \neq \psi \in H_0^1(\Omega)$. By Poincaré's inequality, $\|\psi\|_2 \le c_0 \|\nabla \psi\|_2$ for some constant c_0. So taking the inner product of (26) with ϕ, we have

$$
\begin{aligned}
0 > (k\psi, \psi) &= \|\nabla \psi\|_2^2 + (K_\lambda \psi, \psi) \\
&\ge \|\nabla \psi\|_2^2 - \|K_\lambda \psi\|_2 \|\psi\|_2 \\
&\ge \|\nabla \psi\|_2^2 - \frac{c_0 |\omega_0|_{C^1}}{\lambda} \|\nabla \psi\|_2^2 \\
&> 0,
\end{aligned}
$$

which is a contradiction if $\lambda > \Lambda_0 = c_0 |\omega_0|_{C^1}$. □

3.2. Infinite determinant and an abstract theorem. In this part, we prove the following abstract result using the infinite determinant method developed in [13].

Theorem 3.1. *Consider a continuous family of operators $A_\lambda : \mathcal{H} \to \mathcal{L}$,*

$$
A_\lambda = A + B_\lambda, \left(\lambda \in \mathbf{R}^+\right).
$$

We assume that:

- *(I) B_λ are uniformly bounded operators and A_λ commute with complex conjugation.*
- *(II) The self-adjoint operator $-A$ generates a generalized parabolic semigroup, that is,* $\exp(-tA)$ *is in the trace class and* $A\exp(-tA)$ *is bounded. Furthermore, the embedding $i : (\mathcal{H}, \|.\|_A) \to (\mathcal{L}, \|.\|)$ is compact. Here $\|.\|_A$ is the graph norm of operator A and $\|.\|$ is the norm in $\mathcal{L}$.*
- *(III) When λ is sufficiently large, A_λ has no negative eigenvalue.*
- *(IV) When λ tends to 0, A_λ tends to A_0 strongly in the sense that: for any $u \in \mathcal{H}$,*

$$
A_\lambda u \to A_0 u \ \text{ and } \ A_\lambda^* u \to A_0^* u \text{ strongly}, \ \text{as } \lambda \to 0+ \tag{27}
$$

for any function $u \in \mathcal{H}$.

Then if A_0 has an odd number of negative eigenvalues and no kernel, there must exist some $\lambda_0 > 0$ such that A_{λ_0} has a nontrivial kernel.

Note that here the condition (IV) is weaker than that in [13], where it is required that

$$\|(A_\lambda - A_0)\,\phi\| \leq c\,(\lambda)\,(\|A\phi\| + \|\phi\|)\,, \tag{28}$$

for some function $c\,(\lambda)$ approaching 0 as $\lambda \to 0^+$. Condition (28) can not be proved for the case of Theorem 1.2, and only (27) is available.

Proof. The line of proof follows that of in [13]. So we just sketch it and indicate some differences caused by the weaker assumption (IV). Denote all the distinct eigenvalues of A_λ (arranged with non-decreasing real parts) by

$$\mu_1\,(\lambda)\,, \mu_2\,(\lambda)\,, \cdots, \mu_k\,(\lambda)\,, \cdots,$$

with multiplicities $n_1, n_2, \cdots, n_k, \cdots$. We define

$$d\,(\lambda) = \prod_{k=1}^{\infty} (1 - \exp\,(-\mu_k\,(\lambda)))^{n_k}\,,$$

which is the infinite determinant of the operator $\exp\,(-A_\lambda)$. By assumptions (I) (II), $\exp\,(-A_\lambda)$ is a trace class operator. Since $\mu_k\,(\lambda)$ appears in complex conjugate pairs, $d\,(\lambda)$ is a finite real number. From the definition of $d\,(\lambda)$, we know that the sign of $d\,(\lambda)$ is determined by the number of negative eigenvalues of $A_{\lambda.}$ If this number is odd, then $d\,(\lambda)$ is negative and $d\,(\lambda)$ is positive if this number is even. Here we assume that A_λ has no kernel, since otherwise we have already found the growing mode. The main idea is to keep track of the sign change of $d\,(\lambda)$, especially as λ tends to zero and infinity. By assumption (III), $d\,(\lambda)$ is nonnegative when λ is large. So if $d\,(\lambda)$ is negative for small λ, by the continuity argument as that of [13] we conclude that there exists some λ_0 such that $d\,(\lambda_0) = 0$, which implies the singularity of A_{λ_0}. We show that when λ is small enough, the sign of $d\,(\lambda)$ can be determined by the number of negative eigenvalues of A_0. If the number is odd as assumed in the theorem, then $d\,(\lambda)$ is negative for small λ. So the key issue is to show that the negative spectrum of A_0 is stable when perturbed to A_λ. Since only the weaker convergence (27) is available, the regular perturbation theory as that of in [13] is not applicable. We deal with this issue by using ideas from the asymptotic perturbation theory for Schrödinger operators (see [7]).

First we show the following:

(i) For any eigenvalue $\mu\,(\lambda)$ of A_λ, we have $|\mathrm{Im}\,\mu\,(\lambda)| < M$ (here M is such that $\|B_\lambda\| < M$).

(ii) Let $b > 0$ be such that there are no eigenvalues of A_0 with real part b. There exists positive ε_1, δ_1 such that if $\lambda < \delta_1$, then for any eigenvalue $\mu\,(\lambda)$ of A_λ, we have $|\mathrm{Re}\,\mu\,(\lambda) - b| > \varepsilon_1$.

(iii) Define

$$P\,(A) = \left\{ z | R_k\,(z) = (z - A_\lambda)^{-1} \text{ exists and is uniformly bounded for small } \lambda \right\}$$

and $P\,(A^*)$ is defined in a similar way. Then we have

$$\rho\,(A_0) \subset P\,(A)\,, (\rho\,(A_0))^* \subset P\left(A^*\right).$$

The proof of (i) is obvious by our assumptions.

Now we prove (ii): Supposing it were false, we could find a sequence $\lambda_n \to 0$, μ_n being an eigenvalue of A_{λ_n}, and $\operatorname{Re} \mu_n \to b$. Let u_n be the corresponding eigenfunction and $\|u_n\| = 1$. By (i), $\{\mu_n\}$ is a bounded sequence. We can find a subsequence $\mu_{n_k} \to \mu_0$ and $\operatorname{Re} \mu_0 = b$. For convenience, we still denote the subsequence by $\{\mu_n\}$. It is easy to see that $\|u_n\|_A \le C$ (independent of n). So by assumption (II), there exists some u_0 with $u_n \to u_0$ strongly in $\mathcal{L}$ and $\|u_0\| = 1$. Moreover, $A_{\lambda_n} u_n \to A_0 u_0$ weakly in $\mathcal{L}$. To see that, we take any function $v \in \mathcal{L}$, then

$$\begin{aligned}\lim_{n\to\infty} \left(A_{\lambda_n} u_n, v\right) &= \lim_{n\to\infty} \left(u_n, A^*_{\lambda_n} v\right) = \left(u_0, A^*_0 v\right) \quad \text{(by assumption (IV))}\\ &= \left(A_0 u_0, v\right).\end{aligned}$$

Combining the above with $A_{\lambda_n} u_n = \mu_n u_0$, we get $A_0 u_0 = \mu_0 u_0$ in the limit $n \to \infty$. This is a contradiction since $\operatorname{Re} \mu_0 = b$.

To prove (iii), we note that $z \in P(A)$ is equivalent to the following: there exists some $\varepsilon > 0$ such that

$$\|(z - A_\lambda) u\| \ge \varepsilon, \text{ for small } \lambda \text{ and any } u \in \mathcal{H}. \tag{29}$$

Indeed assuming $z \notin P(A)$, we have

$$\left\|\left(z - A_{\lambda_k}\right) u_k\right\| \to 0$$

for some sequence $\{\lambda_k\} \to 0$ and $u_k \in \mathcal{H}$ with $\|u_k\| = 1$. By the same argument as in the proof of (ii), we have $(z - A_0) u_0 = 0$ for some nontrivial function u_0. This is a contradiction. The proof for $P(A^*)$ is the same since

$$(\rho(A_0))^* = \rho\left(A_0^*\right).$$

This proves (i)–(iii).

Let Λ be the minimum of the real part of eigenvalues of A_λ. The number Λ is finite since A_λ are uniformly bounded from below. Define

$$D = \left\{(x, y) \mid \Lambda - 1 < x < -\frac{\varepsilon_1}{2} + b, \ -M < y < M\right\}$$

and $\Gamma = \partial D$. By taking M, Λ large, we can assume $\Gamma \subset \rho(A_0)$. By claim (ii) just proved, if $\lambda < \delta_1$ then all eigenvalues of A_λ with negative real part lie in D. Define the Riesz projection as

$$P_\lambda = \frac{1}{2\pi i} \oint_\Gamma R_\lambda(k)\, dk \tag{30}$$

and $R(P_\lambda)$ its range. Similarly we define

$$P_0 = \frac{1}{2\pi i} \oint_\Gamma R_0(k)\, dk.$$

Here the Γ-integral is in the counterclockwise sense.

Now we show that

$$\dim\left(R(P_\lambda)\right) = \dim\left(R(P_0)\right), \text{ if } \lambda \text{ is small enough,} \tag{31}$$

which together with (34) implies that

$$\|P_\lambda - P_0\| \to 0 \text{ as } \lambda \to 0 \tag{32}$$

(see [8, Lemma 1.21 of Chapter VIII]).

Let us prove (31). Since $\Gamma \subset \rho\left(A_0\right)$ is compact, by claim (iii) above there exists $\delta > 0$ such that if $\lambda < \delta$ and $k \in \Gamma$, then

$$\|R_\lambda(k)\|, \left\|R_\lambda^*(k)\right\| \leq C_1 \tag{33}$$

for some constant C_1. It follows that $R_\lambda(k)$, $R_\lambda^*(k)$ are strongly continuous at $\lambda = 0$. Indeed for any $u \in \mathcal{H}$, we have

$$\begin{aligned}\|(R_\lambda(k) - R_0(k))\, u\| &= \|R_\lambda(k)\left(A_\lambda - A_0\right) u\| \\ &\leq C_1 \left\|\left(A_\lambda - A_0\right) u\right\| \to 0 \text{ (by (27)).}\end{aligned}$$

The strong continuity of $R_\lambda^*(k)$ can be shown in the same way. So we have

$$P_\lambda \to P_0 \text{ and } P_\lambda^* \to P_0^* \text{ strongly.} \tag{34}$$

Therefore $\dim P_\lambda \geq \dim P_0$ for small λ. To show (31), we only need to prove

$$\dim P_\lambda \leq \dim P_0, \text{ for small } \lambda. \tag{35}$$

Supposing otherwise, then we can find a sequence $\{\lambda_n\} \to 0$ and $\{u_n\} \subset \mathcal{H}$ with $\|u_n\| = 1$, such that

$$P_{\lambda_n} u_n = u_n \text{ and } P_0 u_n = 0. \tag{36}$$

By passing to a subsequence we can assume that $u_n \to u_0$ weakly. By (27), we have $P_{\lambda_n} u_n \to P_0 u_0$ weakly. So passing to the limit in (36), we have

$$P_0 u_0 = u_0 \text{ and } P_0 u_0 = 0,$$

which implies that $u_0 = 0$ and $u_n \to 0$ weakly. But by (33) and the definition of P_λ (30),

$$\left\|A_{\lambda_n} u_n\right\| = \left\|A_{\lambda_n} P_{\lambda_n} u_n\right\| \leq \left\|A_{\lambda_n} P_{\lambda_n}\right\| \leq \text{const, for small } \lambda_n.$$

This implies the bound $\|u_n\|_A \leq$ const, from which we deduce that $u_n \to 0$ strongly in $\mathcal{L}$ by assumption (II). This is a contradiction and ends the proof of (31).

Let $\mu_1, \mu_2, \cdots, \mu_N$ be all the distinct eigenvalues of A_0 in D. Let m_k be the multiplicity of μ_k. For each μ_k, we can pick a small ball $B_k = B\left(\mu_k; r_k\right)$, inside which μ_k is the isolated eigenvalue of A_0. And by taking r_k small enough we can ensure that B_k does not intersect with the imaginary axis if $\operatorname{Re} \mu_k \neq 0$, and B_k does not intersect with the real axis if $\operatorname{Re} \mu_k = 0$. We also assume $\{B_k\}$ does intersect with Γ. The disks $\{B_k\}$ are disjoint and for the conjugate of μ_k we take the disk with the same radius. Then if λ is small enough, by the same proof as that of (31), there are exactly m_k eigenvalues (counting multiplicity) of A_λ in each B_k. Since $\dim\left(R\left(P_\lambda\right)\right) = \dim\left(R(P_{\lambda_0})\right)$, these are all the eigenvalues of A_λ in D. By our construction of B_k, if we multiply all the eigenvalues of A_λ contained in them, the sign is the same as that of A_0. Thus in the definition of $d(\lambda)$ the product corresponding to all the eigenvalue of A_λ with real part smaller than b has the same sign as that of A_0. Thus it is negative if λ is small. Since the other part of the product is always positive, we have proved that $d(\lambda)$ is negative when λ is small. This finishes the proof of the theorem. $\square$

Remark 3.2. If A_0 has a kernel, we denote by $e(A_0)$ the number of null vectors in $\ker A_0$ perturbed to be negative eigenfunctions of A_λ as $\lambda > 0$ is small and $n(A_0)$ the number of negative eigenvalues of A_0. Then the conclusion of the theorem still holds if $n(A_0) + e(A_0)$ is odd. The proof is the same as above.

3.3. Proof of Theorem 1.2. Now we use the abstract theorem above to prove Theorem 1.2. The operator A_λ is defined by (21) with $A = -\Delta$, $B_\lambda = K_\lambda$, $\mathcal{H} = \mathbf{X}$, $\mathcal{L} = \mathbf{L}^2(\Omega)$. Now we check the assumptions in Theorem 3.1. Assumption (I) is proved in Remark 3.1. Assumption (II) is standard for the Laplacian defined in a bounded domain. Assumption (III) is proved in Lemma 3.4. Moreover A_0 has an odd number of negative eigenvalues and no kernel as assumed in Theorem 1.2. So we only need to prove assumption (IV). This is in the following lemma.

Lemma 3.5. *For any $\phi \in \mathbf{X}$,*

$$A_\lambda\phi \to A_0\phi \text{ and } A_\lambda^*\phi \to A_0\phi \text{ strongly in } \mathbf{L}^2(\Omega), \text{ as } \lambda \to 0+.$$

Proof. It is easy to show that

$$A_\lambda^*\psi := -\Delta\psi - g'(\psi_0)\psi + g'(\psi_0)\lambda\int_{-\infty}^0 e^{\lambda s}\psi(\mathbf{X}(-s;x,y), \mathbf{Y}(-s;x,y))\,ds,$$

where $(X(s;x,y), Y(s;x,y))$ is the solution to the characteristic equation (22). So we only need to show the strong convergence of A_λ since the proof for A_λ^* is the same.

For any $\phi \in \mathbf{X}$, denote

$$\phi_\lambda = g'(\psi_0)\lambda\int_{-\infty}^0 e^{\lambda s}\phi(X(s;x,y), Y(s;x,y))\,ds$$

and $\phi_0 = g'(\psi_0)\tilde{P}\phi$, where $\tilde{P}$ is defined in the introduction and given by the formula (17). We have

$$\begin{aligned}\|A_\lambda\phi - A_0\phi\|_2^2 &= (A_\lambda\phi - A_0\phi, A_\lambda\phi - A_0\phi)\\ &= (\phi_\lambda - \phi_0, \phi_\lambda - \phi_0)\\ &= (\phi_\lambda, \phi_\lambda) - 2(\phi_\lambda, \phi_0) + (\phi_0, \phi_0).\end{aligned}$$

We analyze the first term

$$\begin{aligned}(\phi_\lambda, \phi_\lambda) &= \int\int_\Omega \left(g'(\psi_0)\right)^2\lambda^2\int_{-\infty}^0\int_{-\infty}^0 e^{\lambda s}e^{\lambda t}\\ &\quad\cdot\phi(X(s;x,y), Y(s;x,y))\,\phi(\mathbf{X}(t;x,y), \mathbf{Y}(t;x,y))\,dsdtdxdy\\ &= \int\int_\Omega f_\lambda(x,y)\,dxdy,\end{aligned}$$

where

$$\begin{aligned}f_\lambda(x,y) = \left(g'(\psi_0)\right)^2\lambda^2\int_{-\infty}^0\int_{-\infty}^0 & e^{\lambda s}e^{\lambda t}\phi(X(s;x,y), Y(s;x,y))\\ &\cdot\phi(\mathbf{X}(t;x,y), \mathbf{Y}(t;x,y))\,dsdt.\end{aligned}$$

We claim the following:

(i) As $\lambda \to 0+,\ f_\lambda(x, y) \to \phi_0^2$ almost everywhere.
(ii) $\{f_\lambda\}\left(\lambda \in \mathbf{R}^+\right)$ are uniformly integrable.

Proof of claim (i). First we study the characteristic equation (22) for the fluid particle. For each initial position (x, y) not on the critical level sets, we know that the trajectory $(X(s; x, y), Y(s; x, y))$ always lies in some component of $\{\psi_0 = \psi_0(x, y)\}$ which is a closed curve denoted by Γ. If we denote by γ the arc length variable of Γ and by $L(\Gamma)$ the length, then the particle has a periodic trajectory according to the law

$$\frac{d\gamma(s)}{ds} = |\nabla\psi_0|(X(s; x, y), Y(s; x, y))$$

with the period

$$T(\Gamma) = \int_0^{L(\Gamma)} \frac{d\gamma}{|\nabla\psi_0|}.$$

In the following we identify the point $(X(s; x, y), Y(s; x, y))$ with its arc length variable $\gamma(s)$. We recall the following fact proved in [12]: For any T- periodic function $A(x) \in L^1(0, T)$,

$$\lim_{\lambda\to 0^+} \int_{-\infty}^0 e^s A\left(\frac{s}{\lambda}\right) ds = \frac{1}{T}\int_0^T A(s)\, ds\ .$$

Using this, we have

$$\begin{aligned}
&\lim_{\lambda\to 0^+} \lambda \int_{-\infty}^0 e^{\lambda s}\phi(X(s; x, y), Y(s; x, y))\, ds \\
&= \lim_{\lambda\to 0^+} \int_{-\infty}^0 e^s\phi\left(\mathbf{X}\left(\frac{s}{\lambda}; x, y\right), \mathbf{Y}\left(\frac{s}{\lambda}; x, y\right)\right) ds \\
&= \frac{1}{T(\Gamma)} \int_0^{T(\Gamma)} \phi(X(s; x, y), Y(s; x, y))\, ds \\
&= \frac{1}{T(\Gamma)} \int_0^{L(\Gamma)} \frac{\phi(\gamma)\, d\gamma}{|\nabla\psi_0|} \\
&= \frac{\oint_\Gamma \frac{\phi}{|\nabla\psi_0|}}{\oint_\Gamma \frac{1}{|\nabla\psi_0|}}.
\end{aligned}$$

The last expression is exactly the formula (17) for $\tilde{P}\phi$, so claim (i) is proved.

Proof of claim (ii). For any $\delta > 0$, there exists $\varepsilon_0 > 0$, such that for any set $B \subset \Omega$ with $|B| < \varepsilon_0$, we have

$$\int\int_B |\psi|^2\, dxdy < \delta.$$

Then

$$
\begin{aligned}
&\int\int_B |f_\lambda(x,y)|\,dxdy \\
&\quad\le \left|g'(\psi_0)\right|_\infty^2 \int_{-\infty}^0\int_{-\infty}^0 \lambda^2 e^{\lambda s}e^{\lambda t} \\
&\qquad\cdot \int\int_B |\phi(X(s;x,y),Y(s;x,y))\,\phi(\mathbf{X}(t;x,y),\mathbf{Y}(t;x,y))|\,dxdydsdt \\
&\quad\le \left|g'(\psi_0)\right|_\infty^2 \int_{-\infty}^0\int_{-\infty}^0 \lambda^2 e^{\lambda s}e^{\lambda t}\left(\int\int_B |\phi(X(s;x,y),Y(s;x,y))|^2\,dxdy\right)^{\frac12} \\
&\qquad\cdot \left(\int\int_B |\phi(\mathbf{X}(t;x,y),\mathbf{Y}(t;x,y))|^2\,dxdy\right)^{\frac12} dsdt \\
&\quad= \left|g'(\psi_0)\right|_\infty^2 \int_{-\infty}^0\int_{-\infty}^0 \lambda^2 e^{\lambda s}e^{\lambda t}\left(\int\int_{\Phi_s(B)} |\phi(x,y)|^2\,dxdy\right)^{\frac12} \\
&\qquad\cdot \left(\int\int_{\Phi_t(B)} |\phi(x,y)|^2\,dxdy\right)^{\frac12} dsdt \\
&\quad\le \left|g'(\psi_0)\right|_\infty^2 \int_{-\infty}^0\int_{-\infty}^0 \lambda^2 e^{\lambda s}e^{\lambda t}\delta^2 dsdt = \left|g'(\psi_0)\right|_\infty^2 \delta^2.
\end{aligned}
$$

We thus prove claim (ii). Here Φ_s denotes the mapping

$$(x,y)\to (X(s;x,y),Y(s;x,y))$$

and we use the fact $|\Phi_s(B)| = |B| < \varepsilon_0$.

Now by claims (i), (ii) and the Dominant Convergence Theorem, we have

$$
\begin{aligned}
\lim_{\lambda\to0^+}(\phi_\lambda,\phi_\lambda) &= \lim_{\lambda\to0^+}\int\int_\Omega f_\lambda(x,y)\,dxdy \\
&= \int\int_\Omega \lim_{\lambda\to0^+} f_\lambda(x,y)\,dxdy \\
&= \int\int_\Omega \phi_0^2 = (\phi_0,\phi_0).
\end{aligned}
$$

By the same proof we have

$$\lim_{\lambda\to0^+}(\phi_\lambda,\phi_0) = (\phi_0,\phi_0).$$

So

$$
\begin{aligned}
\lim_{\lambda\to0^+}\|A_\lambda\phi - A_0\phi\|_2^2 &= \lim_{\lambda\to0^+}(\phi_\lambda,\phi_\lambda) - 2(\phi_\lambda,\phi_0) + (\phi_0,\phi_0) \\
&= (\phi_0,\phi_0) - 2(\phi_0,\phi_0) + (\phi_0,\phi_0) = 0.
\end{aligned}
$$

□

4. An Enstrophy-Stable but Energy-Unstable Steady Flow

In this section, we consider a bounded simply connected domain $\Omega \subset \mathbf{R}^2$ with a smooth boundary $\partial\Omega$. Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be all the eigenvalues of $-\Delta$ with Dirichlet boundary condition. Denote $\phi_0, \phi_1, \cdots, \phi_n, \cdots$ the corresponding normalized orthogonal eigenfunctions. We show that ϕ_0 is a nonlinearly stable steady state of the 2-D incompressible Euler equation (2) in the $\mathbf{L}^2$-norm of the vorticity (enstrophy). This can be deduced from Theorem 1.1 (i). But in the following we give a direct proof. Denote $\omega_0 = -\lambda_0\phi_0$.

Theorem 4.1. *The steady flow with the stream function ϕ_0 is nonlinearly stable in the following sense: for any $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\|\omega(.,0) - \omega_0\|_{\mathbf{L}^2} < \delta \Longrightarrow \sup_{t>0} \|\omega(.,t) - \omega_0\|_{\mathbf{L}^2} < \varepsilon.$$

Proof. For $\psi \in \mathbf{X}$, we define the following energy-Casimir functional:

$$H(\psi) = \frac{1}{2}\int\int_{\Omega}\left(-|\nabla\psi|^2 + \frac{1}{\lambda_0}|\omega|^2\right)dxdy.$$

Then

$$\begin{aligned}
H(\psi) - H(\phi_0) &= \int\int_{\Omega} -\frac{1}{2}|\nabla(\psi-\phi_0)|^2 - \mathrm{Re}\,\nabla(\psi-\phi_0).\nabla\phi_0 \\
&\quad + \frac{1}{2\lambda_0}|\omega-\omega_0|^2 + \frac{1}{\lambda_0}\mathrm{Re}(\omega-\omega_0)\omega_0 \\
&= \int\int_{\Omega} -\frac{1}{2}|\nabla(\psi-\phi_0)|^2 + \frac{1}{2\lambda_0}|\omega-\omega_0|^2 \\
&\quad + \mathrm{Re}(\omega-\omega_0)\left(\frac{1}{\lambda_0}\omega_0 + \phi_0\right) \\
&= \int\int_{\Omega} -\frac{1}{2}|\nabla(\psi-\phi_0)|^2 + \frac{1}{2\lambda_0}|\omega-\omega_0|^2 \\
&= \int\int_{\Omega} \frac{1}{2}|\nabla(\psi-\phi_0)|^2 - |\nabla(\psi-\phi_0)|^2 + \frac{1}{2\lambda_0}|\omega-\omega_0|^2 \\
&= \int\int_{\Omega} \frac{1}{2}|\nabla(\psi-\phi_0)|^2 + (\omega-\omega_0)(\psi-\phi_0)^* + \frac{1}{2\lambda_0}|\omega-\omega_0|^2 \\
&= \int\int_{\Omega} \frac{1}{2}|\nabla(\psi-\phi_0)|^2 + \frac{1}{2\lambda_0}|(\omega-\omega_0) + \lambda_0(\psi-\phi_0)|^2 \\
&\quad - \frac{1}{2}\lambda_0|\psi-\phi_0|^2.
\end{aligned}$$

So if we denote

$$\psi - \phi_0 = \sum_{i=0}^{\infty} a_i\phi_i,$$

then

$$\omega - \omega_0 = \Delta(\psi - \phi_0) = -\sum_{i=0}^{\infty} \lambda_i a_i \phi_i$$

and

$$\begin{aligned} H(\psi) - H(\phi_0) &\geq \frac{1}{2\lambda_0} \left|(\omega - \omega_0) + \lambda_0 (\psi - \phi_0)\right|^2 \\ &= \frac{1}{2\lambda_0} \sum_{i=1}^{\infty} (\lambda_i - \lambda_0)^2 |a_i|^2 \\ &\geq \frac{1}{2\lambda_0} \left(1 - \frac{\lambda_0}{\lambda_1}\right)^2 \sum_{i=1}^{\infty} \lambda_i^2 |a_i|^2 . \end{aligned}$$

Thus if

$$H(\psi) - H(\phi_0) < b,$$

then

$$\sum_{i=1}^{\infty} \lambda_i^2 |a_i|^2 < \frac{2b\lambda_0}{\left(1 - \frac{\lambda_0}{\lambda_1}\right)^2}. \tag{37}$$

Let

$$C(\omega) = \int\int_{\Omega} |\omega|^2 \, dxdy,$$

then

$$\begin{aligned} C(\omega) - C(\omega_0) &= \int\int_{\Omega} |\omega - \omega_0|^2 + 2\operatorname{Re}(\omega - \omega_0)\,\omega_0 \qquad (38) \\ &= \lambda_0^2 \left(|a_0|^2 + 2\operatorname{Re} a_0\right) + \sum_{i=1}^{\infty} \lambda_i^2 |a_i|^2 . \end{aligned}$$

Notice that $H(\psi)$ and $C(\omega)$ are both invariants of (2).

Now

$$a_0(t) = \int\int_{\Omega} (\psi - \phi_0)\phi_0 dxdy$$

is a continuous function of t, so for $\varepsilon > 0$ small we can find some $d(\varepsilon) > 0$ such that

$$|a_0(0)| < \frac{\varepsilon}{\sqrt{2}\lambda_0}, \left| |a_0(t)|^2 + 2\operatorname{Re} a_0(t) \right| < d(\varepsilon) \Rightarrow |a_0(t)| < \frac{\varepsilon}{\sqrt{2}\lambda_0}. \tag{39}$$

Choose $\delta > 0$ such that $\|\omega(.,0) - \omega_0\|_{\mathbf{L}^2} < \delta$ to satisfy

$$|C(\omega(0)) - c(\omega_0)| < \frac{d(\varepsilon)\ \lambda_0^2}{2}$$

and

$$H(\psi(0)) - H(\phi_0) < \frac{1}{2\lambda_0} \left(1 - \frac{\lambda_0}{\lambda_1}\right)^2 \min\left(\frac{1}{2}\varepsilon^2, \frac{d(\varepsilon)\ \lambda_0^2}{2}\right).$$

Then from (37), (38) and (39) , we see that for all $t > 0$,

$$\|\omega(.,t) - \omega_0\|_{\mathbf{L}^2}^2 = \lambda_0^2 |a_0|^2 + \sum_{i=1}^{\infty} \lambda_i^2 |a_i|^2 < \varepsilon^2.$$

This finishes the stability proof. $\square$

By Theorem 1.1(ii), the steady flow with the stream function ϕ_0 is also linearly stable in the norm $\|\cdot\|_{\mathbf{X}}$. However, in the following we will show that for some domain Ω, this flow is linearly unstable in the $\mathbf{L}^2-$norm of the velocity (energy norm). Starting with Echhoff in the 1970s (see [3]), there are lots of papers using geometric optics or the WKB asymptotic method to treat the local instability in fluid dynamics. Typically, it allows one to estimate from below the growth rate of the solutions of the initial value problem for the linearized equation in terms of the growth rate of solutions of an ODE system. For Euler equation of 3D inviscid incompressible fluid, the ODE system (see e.g. [11]) is

$$\begin{cases} \dot{X} = -\mathbf{U}_0(X), \\ \dot{\mathbf{K}} = \left(\frac{\partial \mathbf{U}_0}{\partial X}\right)^T \mathbf{K}, \\ \dot{\mathbf{a}} = -\frac{\partial \mathbf{U}_0}{\partial X}\mathbf{a} + 2\frac{\partial \mathbf{U}_0}{\partial X}\mathbf{a} \cdot \mathbf{K}\frac{\mathbf{K}}{|\mathbf{K}|^2}, \end{cases}$$

with initial conditions at $t = 0$,

$$X = X_0, \mathbf{K} = \mathbf{K}_0, \mathbf{a} = \mathbf{a}_0,$$

where $\mathbf{K}_0 \cdot \mathbf{a}_0 = 0$. Here $\mathbf{U}_0(X)$ is the steady flow velocity field and the matrix $\partial \mathbf{U}_0/\partial X$ has components $\partial \mathbf{U}_{0i}/\partial X_j$, $i, j = 1, 2, 3$.

Theorem 4.2. *[11, 4]. If*

$$\sup_{\substack{X_0, \mathbf{K}_0, \mathbf{a}_0 \\ |\mathbf{K}_0| = |\mathbf{a}_0| = 1, \mathbf{K}_0 \cdot \mathbf{a}_0 = 0}} \overline{\lim_{t \to \infty}} |\mathbf{a}(t; X_0, \mathbf{K}_0, \mathbf{a}_0)| = \infty, \tag{40}$$

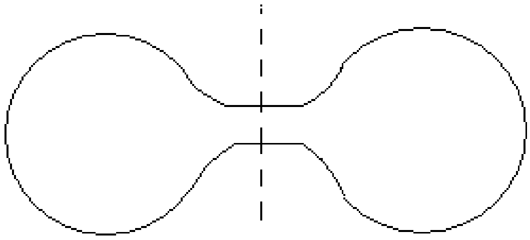
then the steady flow $\mathbf{U}_0(X)$ is linearly unstable in the sense that for suitable initial data, the $\mathbf{L}^2-$norm of the velocity of the corresponding solution of the linearized Euler equation is not bounded in time.

We said a point x_0 is a hyperbolic stagnation point of the flow $\mathbf{U}_0(X)$, if $\mathbf{U}_0(x_0) = 0$ and the matrix $\partial \mathbf{U}_0/\partial X$ has at least one positive real eigenvalue. It is shown by Friedlander and Vishik (see [5]) that

Lemma 4.1. *Let the 3-D flow $\frac{dX}{dt} = \mathbf{U}_0(X)$ have a hyperbolic stagnation point at some point x_0. Then $\mathbf{U}_0(X)$ is linearly unstable in the $\mathbf{L}^2-$norm of the velocity, as a steady flow of an ideal fluid.*

For the 2D case, let the corresponding stream function of $\mathbf{U}_0(X)$ be $\phi_0(X)$. Then for $\mathbf{U}_0(X)$ to have a hyperbolic stagnation point, it is equivalent that ϕ_0 has a saddle point. In the following, we construct a stable flow with a hyperbolic stagnation point.

Lemma 4.2. *There exists some domain Ω with smooth boundary such that the eigenfunction with the lowest eigenvalue of $-\Delta$ on $\mathbf{H}_0^1(\Omega)$ has a saddle point.*



Proof. First we consider a smooth domain Ω_0 composed of two circular disks of radius R smoothly connected by a thin channel of width 2ε (see the graph). We also make Ω_0 symmetric with respect to the middle vertical axis. Let ϕ_0 be the eigenfunction with the lowest eigenvalue λ_0 of $-\Delta$ on $\mathbf{H}_0^1(\Omega_0)$. From the standard theory, we know that ϕ_0 is positive and is symmetric with respect to the axis.

We shall prove the following estimate:

$$\sup_{\bar{\Omega}_0} |\nabla \phi_0| \leq \left(k_0 + \sqrt{k_0^2 + \lambda_0} \right) \sup_{\Omega_0} \phi_0. \tag{41}$$

Here k_0 is a positive number such that the curvature $k \geq -k_0$ at each point of $\partial\Omega_0$. The proof of (41) follows from an idea in [14], see also [16, Chapter 5]. Let $\tau = \sup_{\bar{\Omega}_0} |\nabla \phi_0|$, $M = \sup_{\Omega_0} \phi_0$ and $\alpha = 2k_0\tau$. Define

$$P = |\nabla \phi_0|^2 + \lambda_0^2 \phi_0^2 + \alpha \phi_0.$$

Then by a direct computation

$$\Delta P + \frac{L_i P_i}{|\nabla \phi_0|^2} = (\lambda_0 \phi_0 + \alpha)\, \alpha > 0,$$

here

$$L_i = -P_i - 2\partial_i \phi_0 \left(\lambda_0 \phi_0 + \alpha \right).$$

So by the Maximum principle, the maximum of P is either obtained on the boundary $\partial\Omega_0$ or at some point in Ω_0, where $\nabla \phi_0 = 0$. For the first case, supposing the maximum of P is obtained at $x_0 \in \partial\Omega_0$, we have

$$\frac{\partial P}{\partial n}|_{x_0} > 0 \tag{42}$$

by Hopf's principle. Here n is the outward normal direction. But (see [16, p. 76])

$$\begin{aligned} \frac{\partial P}{\partial n}|_{x_0} &= 2 \frac{\partial \phi_0}{\partial n} \frac{\partial^2 \phi_0}{\partial n^2} + \alpha \frac{\partial \phi_0}{\partial n} \\ &= -2\, |\nabla \phi_0| \left(\alpha + 2k\,(x_0)\, |\nabla \phi_0| \right). \end{aligned}$$

Since

$$\alpha + 2k\,(x_0)\, |\nabla \phi_0| \geq \alpha - 2k_0 \sup_{\bar{\Omega}_0} |\nabla \phi_0| \geq 0,$$

we get a contradiction to (42). Thus P can obtain the maximum only at a point in Ω_0 where $\nabla\phi_0 = 0$. So we get

$$\tau^2 \leq \lambda_0^2 M^2 + \alpha M = \lambda_0^2 M^2 + 2k_0 M\tau$$

from which (41) follows.

For any point D in the channel, the distance to the nearest boundary point is at most ε. So by the mean value theorem

$$\phi_0(D) \leq \varepsilon\tau \leq \varepsilon\left(k_0 + \sqrt{k_0^2 + \lambda_0}\right)\sup_{\Omega_0}\phi_0. \tag{43}$$

Since a disk $B_{R/2}$ with radius $\frac{R}{2}$ is contained in Ω_0, by the monotonicity of the eigenvalues of $-\Delta$ with respect to the domain, we have $\lambda_0 < \lambda\left(B_{R/2}\right)$ (the lowest eigenvalue of $-\Delta$ in $B_{R/2}$). We can make k_0 bounded (with a bound independent of ε) as long as the domain Ω_0 is smooth. So from (43), we can see that if ε is small then ϕ_0 can not obtain its maximum in the channel. But since ϕ_0 is symmetric with respect to the middle vertical axis, ϕ_0 must have at least two maximum points, one on each disk. We cannot ensure the two maximum points we get are non-degenerate. But we can deform Ω_0 to make them non-degenerate in the following way.

We quote a result of K. Uhlenbeck ([18]). First we introduce some notations as in [18]. Let N be a compact n-manifold with boundary which can be embedded in $\mathbf{R}^n$ and $B = Emb_k(N, \mathbf{R}^n)$ be the set of $\mathbf{C}^k$ embedding of N in $\mathbf{R}^n$. We associate with the embedding $F : N \to \mathbf{R}^n$ the Laplace operator on the image of F with Dirichlet boundary condition, which we denote by $\Delta_{\mathrm{Im}(F)}$. Consider the following properties of $\Delta_{\mathrm{Im}(F)}$:

A. One dimensional eigenspaces.

B. Zero is not a critical value of the eigenfunction restricted to the interior of the domain of the operator.

C. The eigenfunctions are Morse functions on the interior of the domain of the operator.

Theorem 9 in [18] is

Lemma 4.3. *Let $k > n + 2$. Then the set*

$$\left\{F \in Emb_k\left(N, \mathbf{R}^n\right) : \textit{properties A, B, and C hold for } \Delta_{\mathrm{Im}(F)}\right\}$$

is residual in $Emb_k(N, \mathbf{R}^n)$.

Using this result, we can deform Ω_0 slightly to get a new domain Ω. This domain Ω is still symmetric to its middle vertical axis with the curvature condition

$$k \geq -2k_0 \text{ on } \partial\Omega,$$

and the first eigenfunction ϕ of $-\Delta$ on Ω is a Morse function. Then by the same argument as above for the domain Ω_0, ϕ still has at least two maximum points. By the strong maximum principle, the normal derivative $\partial\phi/\partial n$ is negative everywhere on $\partial\Omega$. So the vector field $\mathbf{U}_0(x, y) = \left(-\partial_y\phi, \partial_x\phi\right)$ is always nonzero on $\partial\Omega$. It defines a non-degenerate vector field on Ω, tangential to $\partial\Omega$. Denote the number of equilibria of $\mathbf{U}_0$ with index $+1(-1)$ by n_{+1} (n_{-1}). We have

$$n_{+1} - n_{-1} = 1. \tag{44}$$

For the proof of (44), we introduce the double manifold $\tilde{\Omega}$ of Ω, which is obtained from Ω by attaching a second copy of Ω along $\partial\Omega$. By doing so we identify each point on $\partial\Omega$ with its copy in the boundary of the second copy. In this way we get a 2-dimensional manifold $\tilde{\Omega}$ without boundary, which is clearly diffeomorphic to $\mathbf{S}^2$. The vector field $\mathbf{U}_0$ is extended to $\tilde{\mathbf{U}}_0$ on $\tilde{\Omega}$ in the natural way. And we have that the number of equilibria of $\tilde{\mathbf{U}}_0$ with index $+1(-1)$ is $\tilde{n}_{+1}$ $(\tilde{n}_{-1}) = 2n_{+1}$ (n_{-1}). By Hopf's Theorem,

$$\tilde{n}_{+1} - \tilde{n}_{-1} = \chi\left(\tilde{\Omega}\right) = 2,$$

where $\chi\left(\tilde{\Omega}\right)$ is the Euler characteristic of $\tilde{\Omega}$. So (44) follows. Noticing that n_{+1} is the number of maximum and minimum points of ϕ on Ω which is at least 2 and n_{-1} is the number of saddle points of $\mathbf{U}_0$, we conclude that there exists some non-degenerate saddle point of ϕ. □

Combining the above results in this section, we get the following theorem.

Theorem 4.3. *Let ϕ be the eigenfunction with the lowest eigenvalue of $-\Delta$ on $\mathbf{H}_0^1(\Omega)$, where Ω is constructed in Lemma 4.2. Then the steady flow $\mathbf{U}_0(x, y) = \left(-\partial_y\phi, \partial_x\phi\right)$ is nonlinearly stable in the $\mathbf{L}^2-$norm of the vorticity, but linearly unstable in the $\mathbf{L}^2-$norm of the velocity.*

We note that it was proved in [10] that a steady flow with a saddle point is nonlinearly unstable in the $\mathbf{C}^{1,\alpha}$ norm of the velocity. However the nonlinearly stability or instability in the energy space is unknown.

5. Remarks on the Case of a Non-Simply Connected Domain

The results in Theorem 1.1, 1.2 can be generalized to the non-simply connected case. For this case, the boundary conditions for the vorticity equation is now (3a), (3b). So we have to change the function space for the stability and instability results. Define

$$\mathbf{X} := \left\{\psi \in \mathbf{H}^2(\Omega) \,|\, \psi = \Psi_i \text{ on } \Lambda_i\,,\ \oint_{\Lambda_i} \frac{\partial\psi}{\partial n} = 0 \text{ and } \int\int_\Omega \psi = 0\right\}, \tag{45}$$

$$\mathbf{Y} := \left\{\psi \in \mathbf{H}^1(\Omega) \,|\, \psi = \Psi_i \text{ on } \Lambda_i\,,\ \oint_{\Lambda_i} \frac{\partial\psi}{\partial n} = 0 \text{ and } \int\int_\Omega \psi = 0\right\}, \tag{46}$$

$$\mathbf{L}_0^2 := \left\{\psi \in \mathbf{L}^2(\Omega) \,|\, \int\int_\Omega \psi = 0\right\}$$

with

$$\|\psi\|_{\mathbf{X}} = \int\int_\Omega |\Delta\psi|^2\, dxdy,\ \|\psi\|_{\mathbf{Y}} = \int\int_\Omega |\nabla\psi|^2\, dxdy,\ \|\psi\|_{\mathbf{L}_0^2} = \int\int_\Omega |\psi|^2\, dxdy.$$

Here Ψ_i are unspecified constants. We define the functionals a, b on $\mathbf{Y}$ and the operators $A_\lambda, A_0 : \mathbf{X} \to \mathbf{L}_0^2$ in the same way as in the simply connected case. Then the conclusions of Theorem 1.1, 1.2 still hold true. The proofs are similar, so we skip them here. Now we explain why the spaces defined by (45) and (46) are natural for non-simply connected domains. The first condition ($\psi = \Psi_i$ on Λ_i) is the requirement of (3a). The zero integral condition is used to get rid of the arbitrary constant which might add to the stream function ψ. For the zero circulation condition, we recall that for the Euler equation the circulation is invariant, and this is also true for the linearized Euler equation. So for a growing mode $e^{\lambda t}\phi$ satisfying the linearized Euler equation, we must have

$$\oint_{\Lambda_i} \frac{\partial \phi}{\partial n} = 0.$$

So we only need consider function spaces defined by (45) and (46) when studying the existence of growing modes. For the stability study, we can decompose any function ψ satisfying the boundary conditions (3a),(3b) as $\psi = \psi' + \psi_0$, where ψ' is in $\mathbf{X}$ or $\mathbf{Y}$ and ψ_0 is a harmonic function with Λ_i- circulation

$$\oint_{\Lambda_i} \frac{\partial \psi}{\partial n}.$$

When ψ is the stream function for a solution of Euler equation, the circulation is fixed and thus ψ_0 is independent of time. We can use the energy-Casimir method as in [17] to control ψ' under the vorticity norm.

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