UNSTABLE SURFACE WAVES IN RUNNING WATER

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Abstract. We consider the stability of periodic gravity free-surface water waves traveling downstream at a constant speed over a shear flow of finite depth. In case the free surface is flat, a sharp criterion of linear instability is established for a general class of shear flows with inflection points and the maximal unstable wave number is found. Comparison to the rigid-wall setting testifies that free surface has a destabilizing effect. For a class of unstable shear flows, the bifurcation of nontrivial periodic traveling waves of small-amplitude is demonstrated at any wave number. We show the linear instability of small nontrivial waves bifurcated at an unstable wave number of the background shear flow. The proof uses a new formulation of the linearized water-wave problem and a perturbation argument. An example of the background shear flow of unstable small-amplitude periodic traveling waves is constructed for an arbitrary vorticity strength and for an arbitrary depth, illustrating that vorticity has a subtle influence on the stability of water waves.

1. Introduction

The water-wave problem in its simplest form concerns two-dimensional motion of an incompressible inviscid liquid with a free surface, acted on only by gravity. Suppose, for definiteness, that in the (x, y)-Cartesian coordinates gravity acts in the negative y-direction and that the liquid at time t occupies the region bounded from above by the free surface $y = \eta(t; x)$ and from below by the flat bottom $y = 0$. In the fluid region $\{(x, y) : 0 < y < \eta(t; x)\}$, the velocity field $(u(t; x, y), v(t; x, y))$ satisfies the incompressibility condition

$$\partial_x u + \partial_y v = 0 \tag{1.1}$$

and the Euler equation

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x P \\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y P - g, \end{cases} \tag{1.2}$$

where $P(t; x, y)$ is the pressure and $g > 0$ denotes the gravitational constant of acceleration. The flow is allowed to have rotational motions and characterized by the vorticity $\omega = v_x - u_y$. The kinematic and dynamic boundary conditions at the free surface $\{y = \eta(t; x)\}$

$$v = \partial_t \eta + u\partial_x \eta \quad \text{and} \quad P = P_{\text{atm}} \tag{1.3}$$

express, respectively, that the boundary moves with the velocity of the fluid particles at the boundary and that the pressure at the surface equals the constant atmospheric pressure P_{atm}. The impermeability condition at the flat bottom states that

$$v = 0 \qquad \text{at} \quad \{y = 0\}. \tag{1.4}$$

It is a matter of common experience that waves which may be observed on the surface of the sea or on the river are approximately periodic and propagating of permanent form at a constant speed. In case $\omega \equiv 0$, namely in the irrotational setting, waves of the kind are referred to as Stokes waves, whose mathematical treatment was initiated by formal but far-reaching considerations of Stokes [43] himself. The existence theory of Stokes waves dates back to the construction due to Levi-Civita [30] and Nekrasov [40] in the infinite-depth case and due to Struik [44] in the finite-depth case of small-amplitude waves, and it includes the global theory due to Krasovskii [29] and Keady and Norbury [28]. Stokes waves of greatest height exist [47], [37] and are shown to have stagnation at wave crests [4]. While the irrotationality assumption may serve as an approximation under certain circumstances and has been used in a majority of the existing research, surface water-waves typically carry vorticity, e.g. shear currents on a shallow channel and wind-drift boundary layers. Moreover, the governing equations for water waves allow for rotational steady motions. Gersner [21] early in 1802 found an explicit formula for a family of periodic traveling waves on deep water with a particular nonzero vorticity. An extensive existence theory of periodic traveling water waves with vorticity appeared in the construction due to Dubreil-Jacotin [20] of small-amplitude waves. Recently, for a general class of vorticity distributions, Constantin and Strauss [14] in the finite-depth case and Hur [26] in the infinite-depth case accomplished the bifurcation analysis for periodic traveling waves of large amplitude.

Waves of Stokes' kind is one of the few exact solutions of the free-surface water-wave problem, and as such it is important to understand the stability of these solutions. The present purpose is to investigate the linear instability of periodic gravity water-waves with vorticity.

The stability of water waves in case of zero vorticity has been under research much by means of numerical computations and formal analysis, especially in the works of Longuet-Higgins and his coworkers. Numerical studies of Stokes waves under perturbations of the same period, namely the superharmonic perturbations, indicate that [39], [45] instability sets in only when the wave amplitude is large enough to link with wave breaking [18], and thus small-amplitude Stokes waves are found to be linearly stable under the same-period perturbations. MacKay and Saffman [36] also considered linear stability of small-amplitude Stokes waves by means of general results of the Hamiltonian system. The Hamiltonian formulation in terms of the velocity potential, however, does not avail in the presence of effects of vorticity. The analysis of Benjamin and Feir [9] showed that there is a "sideband" instability for small Stokes waves, meaning that the perturbation has a different period than the steady wave. The Benjamin-Feir instability was made mathematically rigorous by Bridges and Mielke [10].

Analytical works on the stability of water waves with vorticity, on the other hand, are quite sparse. A recent contribution due to Constantin and Strauss [15] concerns two different kinds of formal stability of periodic traveling water-waves with vorticity under perturbations of the same period. First, in case when the vorticity decreases with depth an energy-Casimir functional $\mathcal{H}$ is constructed as a temporal invariant of the nonlinear water-wave problem, whose first variation gives the exact equations for steady waves. Its second variation is positive and the water-wave system is $\mathcal{H}$-formally stable under some special perturbations. Their second approach uses another functional $\mathcal{J}$, which is essentially the dual of $\mathcal{H}$ in the transformed variables but not an invariant. Its first variation gives the exact equations for steady waves in the transformed variables, which served as a basis in [14] of the existence theory for traveling waves. The $\mathcal{J}$-formal stability then means the positivity of its second variation. The "exchange of stability" theorem due to Crandall and Rabinowitz [16] applies to conclude that the $\mathcal{J}$-formal stability of the trivial solutions switches exactly at the bifurcation point and that steady waves along the curve of local bifurcation are $\mathcal{J}$-formally stable provided that both the depth and the vorticity strengh are sufficiently small.

The main results. As a preliminary step toward the stability and instability of nontrivial periodic waves, we examine the linear stability and instability of flat-surface shear flows. The stability of shear flows in the rigid-wall setting is a classical problem [19], whose theories date back to the necessary condition due to Rayleigh [42]. Recently, Lin [32] obtained instability criteria for several classes of shear flows in a channel with rigid walls, and our results generalize these to the free-surface setting. More specifically, our conclusions include: (1) The linear stability of shear flows with no inflection points (Theorem 6.4), which generalizes Rayleigh's criterion in the rigid-wall setting [42] to the free-surface setting; (2) A sharp criterion of linear instability for a class of shear flows with one inflection value (Theorem 4.2); and (3) A sufficient condition of linear instability for a class of shear flows with multiple inflection values (Theorem 6.1) including monotone flows. Our result testifies that free surface has a destabilizing effect compared to rigid walls.

Our next step is to understand the local bifurcation of small-amplitude periodic traveling waves in the physical space. While our setting is similar to that in [15] in that it hinges on the existence results of periodic waves in [14] via local bifurcation, the choice of bifurcation parameter and the dependence of other parameters on the bifurcation parameter and free parameters in the description of the background shear flow are different. In our setting, it is natural to consider that the shear profile and the channel depth are given and that the speed of wave propagation is chosen to ensure local bifurcation. The relative flux and the vorticity-stream function relation are then computed. In contrast, in the bifurcation analysis [14] in the transformed variables, the wave speed is given arbitrary and the relative flux and the vorticity-stream function relation are held fixed. In turn, the shear profile and the channel depth vary along the bifurcation curve. Lemma 2.3 establishes the equivalence between the bifurcation equation (2.7) in the transformed variables and the Rayleigh system (2.9)–(2.10) to obtain the bifurcation results for a large class of shear flows. In addition, our result helps to clarify that the nature of the local bifurcation of periodic traveling water-waves does not involve the exchange of stability of trivial solutions (Remark 4.14).

Our third step is to show under some technical assumptions that the linear instability of the background shear flow persists along the local curve of bifurcation of small-amplitude periodic traveling waves (Theorem 5.1). An example of such an unstable shear flow is

$$U(y) = a \sin b(y - h/2) \qquad \text{for} \quad y \in [0, h],$$

where $h, b > 0$ satisfy $hb \leqslant \pi$ and $a > 0$ is arbitrary (Remark 5.2). In particular, by choosing a and h to be arbitrarily small, we can construct linearly unstable small periodic traveling water-waves with an arbitrarily small vorticity strength and an arbitrarily small channel depth. This indicates that the formal stability of the second kind in [15] is quite different from the linear stability of the physical water wave problem. Our example also shows that adding an arbitrarily small vorticity to the water-wave system may affect the superharmonic stability of small-amplitude periodic irrotational waves in a water of arbitrary depth. Thus, it is important to take into account of the effects of vorticity in the study of the stability of water waves.

Temporal invariants of the linearized water-wave problem are derived and their implications for the stability of the water-wave system are discussed (Section 3.3). In case when the vorticity-stream function relation is monotone, the energy functional $\partial^2\mathcal{H}$ in [15] is indeed an invariant of the linearized water-wave problem. Other invariants are derived, and one may study the stability of the water-wave problem via the energy method. However, even with these additional invariants as constraints, the quadratic form $\partial^2\mathcal{H}$ is in general indefinite, indicating that that a steady (gravity) water-wave may be an energy saddle. A successful proof of the stability for the water-wave problem therefore would require to exhaust the full equations instead of using a few of its invariants.

Ideas of the proofs. Our approach in the proof of the linear instability of free-surface shear flows uses the Rayleigh system (4.1)–(4.2), which is related to that in the rigid-wall setting [32]. The main difference from [32] lies in the complicated boundary condition (4.2) on the free surface, which renders the analysis more involved. The instability property depends on the wave number, which is considered as a parameter. As in the rigid-wall setting [32], a key to a successful instability analysis is to locate the neutral limiting modes, which are neutrally stable solution of the Rayleigh system and contiguous to unstable modes. For certain classes of flows, neutral limiting modes in the free-surface setting are characterized by the inflection values. This together with the local bifurcation of unstable solutions from each neutral limiting wave number gives a complete knowledge on the instability at all wave numbers.

The instability analysis of small-amplitude nontrivial waves taken here is based on a new formulation which directly linearizes the Euler equation and the kinematic and dynamic boundary conditions on the free surface around a periodic traveling wave. Its growing-mode problem then is written as an operator equation for the stream function perturbation restricted on the steady free-surface. The mapping by the action-angle variables is employed to prove the continuity of the operator with respect to the amplitude parameter. In addition, in the action-angle variables, the equation of the particle trajectory becomes simple. The persistence of instability along the local curve of bifurcation is established by means of Steinberg's eigenvalue perturbation theorem [41] for operator equations. In addition, growing-mode solutions of the operator equation acquire regularity.

This paper is organized as follows. Section 2 is the discussion on the local bifurcation of periodic traveling water-waves when a background shear flow in the physical space is given. Section 3 includes the formulation of the linearized periodic water-wave problem and the derivation of its invarints. Section 4 is devoted to the linear instability of shear flows with one inflection value, and subsequently, Section 5 is to the linear instability of small-amplitude periodic waves over an unstable shear flow. Section 6 revisits the linear instability of shear flows in a more general class.

2. Existence of small-amplitude periodic traveling water-waves

We consider a traveling-wave solution of (1.1)–(1.4), that is, a solution for which the velocity field, the wave profile and the pressure have space-time dependence $(x-\underline{c}t, y)$, where $\underline{c} > 0$ is the speed of wave propagation. With respect to a frame of reference moving with the speed $\underline{c}$, the wave profile appears to be stationary and the flow is steady. The traveling-wave problem for (1.1)–(1.4) is further supplemented with the periodicity condition that the velocity field, the wave profile and the pressure are $2\pi/\alpha$-periodic in the x-variable, where $\alpha > 0$ is the wave number.

It is traditional in the traveling-wave problem to define the relative stream function $\psi(x, y)$:

$$\psi_x = -v, \qquad \psi_y = u - \underline{c} \tag{2.1}$$

and $\psi(0, \eta(0)) = 0$. This reduces the traveling-wave problem for (1.1)–(1.4) to a stationary elliptic boundary value problem [14, Section 2]:

For a real parameter B and a function $\gamma \in C^{1+\beta}([0, |p_0|])$, $\beta \in (0, 1)$, find $\eta(x)$ and $\psi(x, y)$ which are $2\pi/\alpha$-periodic in the x-variable, $\psi_y(x, y) < 0$ in $\{(x, y) : 0 < y < \eta(x)\}$[1], and

$$-\Delta\psi = \gamma(\psi) \quad \text{in} \quad 0 < y < \eta(x), \tag{2.2a}$$

$$\psi = 0 \quad \text{on} \quad y = \eta(x), \tag{2.2b}$$

$$|\nabla\psi|^2 + 2gy = B \quad \text{on} \quad y = \eta(x), \tag{2.2c}$$

$$\psi = -p_0 \quad \text{on} \quad y = 0, \tag{2.2d}$$

where

$$p_0 = \int_0^{\eta(x)} \psi_y(x, y)dy \tag{2.3}$$

is the relative total flux[2].

The vorticity function γ gives the vorticity-stream function relation, that is, $\omega = \gamma(\psi)$. The assumption of no stagnation, i.e. $\psi_y(x, y) < 0$ in the fluid region $\{(x, y) : 0 < y < \eta(x)\}$, guarantees that such a function is well-defined globally; See [14]. Furthermore, under this physically motivated stipulation, interchanging the roles of the y-coordinates and ψ offers an alternative formulation to (2.2) in a fixed strip, which serves as the basis of the existence theories in [20], [14], [26]. The nonlinear boundary condition (2.2c) at the free surface $y = \eta(x)$ expresses Bernoulli's law. The steady hydrostatic pressure in the fluid region is given by

$$P(x, y) = B - \tfrac{1}{2}|\nabla\psi(x, y)|^2 - gy - \int_0^{\psi(x,y)} \gamma(-p)dp. \tag{2.4}$$

In this setting, α and B are considered as parameters whose values form part of the solution. The wave number α in the existence theory is independent of other physical parameters and hence is held fixed, while in the stability analysis in Section 4 it serves as parameter. The Bernoulli constant B measures the total mechanical energy of the flow and varies along a solution branch.

[1] In other words, there is no stagnation in the fluid region. Field observations [31] as well as laboratory experiments [46] indicate that for wave patterns which are not near the spilling or breaking state, the speed of wave propagation is in general considerably larger than the horizontal velocity of any water particle.

[2] $p_0 < 0$ is independent of x.

2.1. **The local bifurcation theorem in** [14]**.** This subsection contains a summary of the existence result in [14] via the local bifurcation theorem of small-amplitude travelling-wave solutions to (2.2) provided that the total flux p_0 and the vorticity-stream function relation γ are given.

A preliminary result for local bifurcation is to find a curve of trivial solutions, which correspond to horizontal shear flows under a flat surface. As in [14, Section 3.1], let

$$\Gamma(p) = \int_0^p \gamma(-p')dp', \qquad \Gamma_{\min} = \min_{[p_0,0]} \Gamma(p) \leqslant 0.$$

Lemma 2.1 ([14], Lemma 3.2)**.** *Given $p_0 < 0$ and $\gamma \in C^{1+\beta}([0,|p_0|])$, $\beta \in (0,1)$, for each $\mu \in (-2\Gamma_{\min}, \infty)$ the system (2.2) has a solution*

$$y(p) = \int_{p_0}^p \frac{dp'}{\sqrt{\mu + 2\Gamma(p')}},$$

which corresponds to a parallel shear flow in the horizontal direction

$$u(t;x,y) = U(y;\mu) = \underline{c} - \sqrt{\mu + 2\Gamma(p(y))} \tag{2.5}$$

and $v(t;x,y) \equiv 0$ in the channel $\{(x,y) : 0 < y < h(\mu)\}$, where

$$h(\mu) = \int_{p_0}^0 \frac{dp}{\sqrt{\mu + 2\Gamma(p)}}.$$

The hydrostatic pressure is $P(y) = -gy$ for $y \in [0, h(\mu)]$. Here, $p(y)$ is the inverse of $y = y(p)$ and determines the stream function $\psi(y;\mu) = -p(y;\mu)$; $\underline{c} > 0$ is arbitrary.

In the statement of Theorem 2.2 below, instead of B the squared (relative) upstream flow speed $\mu = (U(h) - \underline{c})^2$ of a trivial shear flow (2.5) serves as a bifurcation parameter. For each $\mu \in (-2\Gamma_{\min}, \infty)$ the Bernoulli constant B is determined uniquely in terms of μ as

$$B = \mu + 2g \int_{p_0}^0 \frac{dp}{\sqrt{\mu + 2\Gamma(p)}}. \tag{2.6}$$

The following theorem [14, Theorem 3.1] states the existence result of a one-parameter curve of small-amplitude periodic water-waves for a general class of vorticities and their properties, in a form convenient for our purposes.

Theorem 2.2 (Existence of small-amplitude periodic water-waves)**.** *Let the speed of wave propagation $\underline{c} > 0$, the flux $p_0 < 0$, the vorticity function $\gamma \in C^{1+\beta}([0,|p_0|])$, $\beta \in (0,1)$, and the wave number $\alpha > 0$ be given such that the system*

$$\begin{cases} (a^3(\mu)M_p)_p = \alpha^2 a(\mu) M & \text{for} \quad p \in (p_0, 0) \\ \mu^{3/2} M_p(0) = gM(0) \\ M(p_0) = 0 \end{cases} \tag{2.7}$$

admits a nontrivial solution for some $\mu_0 \in (-2\Gamma_{\min}, \infty)$, where $a(\mu) = a(\mu;p) = \sqrt{\mu + 2\Gamma(p)}$.

Then, for $\epsilon \geqslant 0$ sufficiently small there exists a one-parameter curve of steady solution-pair μ_ϵ of (2.6) and $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ of (2.2) such that $\eta_\epsilon(x)$ and $\psi_\epsilon(x,y)$ are $2\pi/\alpha$-periodic in the x-variable, of $C^{3+\beta}$ class, where $\beta \in (0,1)$, and $\psi_{\epsilon y}(x,y) < 0$ throughout the fluid region.

At $\epsilon = 0$ the solution corresponds to a trivial shear flow under a flat surface:

(i0) *The flat surface is given by $\eta_0(x) \equiv h(\mu_0) =: h_0$ and the velocity field is*

$$(\psi_{0y}(x,y), -\psi_{0x}(x,y)) = (U(y) - \underline{c}, 0),$$

where $U(y)$ is determined in (2.5);

(ii0) *The pressure is given by the hydrostatic law $P_0(x,y) = -gy$ for $y \in [0, h_0]$.*

At each $\epsilon > 0$ the corresponding nontrivial solution enjoys the following properties:

(iϵ) *The bifurcation parameter has the asymptotic expansion*

$$\mu_\epsilon = \mu_0 + O(\epsilon) \qquad \text{as} \quad \epsilon \to 0$$

and the wave profile is given by

$$\eta_\epsilon(x) = h_\epsilon + \alpha^{-1}\delta_\gamma \epsilon \cos \alpha x + O(\epsilon^2) \qquad \text{as} \quad \epsilon \to 0,$$

where $h_\epsilon = h(\mu_\epsilon)$ is given in Lemma 2.1 and δ_γ depends only on γ and p_0; The mean height satisfies

$$h_\epsilon = h_0 + O(\epsilon) \qquad \text{as} \quad \epsilon \to 0;$$

Furthermore, the wave profile is of mean-zero; That is,

$$\int_0^{2\pi/\alpha} (\eta_\epsilon(x) - h_\epsilon) dx = 0; \tag{2.8}$$

(iiϵ) *The velocity field $(\psi_{\epsilon y}(x,y), -\psi_{\epsilon x}(x,y))$ in the steady fluid region $\{(x,y) : 0 < x < 2\pi/\alpha\, ,\, 0 < y < \eta_\epsilon(x)\}$ is given by*

$$\psi_{\epsilon x}(x,y) = \epsilon \psi_{*x}(y) \sin \alpha x + O(\epsilon^2),$$
$$\psi_{\epsilon y}(x,y) = U(y) - \underline{c} + \epsilon \psi_{*y}(y) \cos \alpha x + O(\epsilon^2)$$

as $\epsilon \to 0$, where ψ_{*x} and ψ_{*y} are determined from the linear theory;

(iiiϵ) *The hydrostatic pressure has the asymptotic expansion*

$$P_\epsilon(x,y) = -gy + O(\epsilon) \qquad \text{as} \quad \epsilon \to 0.$$

The condition that the system (2.7) admits a nontrivial solution for some $\mu_0 \in (-2\Gamma_{\min}, \infty)$ is necessary and sufficient for local bifurcation [14, Section 3]. A sufficient condition ([14]) for the solvability of (2.7) and therefore the local bifurcation is

$$\int_{p_0}^0 \left(\alpha^2 (p - p_0)^2 (2\Gamma(p) - 2\Gamma_{\min})^{1/2} + (2\Gamma(p) - 2\Gamma_{\min})^{3/2} \right) dp < g p_0^2,$$

which is satisfied when p_0 is sufficiently small.

2.2. **The bifurcation condition for shear flows.** Our instability analyses in Section 4 and Section 5 are carried out in the physical space, where a shear-flow profile and the depth are held fixed. The bifurcation analysis in the proof of Theorem 2.2, on the other hand, is carried out in the space of transformed variables, where the relative flux p_0 and the vorticity-stream function relation γ are held fixed, and the relative flow speed $U(h) - \underline{c}$ and the channel height h vary along the curve of local bifurcation. In this subsection, we undertake the study of the local bifurcation in the physical space, which is relevant to the future instability analyses. The natural choice for parameters is the speed of wave propagation $\underline{c} > \max U$ and the wave number k.

Our first task is to relate the bifurcation equation (2.7) in transformed variables with the Rayleigh system in the physical variables.

Lemma 2.3. *For the shear flow $U(y)$ for $y \in [0,h]$ defined via Lemma 2.1, the bifurcation equation (2.7) is equivalent to that the following Rayleigh equation*

$$(U-c)(\phi''-k^2)\phi - U''\phi = 0 \qquad \text{for} \quad y \in (0,h) \tag{2.9}$$

with the boundary conditions

$$\phi'(h) = \left(\frac{g}{(U(h)-c)^2} + \frac{U'(h)}{U(h)-c}\right)\phi(h) \quad \text{and} \quad \phi(0)=0, \tag{2.10}$$

where $(c,k) = (\underline{c},\alpha)$, $\underline{c} > \max U$, and $\phi(y) = (\underline{c} - U(y))M(p(y))$. Here and in the sequel, the prime denotes the differentiation in the y-variable.

Proof. Notice that $\underline{c} > \max U$. Indeed,

$$a(\mu;p) = \sqrt{\mu + 2\Gamma(p)} = -(U(y(p)) - \underline{c}) > 0.$$

Since $\frac{\partial p}{\partial y} = -\psi'(y) = -(U(y)-\underline{c})$, it follows that $\partial_p = \partial_y \frac{\partial y}{\partial p} = -\frac{1}{U-\underline{c}}\partial_y$. Let $M(p(y)) = \Phi(y)$, then (2.7) is written as

$$((U-\underline{c})^2\Phi')' - \alpha^2(U-\underline{c})^2\Phi = 0 \qquad \text{for} \quad y \in (0,h),$$
$$(U-\underline{c})^2\Phi'(h) = g\Phi(h) \quad \text{and} \quad \Phi(0)=0.$$

Let $\phi(y) = (\underline{c} - U(y))\Phi(y)$, and the above system becomes (2.9)–(2.10). □

Remark 2.4. We illustrate how to construct downstream-traveling periodic waves of small-amplitude bifurcating from a fixed background shear-flow $U(y)$ for $y \in [0,h]$. First, one finds the parameter values $(c,k) = (\underline{c},\alpha)$ with $\underline{c} > \max U$ and $\alpha > 0$ such that the Rayleigh system (2.9)– (2.10) admits a nontrivial solution. The wave speed then determines the bifurcation parameter as $\mu_0 = (U(h)-\underline{c})^2$. the flux and the vorticity function determined as

$$p_0 = \int_0^h (U(y)-\underline{c})dy \quad \text{and} \quad \gamma(p) = U'(y(p)), \tag{2.11}$$

respectively. In view of Lemma 2.3 the bifurcation equation (2.7) with μ_0, p_0 and γ are as above has a nontrivial solution. Moreover, each $U(h) - \underline{c}$ for $\underline{c} > \max U$ corresponds to a trivial solution in Lemma 2.1. Indeed, μ and $\underline{c}$ has a one-to-one correspondence $\mu = (U(h)-\underline{c})^2$; The (relative) stream function defined as

$$\psi(y) = -\int_y^h (U(h) - \underline{c})dy$$

is monotone and its inverse $y = y(-\psi)$ is well-defined. Then, Theorem 2.2 applies in the setting above to obtain the local curve of bifurcation of periodic waves.

The lemma below obtains for a large class of shear flows the local bifurcation by showing that the Rayleigh system (2.9)–(2.10) has a nontrivial solution.

Lemma 2.5. *If*

$$U \in C^2([0,h]), \quad U''(h) < 0 \quad \text{and} \quad U(h) > U(y) \quad \text{for} \quad y \neq h, \tag{2.12}$$

then for any wave number $k > 0$ there exists $c(k) > U(h) = \max U$ such that the system (2.9)–(2.10) has a nontrivial solution ϕ with $\phi > 0$ in $(0,h]$.

Proof. For $c \in (U(h), \infty)$ and $k > 0$ let ϕ_c be the solution of (2.9), or equivalently,

$$((U-c)\phi_c' - U'\phi_c)' - k^2(U-c)\phi_c = 0 \qquad \text{for} \quad y \in (0,h) \tag{2.13}$$

with $\phi_c(0) = 0$ and $\phi_c'(0) = 1$. Integrating the above equation on the interval $[0, h]$ yields that

$$(U(h)-c)\phi_c'(h) - (U(0)-c) - \phi_c(h)U'(h) - k^2 \int_0^h (U-c)\phi_c dy = 0.$$

Note that the bifurcation condition (2.10) is fulfilled if and only if the function

$$f(c) = c - U(0) + k^2 \int_0^h (c-U)\phi_c dy - \frac{g}{c-U(h)}\phi_c(h) \tag{2.14}$$

has a zero at some $c(k) > U(h)$. It is easy to see that f is a continuous function of c whenever $c > U(h)$.

We claim that $\phi_c(y) > 0$ for $y \in (0, h]$. Suppose, on the contrary, that $\phi_c(y_0) = 0$ for some $y_0 \in (0, h]$. Note that (2.9) is written as a Sturm-Liouville equation as

$$\phi_c'' - k^2\phi_c - \frac{U''}{U-c}\phi_c = 0. \tag{2.15}$$

Since

$$(c-U)'' + \frac{U''}{c-U}(c-U) = 0 \qquad \text{for} \quad y \in (0,h),$$

Sturm's comparison theorem applies to assert that the function $c - U(y)$ must have a zero on the interval $(0, y_0)$. A contradiction then proves the claim. Our goal is to show that $f(c) > 0$ for c large enough and $f(c) < 0$ as $c \to U(h)+$.

First, as $c \to \infty$ the sequence of a solution ϕ_c of (2.9), or equivalently (2.15) converges in C^2, that is $\phi_c \to \phi_\infty$. By continuity, ϕ_∞ satisfies the boundary value problem

$$\phi_\infty'' - k^2\phi_\infty = 0 \qquad \text{for} \quad y \in (0,h)$$

with $\phi_\infty(0) = 0$ and $\phi_\infty'(0) = 1$. Therefore, ϕ_∞ is bounded, continuous, and positive on $(0, h]$. By the definition (2.14) then it follows that $f(c) \to \infty$ as $c \to \infty$.

Next is to examine $f(c)$ as $c \to U(h)+$. Let us denote $\varepsilon = c - U(h) > 0$ be the small parameter. We claim that $\phi_c(h) \geqslant C_1 > 0$ for $\varepsilon > 0$ sufficiently small, where $C_1 > 0$ is independent of ε. To see this, it is convenient to write (2.13) as

$$((c-U)\phi_c' - (c-U)'\phi_c)' = k^2(c-U)\phi_c > 0 \qquad \text{for} \quad y \in (0,h),$$

whence

$$(c-U(y))\phi_c'(y) - (c-U(y))'\phi_c(y) > c - U(0) > 0 \qquad \text{for} \quad y \in (0,h).$$

Integrating the above yields that

$$\phi_c(h) > (c-U(h)) \int_0^h \frac{c-U(0)}{(\underline{c}-U(y))^2} dy.$$

This uses that $(c-U)\phi_c' - (c-U)'\phi_c = (c-U)^2 \left(\frac{\phi_c}{c-U}\right)'$. Our assumption on $U(y)$ asserts that $0 \leqslant U(h) - U(y) \leqslant \beta(h-y)$ for $y \in [h-\delta, h]$ for $\delta > 0$ small, where

$\beta > 0$ is a constant. Thus,

$$\phi_c(h) > \varepsilon(c-U(0))\int_{h-\delta}^{h}\frac{1}{(\varepsilon+\beta(h-y))^2}dy$$
$$=(c-U(0))\int_0^{(h-\delta)/\varepsilon}\frac{1}{(1+\beta x)^2}dx \geqslant C_1 > 0,$$

where $C_1 > 0$ is independent of $\varepsilon > 0$. This proves the claim. The treatment of the second term in the definition of (2.14) is divided into two cases.

First, in case $\max \phi_c(y) = \phi_c(h)$ it follows that

$$f(c) \leqslant c - U(0) + C_1\left(k^2\int_0^h (c-U)dy - \frac{g}{c-U(h)}\right) < 0$$

provided that $c - U(h) = \varepsilon > 0$ is small enough.

Next, in case $\max \phi_c = \phi_c(y_c)$, where $y_c \in (0,h)$, since $\phi_c''(y_c) \leqslant 0$ and $\phi_c(y_c) > 0$ by (2.15) it follows that $U''(y_c) > 0$. On the other hand, $U''(h) \leqslant 0$ and hence $y_c \in [0, h-\delta]$ for some $\delta > 0$. Note that $c - U(y)$ is bounded away from zero on the interval $y \in [0, h-\delta]$. Since the coefficients of (2.15) is uniformly bounded for c on $y \in [0, h-\delta]$, the solution ϕ_c is bounded on $y \in [0, h-\delta]$. In particular, $0 < \phi_c(y_c) \leqslant C_2$ independently for c. Therefore,

$$f(c) \leqslant c - U(0) + k^2 h C_2 \max_{[0,h]}(c-U(y)) - \frac{gC_1}{c-U(h)} < 0$$

for $\varepsilon = c - U(h)$ is small enough, where $C_1, C_2 > 0$ are independent of ε. This completes the proof. □

Lemma 2.3 and Remark 2.4 ensure the local bifurcation for a shear flow satisfying (2.12) at any wave number $k > 0$.

Theorem 2.6. *If $U \in C^2([0,h])$, $U''(h) < 0$ and $U(h) > U(y)$ for $y \neq h$ then for an arbitrary wavelength $2\pi/k$, where $k > 0$, there exist small-amplitude periodic waves bifurcating in the sense as in Theorem 2.2 from the flat-surface shear flow $U(y)$, where $c(k) > \max U$.*

In the irrotational setting, i.e. $U \equiv 0$, the parameter values c and k for which the Rayleigh system (2.9)– (2.10) is solvable give the dispersion relation ([17], for instance)

$$c^2 = \frac{g\tanh(kh)}{k}.$$

In case with a nonzero background shear flow, on the other hand, such an explicit algebraic relation is no longer available. Still, the Rayleigh system (2.9)–(2.10) may be considered to give a generalized dispersion relation. Moreover, the following quantitative information about $c(k)$ may be derived.

Lemma 2.7. *Given a shear flow $U(y)$ in $[0,h]$, let k and $c(k) > \max U$ be such that (2.9)–(2.10) has a nontrivial solution. Then,*

(a) $c(k)$ is bounded for $k > 0$;

(b) If $k_1 \neq k_2$ then $c(k_1) \neq c(k_2)$;

(c) In the long wave limit $k \to 0+$, the limit of the wave speed $c(0)$ satisfies Burns condition [12]

$$\int_0^h \frac{dy}{(U-c(0))^2} = \frac{1}{g}.$$

Proof. (a) The proof of Lemma 2.5 implies that for each $A > 0$ there exists C_A such that $f(c)$ defined in (2.14) is positive when $c > C_A$ and $k < A$. In interpretation, $c(k) \leqslant C_A$ for $k < A$. Thus, it suffices to show that $f(c) > 0$ when c and k are large enough. Indeed, let $c > \max U$ be large enough so that $\left|\frac{U''}{c-U}\right| \leqslant 1$ and let us denote by ϕ_1 and ϕ_2 the solutions of

$$\phi_1'' + (1-k^2)\phi_1 = 0 \quad \text{and} \quad \phi_2'' + (-1-k^2)\phi_2 = 0,$$

respectively, with $\phi_j(0) = 0$ and $\phi_j'(0) = 1$, where $j = 1, 2$. It is straightforward to see that

$$\phi_1(y) = \frac{1}{\sqrt{k^2-1}} \sinh(\sqrt{k^2-1})y \quad \text{and} \quad \phi_2(y) = \frac{1}{\sqrt{k^2+1}} \sinh(\sqrt{k^2+1})y.$$

Sturm's second comparison theorem [25] then asserts that the solution $\phi_{c,k}$ of (2.15) and ϕ_1 ϕ_2 satisfy that

$$\frac{\phi_1'}{\phi_1} \leqslant \frac{\phi_{c,k}'}{\phi_{c,k}} \leqslant \frac{\phi_2'}{\phi_2},$$

and thus $\phi_1 \leqslant \phi_{c,k} \leqslant \phi_2$. It is then easy to see that $f(c) > 0$ when k is big enough.

(b) Suppose on the contrary that $c(k_1) = c(k_2) = c$ for $k_1 < k_2$. Let us denote by ϕ_{c,k_1} and ϕ_{c,k_2} be the corresponding nontrivial solutions of (2.9)–(2.10). By Sturm's second comparison theorem [25] follows that

$$\frac{\phi_{c,k_1}'(h)}{\phi_{c,k_1}(h)} < \frac{\phi_{c,k_2}'(h)}{\phi_{c,k_2}(h)}.$$

This contradicts the boundary condition (2.10).

(c) The Rayleigh equation (2.13) for $k = 0$ implies that

$$(c - U(h))\phi_c'(h) + U'(h)\phi_c(h) = m,$$

where m is a constant. On the other hand, an integration of (2.13) yields that

$$m = \frac{g}{(c - U(h))}\phi_c(h),$$

and in turn

$$\phi_c(h) = (c - U(h)) \int_0^h \frac{m}{(c - U(y))^2} dy.$$

These proves the assertion. □

The limiting parameter value μ which corresponds to the limiting wave speed $c(0)$ gives the lowest hydraulic head B defined in (2.6); see [14, Section 3] for detail. The limiting wave speed $c(0)$ is the critical value of parameter near which solitary waves of elevation exist [27].

3. Linearization of the periodic gravity water-wave problem

This section includes the detailed account of the linearization of the water-wave problem (1.1)–(1.4) around a periodic traveling wave which solves (2.2). The growing-mode problem is formulated as a set of operator equations. Invariants of the linearized problem are derived, and their implications in the stability of water waves are discussed. Our derivations are free from restrictions on the amplitude of the steady solution.

3.1. **Derivation of the linearized problem of periodic water waves.** A periodic traveling-wave solution of (2.2) is held fixed, as such it serves as the undisturbed state about which the system (1.1)–(1.4) is linearized. The derivation is performed in the moving frame of references, in which the wave profile appears to be stationary and the flow is steady. Let us denote the undisturbed wave profile and (relative) stream function by $\eta_e(x)$ and $\psi_e(x,y)$, respectively, which satisfy the system (2.2). The steady (relative) velocity field $(u_e(x,y)-\underline{c}, v_e(x,y)) = (\psi_{ey}(x,y), -\psi_{ex}(x,y))$ is given by (2.1), and the hydrostatic pressure $P_e(x,y)$ is determined through (2.4). Let

$$\mathcal{D}_e = \{(x,y) : 0 < x < 2\pi/\alpha\,,\ 0 < y < \eta_e(x)\} \quad \text{and} \quad \mathcal{S}_e = \{(x,\eta_e(x)) : 0 < x < 2\pi/\alpha\}$$

denote, respectively, the undisturbed fluid domain of one period and the steady wave profile. The steady vorticity $\omega_e(x,y)$ may be expressed as $\omega_e = -\Delta\psi_e = \gamma(\psi_e)$.

The linearization concerns a slightly-perturbed time-dependent solution of the nonlinear problem (1.1)–(1.4) near the steady state $(\eta_e(x), \psi_e(x,y))$. Let us denote the small perturbation of the wave profile, the velocity field and the pressure by $\eta_e(x)+\eta(t;x)$, $(u_e(x,y)-\underline{c}+u(t;x,y), v_e(x,y)+v(t;x,y))$ and $P_e(x,y)+P(t;x,y)$, respectively. The idea is to expand the nonlinear equations in (1.1)–(1.4) around the steady state in the order of small perturbations and restrict the first-order terms to the unperturbed domain and the steady boundary to obtain linearized equations for the the deviations $\eta(t;x)$, $(u(t;x,y), v(t;x,y))$, and $P(t;x,y)$ in the wave profile, the velocity field and the pressure from those of the undisturbed state.

In the steady fluid domain $\mathcal{D}_e$, the velocity deviation (u,v) satisfies the incompressibility condition

$$\partial_x u + \partial_y v = 0 \tag{3.1}$$

and the linearized Euler equation

$$\begin{cases} \partial_t u + (u_e - \underline{c})\partial_x u + u_{ex}u + v_e\partial_y u + v_{ey}v = -\partial_x P \\ \partial_t v + (u_e - \underline{c})\partial_x v + v_{ex}u + v_e\partial_y v + v_{ey}v = -\partial_y P, \end{cases} \tag{3.2}$$

where P is the pressure deviation. Equation (3.1) allows us to introduce the stream function $\psi(t;x,y)$ for the velocity deviation $(u(t;x,y), v(t;x,y))$:

$$\partial_x\psi = -v \quad \text{and} \quad \partial_y\psi = u.$$

Let us denote by $\omega(t;x,y)$ the deviation in vorticity from that of the steady flow ω_e. By definition, $\omega = -\Delta\psi$. This motivates us to write (3.2) into the linearized vorticity equation as

$$\partial_t\omega + (\psi_{ey}\partial_x\omega - \psi_{ex}\partial_y\omega) + (\omega_{ex}\partial_y\psi - \omega_{ey}\partial_x\psi) = 0. \tag{3.3}$$

Since $\omega_e = \gamma(\psi_e)$, the last term can be written as $-\gamma'(\psi_e)(\psi_{ey}\partial_x\psi - \psi_{ex}\partial_y\psi)$.

The linearized kinematic and dynamic boundary conditions restricted to the steady free boundary $\mathcal{S}_e$ are

$$v + v_{ey}\eta = \partial_t\eta + (u_e - \underline{c})\partial_x\eta + (u + u_{ey}\eta)\eta_{ex} \tag{3.4}$$

and

$$P + P_{ey}\eta = 0,$$

respectively. In terms of the stream function, (3.4) is further written as

$$\begin{aligned} 0 &= \partial_t \eta + \psi_{ey}\partial_x \eta + (\partial_x \psi + \eta_{ex}\partial_y \psi) + (\psi_{exy} + \eta_{ex}\psi_{eyy})\eta \\ &= \partial_t \eta + \psi_{ey}\partial_x \eta + \partial_\tau \psi + \partial_\tau \psi_{ey}\eta \\ &= \partial_t \eta + \partial_\tau(\psi_{ey}\eta) + \partial_\tau \psi, \end{aligned}$$

where

$$\partial_\tau f = \partial_x f + \eta_{ex}\partial_y f$$

denotes the tangential derivative of a function f defined on the curve $\{y = \eta_e(x)\}$. Alternatively, $\partial_\tau f(x) = \partial_x f(x, \eta_e(x))$. The bottom boundary condition of the linearized motion is

$$\partial_x \psi = 0 \qquad \text{on} \quad \{y = 0\}.$$

Our next task is to examine the time-evolution of ψ on the steady free surface $\mathcal{S}_e$. This links the tangential derivative of the pressure deviation P on the steady free surface $\mathcal{S}_e$ with ψ and η on $\mathcal{S}_e$. For a function f defined on $\mathcal{S}_e$, let us denote by

$$\partial_n f = \partial_y f - \eta_{ex}\partial_x f$$

the normal derivative of f on the curve $\{y = \eta_e(x)\}$.

Lemma 3.1. *On the steady free surface $\mathcal{S}_e$, the normal derivative $\partial_n \psi$ satisfies*

$$\partial_t \partial_n \psi + \partial_\tau(\psi_{ey}\partial_n \psi) + \Omega \partial_\tau \psi + \partial_\tau P = 0,$$

where $\Omega = \omega_e(x, \eta_e(x))$ is the (constant) value of steady vorticity on $\mathcal{S}_e$.

Proof. The linearized Euler equation (3.2) may be rewritten in the vector form as

$$-\nabla P = \partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \nabla((u_e - \underline{c})u + v_e v) - \omega_e \begin{pmatrix} v \\ -u \end{pmatrix} - \omega \begin{pmatrix} v_e \\ -u_e \end{pmatrix}$$

Restricting the above to the steady free surface $\mathcal{S}_e$ and computing the tangential derivative yield that

$$\begin{aligned} -\partial_\tau P &= \partial_t(u + \eta_{ex}v) + \partial_\tau((u_e - \underline{c})u + v_e v) - \Omega(v - \eta_{ex}u) \\ &= \partial_t(\partial_y \psi - \eta_{ex}\partial_x \psi) + \partial_\tau(\psi_{ey}\partial_y \psi - \psi_{ey}\eta_{ex}\partial_x \psi) - \Omega(-\partial_x \psi - \eta_{ex}\partial_y \psi) \\ &= \partial_t \partial_n \psi + \partial_\tau(\psi_{ey}\partial_n \psi) + \Omega \partial_\tau \psi. \end{aligned}$$

The second equality uses the kinematic boundary condition where in the above derivation we use the steady kinematic equation for the steady state $\psi_{ex} = \psi_{ey}\eta_{ex}$ on $\mathcal{S}_e$. ☐

In summary, there results in the linearized water-wave problem:

(3.5a) $$\partial_t \omega + (\psi_{ey}\partial_x \omega - \psi_{ex}\partial_y \omega) = \gamma'(\psi_e)(\psi_{ey}\partial_x \psi - \psi_{ex}\partial_y \psi) \qquad \text{in} \quad \mathcal{D}_e,$$

where $\omega = -\Delta \psi$;

(3.5b) $$\partial_t \eta + \partial_\tau(\psi_{ey}\eta) + \partial_\tau \psi = 0 \qquad \text{on} \quad \mathcal{S}_e;$$

(3.5c) $$P + P_{ey}\eta = 0 \qquad \text{on} \quad \mathcal{S}_e;$$

(3.5d) $$\partial_t \partial_n \psi + \partial_\tau(\psi_{ey}\partial_n \psi) + \Omega \partial_\tau \psi + \partial_\tau P = 0 \qquad \text{on} \quad \mathcal{S}_e;$$

(3.5e) $$\partial_x \psi = 0 \qquad \text{on} \quad \{y = 0\}.$$

Note that the above linearized system may be viewed as one for $\psi(t; x, y)$ and $\eta(t; x)$. Indeed, $P(t; x, \eta_e(x))$ is determined through (3.5c) in terms of $\eta(t; x)$ and other physical quantities are similarly determined in terms of $\psi(t; x, y)$ and $\eta(t; x)$.

3.2. **The growing-mode problem.** A *growing mode* refers to a solution to the linearized water-wave problem (3.5) of the form

$$(\eta(t;x), \psi(t;x,y)) = (e^{\lambda t}\eta(x), e^{\lambda t}\psi(x,y))$$

and $P(t;x,\eta_e(x)) = e^{\lambda t}P(x,\eta_e(x))$ with Re $\lambda > 0$. For such a solution, the linearized vorticity equation (3.5a) further reduces to

$$\lambda\omega + (\psi_{ey}\partial_x\omega - \psi_{ex}\partial_y\omega) - \gamma'(\psi_e)(\psi_{ey}\partial_x\psi - \psi_{ex}\partial_y\psi) = 0 \quad \text{in } \mathcal{D}_e, \tag{3.6}$$

where $\omega = -\Delta\psi$. Let $(X_e(s;x,y), Y_e(s;x,y))$ be the particle trajectory of the steady flow

$$\begin{cases} \dot{X}_e = \psi_{ey}(X_e, Y_e) \\ \dot{Y}_e = -\psi_{ex}(X_e, Y_e) \end{cases} \tag{3.7}$$

with the initial position $(X_e(0), Y_e(0)) = (x,y)$. Here, the dot above a variable denotes the differentiation in the s-variable. Integration of (3.6) along the particle trajectory $(X_e(s;x,y), Y_e(s,x,y))$ for $s \in (-\infty, 0)$ yields [34, Lemma 3.1]

$$\Delta\psi + \gamma'(\psi_e)\psi - \gamma'(\psi_e)\int_{-\infty}^{0} \lambda e^{\lambda s}\psi(X_e(s), Y_e(s))ds = 0 \quad \text{in } \mathcal{D}_e. \tag{3.8a}$$

For a growing mode the boundary conditions (3.5b), (3.5c) and (3.5d) on $\mathcal{S}_e$ become

$$\lambda\eta(x) + \frac{d}{dx}\big(\psi_{ey}(x,\eta_e(x))\eta(x)\big) = -\frac{d}{dx}\psi(x,\eta_e(x)), \tag{3.8b}$$

$$P(x,\eta_e(x)) + P_{ey}(x,\eta_e(x))\eta(x) = 0, \tag{3.8c}$$

$$\lambda\psi_n(x) + \frac{d}{dx}\big(\psi_{ey}(x,\eta_e(x))\psi_n(x)\big) = -\frac{d}{dx}P(x,\eta_e(x)) - \Omega\frac{d}{dx}\psi(x,\eta_e(x)). \tag{3.8d}$$

The kinematic boundary condition (3.5e) at the flat bottom $\{y=0\}$ says that $\psi(x,0)$ is a constant. Observe that (3.8a)-(3.8d) remain unchanged by adding a constant on ψ, whereby the boundary condition for a growing mode at the bottom $\{y=0\}$ reduces to

$$\psi(x,0) = 0. \tag{3.8e}$$

In summary, the growing-mode problem for periodic traveling water-waves is to find a nontrivial solution of (3.8a)-(3.8e) with Re $\lambda > 0$.

3.3. **Invariants of the linearized water-wave problem.** In this subsection, the invariants of the linearized water-wave problem (3.5a)–(3.5e) are derived, and their implications in the stability of water waves are discussed.

With the introduction of the Poisson bracket, defined as

$$[f_1, f_2] = \partial_x f_1 \partial_y f_2 - \partial_y f_1 \partial_x f_2,$$

the linearized vorticity equation (3.5a) is further written as

$$\partial_t\omega - [\psi_e, \omega] + \gamma'(\psi_e)[\psi_e, \psi] = 0 \quad \text{in } \mathcal{D}_e, \tag{3.9}$$

where $\omega = -\Delta\psi$. Recall that

$$\partial_\tau f = \partial_x f + \eta_{ex}\partial_y f \quad \text{and} \quad \partial_n f = \partial_y f - \eta_{ex}\partial_x f,$$

where f is a function defined on $\mathcal{S}_e$.

Our first task is to find the energy functional for the linearized problem which, in case that the vorticity-stream function relation is monotone, is an invariant.

Lemma 3.2. *Provided that either $\gamma'(p) < 0$ or $\gamma'(p) > 0$ on $[0, |p_0|]$, then for any solution (η, ψ) of the linearized water-wave problem (3.5), we have $\frac{d}{dt}\mathcal{E}(\eta, \psi) = 0$, where*

$$
\begin{aligned}
\mathcal{E}(\eta,\psi) &= \iint_{\mathcal{D}_e} \tfrac{1}{2}|\nabla\psi|^2 dydx - \iint_{\mathcal{D}_e} \tfrac{1}{2}\gamma'(\psi_e)^{-1}|\omega|^2 dydx \\
&\quad - \int_{\mathcal{S}_e} \tfrac{1}{2}P_{ey}|\eta|^2 dx + \mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx - \Omega\int_{\mathcal{S}_e} \tfrac{1}{2}\psi_{ey}|\eta|^2 dx.
\end{aligned}
\tag{3.10}
$$

In the irrotational setting, i.e. $\gamma \equiv 0$, the invariant functional reduces to

$$
\mathcal{E}(\eta,\psi) = \iint_{\mathcal{D}_e} \tfrac{1}{2}|\nabla\psi|^2 dydx - \int_{\mathcal{S}_e} \tfrac{1}{2}P_{ey}|\eta|^2 dx + \mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx. \tag{3.11}
$$

Proof. In view of the divergence theorem it follows that

$$
\begin{aligned}
\frac{d}{dt}\iint_{\mathcal{D}_e} \tfrac{1}{2}|\nabla\psi|^2 dydx &= \mathrm{Re}\iint_{\mathcal{D}_e} \partial_t(\nabla\psi)\cdot\nabla\psi^* dydx \\
&= \mathrm{Re}\iint_{\mathcal{D}_e} \partial_t\omega\psi^* dydx + \int_{\mathcal{S}_e} \partial_t(\partial_n\psi)\psi^* dx + \int_{\{y=0\}} \partial_t\partial_y\psi\psi^* dx \\
&:= (I) + (II) + (III).
\end{aligned}
$$

A substitution of $\partial_t\omega$ by the linearized vorticity equation (3.9) yields that

$$
\begin{aligned}
(I) &= \mathrm{Re}\iint_{\mathcal{D}_e} \left(-\gamma'(\psi_e)[\psi_e,\psi] + [\psi_e,\omega]\right)\psi^* dydx \\
&= \mathrm{Re}\iint_{\mathcal{D}_e} \left(-\tfrac{1}{2}[\psi_e,\gamma'(\psi_e)|\psi|^2] + [\psi_e,\omega\psi^*] - [\psi_e,\psi^*]\omega\right) dxdy.
\end{aligned}
$$

The second equality in the above uses that

$$
\begin{aligned}
&\tfrac{1}{2}[\psi_e,\gamma'(\psi_e)|\psi|^2] \\
&\qquad = \tfrac{1}{2}[\psi_e,\gamma'(\psi_e)]|\psi|^2 + \tfrac{1}{2}\gamma'(\psi_e)([\psi_e,\psi]\psi^* + [\psi_e,\psi^*]\psi) \\
&\qquad = \gamma'(\psi_e)\,\mathrm{Re}[\psi_e,\psi]\psi^*,
\end{aligned}
$$

which follows since $[\psi_e, f(\psi_e)] = 0$ for any f. With the observation that

$$
\begin{aligned}
\iint_{\mathcal{D}_e} [\psi_e, f] dydx &= \iint_{\mathcal{D}_e} \nabla\cdot((\psi_{ey}, -\psi_{ex})f) dydx \\
&= \int_{\mathcal{S}_e} (\psi_{ey}, -\psi_{ex})\cdot(-\eta_{ex}, 1) f dx + \int_{y=0} f\psi_{ex} dx = 0
\end{aligned}
$$

for any function f, the integral (I) further reduces to

$$
(I) = \mathrm{Re}\iint_{\mathcal{D}_e} -[\psi_e,\psi]^*\omega dydx.
$$

A simple substitution of $[\psi_e, \psi]$ by the linearized vorticity equation (3.9) then yields that

$$
\begin{aligned}
(I) &= \mathrm{Re}\iint_{\mathcal{D}_e} \gamma'(\psi_e)^{-1}(\partial_t\omega^* - [\psi_e,\omega]^*)\omega dydx \\
&= \mathrm{Re}\iint_{\mathcal{D}_e} (\tfrac{1}{2}\gamma'(\psi_e)^{-1}\partial_t|\omega|^2 - \tfrac{1}{2}[\psi_e,\gamma'(\psi_e)^{-1}|\omega|^2]) dydx \\
&= \frac{d}{dt}\iint_{\mathcal{D}_e} \tfrac{1}{2}\gamma'(\psi_e)^{-1}|\omega|^2 dydx.
\end{aligned}
\tag{3.12}
$$

This uses that

$$\tfrac{1}{2}[\psi_e, \gamma'(\psi_e)^{-1}|\omega|^2] = \gamma'(\psi_e)^{-1}\,\mathrm{Re}[\psi_e,\omega]^*\omega.$$

Next is to examine the surface integral (II). Simple substitutions by the linearized boundary conditions (3.5b), (3.5c) and (3.5d) into (II) and an integration by parts yield that

$$\begin{aligned}(II) &= \mathrm{Re}\int_{\mathcal{S}_e} \partial_\tau(P_{ey}\eta - \psi_{ey}\partial_n\psi - \Omega\psi)\psi^* dx \\ &= \mathrm{Re}\int_{\mathcal{S}_e} (P_{ey}\eta - \psi_{ey}\partial_n\psi)(-\partial_\tau\psi^*)dx \\ &= \mathrm{Re}\int_{\mathcal{S}_e} (P_{ey}\eta - \psi_{ey}\partial_n\psi)(\partial_t\eta^* + \partial_\tau(\psi_{ey}\eta^*))dx \\ &= \frac{d}{dt}\int_{\mathcal{S}_e} \tfrac{1}{2}P_{ey}|\eta|^2 dx - \mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\partial_t\eta^* dx + \mathrm{Re}\int_{\mathcal{S}_e} \partial_\tau(P + \psi_{ey}\partial_n\psi)\psi_{ey}\eta^* dx.\end{aligned}$$

The second equality uses that

$$\mathrm{Re}\int_{\mathcal{S}_e} (\partial_\tau\psi)\psi^* dx = \frac{1}{2}\int_{\mathcal{S}_e} \frac{d}{dx}\left|\psi(x,\eta_e(x))\right|^2 dx = 0.$$

More generally, $\mathrm{Re}\int_{\mathcal{S}_e}(\partial_\tau f)f^* dx = 0$ for any function f defined on $\mathcal{S}_e$. With another simple substitution by the boundary condition (3.5d), the last term in the computation of (II) is written as

$$\begin{aligned}\mathrm{Re}&\int_{\mathcal{S}_e} \partial_\tau(P + \psi_{ey}\partial_n\psi)\psi_{ey}\eta^* dx \\ &= -\,\mathrm{Re}\int_{\mathcal{S}_e} (\partial_t\partial_n\psi + \Omega\partial_\tau\psi)\psi_{ey}\eta^* dx \\ &= -\,\mathrm{Re}\int_{\mathcal{S}_e} (\partial_t(\partial_n\psi)\psi_{ey}\eta^* dx + \Omega\,\mathrm{Re}\int_{\mathcal{S}_e} (\partial_t\eta + \partial_\tau(\psi_{ey}\eta))\psi_{ey}\eta^*)dx \\ &= -\,\mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_t(\partial_n\psi)\eta^* dx + \Omega\frac{d}{dt}\int_{\mathcal{S}_e} \tfrac{1}{2}\psi_{ey}|\eta|^2 dx.\end{aligned}$$

The last equality uses that $\mathrm{Re}\int_{\mathcal{S}_e}\partial_\tau(\psi_{ey}\eta)(\psi_{ey}\eta)^* dx = 0$. Therefore,

$$(II) = \frac{d}{dt}\int_{\mathcal{S}_e} \tfrac{1}{2}P_{ey}|\eta|^2 dx - \mathrm{Re}\,\frac{d}{dt}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx + \Omega\frac{d}{dt}\int_{\mathcal{S}_e} \tfrac{1}{2}\psi_{ey}|\eta|^2 dx. \tag{3.13}$$

Finally, it is straightforward to see that

$$(III) = \psi^*(x,0)\int_{\{y=0\}} \partial_t(\partial_y\psi)dx = 0.$$

This, together with (3.12) and (3.13) asserts that $\mathcal{E}(\eta,\psi)$ is an invariant. In the irrotational setting, i.e. $\gamma = 0$, the area integral (I) is zero, $\Omega = 0$, and the other terms remain the same. This completes the proof. □

Remark 3.3. Our energy functional $\mathcal{E}$ agrees with the second variation $\partial^2\mathcal{H}$ of the energy-Casimir functional in [15]. Recall that the hydrostatic pressure of the steady solution is given in (2.4) as

$$P_e(x,y) = B - \tfrac{1}{2}|\nabla\psi_e(x,y)|^2 - gy + \int_0^{\psi_e(x,y)} \gamma(-p)dp,$$

where B is the Bernoulli constant. Differentiation of the above and restriction on $\mathcal{S}_e$ then yield that

$$-P_{ey} - \Omega\psi_{ey} = \tfrac{1}{2}\partial_y|\nabla\psi_e|^2 + g,$$

where $\Omega = \gamma(\psi = 0) = \omega_e(x, \eta_e(x))$. Thus, in case γ is monotone, it follows that

$$\begin{aligned}2\mathcal{E}(\eta,\psi) = &\iint_{\mathcal{D}_e} |\nabla\psi|^2 dydx - \iint_{\mathcal{D}_e} \gamma'(\psi_e)^{-1}|\omega|^2 dydx \\ &+ \int_{\mathcal{S}_e} \left(\tfrac{1}{2}\partial_y|\nabla\psi_e|^2 + g\right)|\eta|^2 dx + 2\,\mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx.\end{aligned}$$

This is exactly the expression of the second variation $\partial^2\mathcal{H}$ in [15] of the enery-Casimir functional, which in our notations

$$\mathcal{H}(\eta,\psi) = \iint_{\mathcal{D}_\eta} \left(\frac{|\nabla(\psi - \underline{c}y)|^2}{2} + gy - B - F(\omega)\right) dydx, \tag{3.14}$$

around the steady state (η_e, ψ_e). Here, $\mathcal{D}_\eta = \{(x,y) : 0 < y < \eta(t;x)\}$ and $(F')^{-1} = \gamma$. The quadratic form $\partial^2\mathcal{H}$ is used in [15] to study formal stability.

Our next invariant is the linearized horizontal momentum. The result is free from restrictions on γ.

Lemma 3.4. *For any solution (η, ψ) of the linearized problem of* (3.5)*, the identity*

$$\frac{d}{dt}\int_{\mathcal{S}_e} (\psi + \psi_{ey}\eta)dx = 0$$

holds true.

Proof. We integrate over the steady fluid region $\mathcal{D}_e$ of the linearized equation for the horizontal velocity

$$\partial_t\partial_y\psi + (\psi_{ey}, -\psi_{ex}) \cdot \nabla(\partial_y\psi) + (\partial_y\psi, -\partial_x\psi) \cdot \nabla\psi_{ey} = -P_x$$

and apply the divergence theorem to arrive at

$$\frac{d}{dt}\iint_{\mathcal{D}_e} \partial_y\psi\, dydx + \int_{\mathcal{S}_e} (\partial_y\psi, -\partial_x\psi) \cdot (-\eta_{ex}, 1)\psi_{ey} dx = -\iint_{\mathcal{D}_e} P_x dydx. \tag{3.15}$$

It is straightforward to see that

$$\iint_{\mathcal{D}_e} \partial_y\psi dydx = \int_{\mathcal{S}_e} \psi\, dx.$$

In view of (3.5b), the second term on the left hand side of (3.15) is written as

$$\begin{aligned}\int_{\mathcal{S}_e} (\partial_\tau\psi)\psi_{ey} dx &= \int_{\mathcal{S}_e} (\partial_t\eta + \partial_\tau(\psi_{ey}\eta))\psi_{ey} dx \\ &= \frac{d}{dt}\int_{\mathcal{S}_e} \psi_{ey}\eta\, dx - \int_{\mathcal{S}_e} \psi_{ey}\partial_\tau(\psi_{ey})\eta dx.\end{aligned}$$

With the use of Stokes' theorem and the dynamic boundary condition (3.1) the right side of (3.15) becomes

$$\iint_{\mathcal{D}_e} P_x dydx = \int_{\mathcal{S}_e} P\eta_{ex} dx = \int_{\mathcal{S}_e} P_{ey}\eta\eta_{ex} dx.$$

On the other hand, the steady Euler equation restricted to the steady free-surface $\mathcal{S}_e$ yields that

$$\begin{aligned}-P_{ex} &= \psi_{ey}\psi_{exy} - \psi_{ex}\psi_{eyy}\\ &= \psi_{ey}(\psi_{exy} + \eta_{ex}\psi_{eyy}) = \psi_{ey}\partial_\tau\psi_{ey}\end{aligned}$$

Since $P_{ex} + P_{ey}\eta_{ex} = 0$ on $\mathcal{S}_e$, a simple substitution then proves the assertion. □

Next, integration of (3.5b) on $\mathcal{S}_e$ also yields that $\int_{\mathcal{S}_e} \eta\, dx$ is invariant. Finally, multiplication on the linearized vorticity equation (3.9) by ξ and then integration yield that

$$\iint_{\mathcal{D}_e} \omega\xi dydx$$

is an invariant, for any function

$$\xi \in \ker(\psi_{ey}\partial_x - \psi_{ex}\partial_y) \subset L^2(\mathcal{D}_e).$$

We summarize our results.

Proposition 3.5. *The linearized problem* (3.5) has the energy invariant:

$$\begin{aligned}\mathcal{E}(\eta,\psi) = &\iint_{\mathcal{D}_e} \tfrac{1}{2}|\nabla\psi|^2 dydx - \iint_{\mathcal{D}_e} \tfrac{1}{2}\gamma'(\psi_e)^{-1}|\omega|^2 dydx\\ &+ \int_{\mathcal{S}_e} \tfrac{1}{2}\left(\partial_y(\tfrac{1}{2}|\nabla\psi_e|^2) + g\right)|\eta|^2 dx + \mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx\end{aligned}$$

if γ is monotone and

$$\begin{aligned}\mathcal{E}(\eta,\psi) = &\iint_{\mathcal{D}_e} \tfrac{1}{2}|\nabla\psi|^2 dydx\\ &+ \int_{\mathcal{S}_e} \tfrac{1}{2}\left(\partial_y(\tfrac{1}{2}|\nabla\psi_e|^2) + g\right)|\eta|^2 dx + \mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx\end{aligned}$$

in case $\gamma \equiv 0$. In addition, (3.5) has the following invariants:

$$\begin{aligned}\mathcal{M}(\eta,\psi) &= \int_{\mathcal{S}_e} (\psi + \psi_{ey}\eta)dx,\\ m(\eta,\psi) &= \int_{\mathcal{S}_e} \eta dx\\ \mathcal{F}(\eta,\psi) &= \iint_{\mathcal{D}_e} \omega\xi dydx\end{aligned}$$

for any $\xi \in \ker(\psi_{ey}\partial_x - \psi_{ex}\partial_y)$.

The nonlinear water-wave problem (1.1)–(1.4) has the following conservation laws [15, Section 2]: Let

$$\mathcal{D}(t) = \{(x,y) : 0 < x < 2\pi/\alpha\,,\, 0 < y < \eta(t;x)\}$$

be the fluid domain at time t of one wave length, and

$$\begin{aligned}
\mathfrak{E} &= \iint_{\mathcal{D}(t)} \left(\tfrac{1}{2}(u^2+v^2)+gy\right) dydx && \text{(energy)},\\
\mathfrak{M} &= \iint_{\mathcal{D}(t)} u\, dydx && \text{(horizontal momentum)},\\
\mathfrak{m} &= \iint_{\mathcal{D}(t)} dydx && \text{(mass)},\\
\mathfrak{F} &= \iint_{\mathcal{D}(t)} f(\omega) dydx && \text{(Casimir invariant)},
\end{aligned}$$

where the function f is arbitrary such that the integral $\mathfrak{F}$ exists.

The invariants of the linearized problem $\mathcal{E}$, $\mathcal{M}$, m, and $\mathcal{F}$ in Proposition 3.5 can be obtained by expanding the invariants of the nonlinear problem $\mathfrak{E}$, $\mathfrak{M}$, $\mathfrak{m}$ and $\mathfrak{F}$, respectively, around the steady wave $(\eta_e(x), \psi_e(x,y))$. The quadratic form $\mathcal{E}(\eta,\psi)$ is the second variation of the energy functional $\mathcal{H}$ given in (3.14) as is shown in Remark 3.3, which is a combination of $\mathfrak{E}$, $\mathfrak{M}$, $\mathfrak{m}$ and $\mathfrak{F}$. The invariants $\mathcal{M}$ and m of the linear problem are the first variations of $\mathfrak{M}$ and $\mathfrak{m}$, respectively. Finally, $\mathcal{F}$ is the first variation of $\mathfrak{F}$ sinceby the assumption of no stagnation and the monotonicity assumption it follows that

$$\ker(\psi_{ey}\partial_x - \psi_{ex}\partial_y) = \{f(\psi_e) : f \text{ is arbitrary}\} = \{f(\omega_e) : f \text{ is arbitrary}\}.$$

We now make some comments on the implications of these invariants on the stability of water waves. A traditional approach to the stability of conservative systems is the so-called energy method, for which one tries to show that a steady state is an energy minimizer under the constraints of other invariants such as momentum, mass, etc. This method has been widely used in the stability analysis of various approximate equations such as the KdV equation [5], [8] and the water-wave problem with a nonzero coefficient of surface tension [38]. Nonetheless, a steady solution of the fully-nonlinear gravity water-wave problem in general is not expect to be an energy minimizer, as is discussed below. Note that if a steady gravity water-wave is an energy minimizer under constraints of fixed $\mathfrak{M}$, $\mathfrak{m}$ and $\mathfrak{F}$, then in the linearized level the second variation $\mathcal{E}$ should be positive under the constraints that the variations $\mathcal{M}$, m and $\mathcal{F}$ are zero.

In the irrotational setting, the first two terms in the expression (3.11) of $\mathcal{E}(\eta,\psi)$ yield some positive contribution since they are equivalent to

$$\iint_{\mathcal{D}_e} |\nabla\psi|^2 dydx + \int_{\mathcal{S}_e} |\eta|^2 dx. \tag{3.16}$$

(Indeed, since $\Delta P_e = -2\psi_{exy}^2 - \psi_{exx}^2 - \psi_{eyy}^2 \leqslant 0$ and $P_{ey}(x,0) = -g$ by the maximum principle P_e attains its minimum at the free surface $\mathcal{S}_e$. Furthermore, since P_e takes a constant on $\mathcal{S}_e$ by the Hopf lemma $P_{ey} < 0$ on $\mathcal{S}_e$.) The last term $\mathrm{Re}\int_{\mathcal{S}_e} \psi_{ey}\partial_n\psi\eta^* dx$ of the right side of (3.11), however, does not have a definite sign and contains a 1/2-higher derivative than that of (3.16). Thus, it cannot be bounded by elements in the positive contribution (3.16), regardless of the constraints $\mathcal{M} = m = 0$. Consequently, the quadratic form $\mathcal{E}(\eta,\psi)$ is indefinite unless $\psi_{ey} \equiv 0$, that is, the steady flow is uniform and the steady surface is flat. In other words, a steady gravity water-wave in the irrotational setting is in general not an energy minimizer.

Indeed, in [23], [11], steady water-waves were constructed as an energy saddle by means of variational methods.

With a nonzero vorticity, a control of the mixed-type term in the energy functional by other positive terms fails for the same reason as in the irrotational setting, even for a vorticity which would amplify positivity property. Let us consider a vorticity function which decreases with the flux, that is, $\gamma' < 0$. Under some additional assumptions, it is shown in [15] that the first three terms in the expression (3.10) of $\mathcal{E}(\eta, \psi)$ give rise a positive norm

$$\iint_{\mathcal{D}_e} (|\nabla \psi|^2 + |\omega|^2) dy dx + \int_{\mathcal{S}_e} |\eta|^2 dx. \tag{3.17}$$

A successful bound of the mixed-type term Re $\int_{\mathcal{S}_e} \psi_{ey} \partial_n \psi \eta^* dx$ by terms in (3.17) would entail a control of ψ on $\mathcal{S}_e$ in terms of positive contribution in (3.17), which does not seem to follow from the consideration of elliptic regularity. In [15], several classes of perturbations were introduced to make the mixed-type term controllable by the elements in (3.17) and thus to ensure the positivity of the energy $\mathcal{E}(\eta, \psi)$ in these classes; The formal stability of the first kind in this regard corresponds to the positivity of $\mathcal{E}(\eta, \psi)$. However, it is difficult to show that these special classes are invariant under the evolution process of the water-wave problem, and it remains unclear how to pass from the formal stability of the first kind to the genuine stability even under some special perturbations. The lack of a control of the mixed-type term in the energy expression by other positive terms is not amendable by taking the other invariants $\mathcal{M}, m$, and $\mathcal{F}$ into consideration as constraints while one may relax the assumptions on the vorticity to obtain the positivity of (3.17). In conclusion, the quadratic form $\mathcal{E}(\eta, \psi)$ is in general indefinite in the rotational setting, and rotational steady water-wave is again expected to be an energy saddle.

The above discussions of steady water-waves as energy saddles do not imply that steady water-waves are necessarily unstable. Indeed, as mentioned before, the small Stokes waves are believed to be stable [39], [45] under perturbations of the same period. For the rotational case, under the assumption of a monotone γ, the corresponding trivial solutions with shear flows defined in Lemma 2.1 can not have an inflection point since $\gamma'(\psi_e(y)) = -U''(y)/U(y) \neq 0$ and the no-stagnation assumption ensures that $U < 0$. Thus by Theorem 6.4, such shear flows are linearly stable to perturbations of any period. Since small-amplitude waves with any monotone vorticity relation γ bifurcate from these strongly stable shear flows, they are likely to be stable. But, a successful stability analysis of general steady gravity water-waves would require to use the full set of equations instead of a few of invariants.

4. Linear instability of shear flows with free surface

This section is devoted to the study of the linear instability of a free-surface shear flow $(U(y), 0)$ in $y \in [0, h]$, a steady solution of the water-wave problem (1.1)–(1.4) with $P(x, y) = -gy$. In the frame of reference moving with the speed $\underline{c} > \max U$, this may be recognized as a trivial solution of the traveling-wave problem (2.2):

$$\eta_e(x) \equiv h \quad \text{and} \quad (\psi_{ey}(x, y), -\psi_{ex}(x, y)) = (U(y) - \underline{c}, 0).$$

Throughout this section, we write U for $U - \underline{c}$ for simplicity of the presentation.

We seek for normal mode solutions of the form

$$\eta(t;x) = \eta_h e^{i\alpha(x-ct)}, \qquad \psi(t;x,y) = \phi(y)e^{i\alpha(x-ct)}$$

and $P(t;x,y) = P(y)e^{i\alpha(x-ct)}$ to the linearized water-wave problem (3.5). Here, $\alpha > 0$ is the wave number and c is the complex phase speed. It is equivalent to find solutions to the growing-mode problem (3.8) of the form $\lambda = -i\alpha c$ and

$$\eta(x) = \eta_h e^{i\alpha x}, \qquad \psi(x,y) = \phi(y)e^{i\alpha x}$$

and $P(x, \eta_e(x)) = P_h e^{i\alpha x}$. Note that $\operatorname{Re}\lambda > 0$ if and only if $\operatorname{Im} c > 0$.

Since $\gamma'(\psi_e(y)) = \omega_{ey}(y)/\psi_{ey}(y) = -U''(y)/U(y)$ and $(X_e(s), Y_e(s)) = (x + U(y)s, y)$, the linearized vorticity equation (3.8a) translates into the Rayleigh equation

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0 \quad \text{for} \quad y \in (0,h). \tag{4.1}$$

Here and elsewhere the prime denote the differentiation in the y-variable. The boundary conditions (3.8b), (3.8c) and (3.8d) on the free surface are simplified to be

$$(c - U(h))\eta_h = \phi(h), \qquad P_h - g\eta_h = 0$$

and

$$(c - U(h))\phi'(h) = P_h + U'(h)\phi(h),$$

respectively. Eliminating η_h and P_h from the above yields that

$$(U(h) - c)^2\phi'(h) = (g + U'(h)(U(h) - c))\,\phi(h).$$

The bottom boundary condition (3.8e) becomes $\phi(0) = 0$. In summary, there results in the Rayleigh equation (4.1) with the boundary condition

$$\begin{cases} (U(h) - c)^2\phi'(h) = (g + U'(h)(U(h) - c))\,\phi(h) \\ \phi(0) = 0. \end{cases} \tag{4.2}$$

A shear profile U is said to be *linearly unstable* if there exists a nontrivial solution of (4.1)-(4.2) with $\operatorname{Im} c > 0$.

The Rayleigh system (4.1)–(4.2) is alternatively derived in [48] by linearizing directly the water-wave problem (1.1)–(1.4) around $(U(y), 0)$ in $y \in [0,h]$. Note that with $c > \max U$ the Rayleigh system (4.1)–(4.2) is the bifurcation equation (2.9)–(2.10).

In case of rigid walls at $y = h$ and $y = 0$, that is the Dirichlet boundary conditions $\phi(h) = 0 = \phi(0)$ in place of (4.2), the instability of a shear flow is a classical problem, which has been under extensive research since Lord Rayleigh [42]. Recently, by a novel analysis of neutral modes, Lin [32] established a sharp criterion for linear instability in the rigid-wall setting of a general class of shear flows. Our objective in this section is to obtain an analogous result in the free-surface setting.

Below is the definition of the class of shear flows studied in this section. By an inflection value we mean the value of U at an inflection point.

Definition 4.1. *A function $U \in C^2([0,h])$ is said to be in the class $\mathcal{K}^+$ if U has a unique inflection value U_s and*

$$K(y) = -\frac{U''(y)}{U(y) - U_s} \tag{4.3}$$

is bounded and positive on $[0,h]$.

Typical example of $\mathcal{K}^+$-flows include $U(y) = \cos my$ and $U(y) = \sin my$.

For $U \in \mathcal{K}^+$ let us consider the Sturm-Liouville equation

$$\phi'' - \alpha^2\phi + K(y)\phi = 0 \qquad \text{for} \quad y \in (0,h) \tag{4.4}$$

with the boundary conditions

$$\begin{cases} \phi'(h) = g_r(U_s)\phi(h) & \text{if} \quad U(h) \neq U_s \\ \phi(h) = 0 & \text{if} \quad U(h) = U_s, \end{cases} \tag{4.5}$$

$$\phi(0) = 0. \tag{4.6}$$

Here,

$$g_r(c) = \frac{g}{(U(h)-c)^2} + \frac{U'(h)}{U(h)-c}. \tag{4.7}$$

The following theorem states a sharp instability criterion of free-surface gravity shear-flows in class $\mathcal{K}^+$.

Theorem 4.2 (Linear instability of free-surface shear flows in $\mathcal{K}^+$)**.** *For $U \in \mathcal{K}^+$, let the lowest eigenvalue $-\alpha_{\max}^2$ of the ordinary differential operator $-\frac{d^2}{dy^2} - K(y)$ on the interval $y \in (0,h)$ with the boundary conditions (4.5)–(4.6) exists and be negative. Then, to each $\alpha \in (0, \alpha_{\max})$ corresponds an unstable solution-triple (ϕ, α, c) (with $\operatorname{Im} c > 0$) of (4.1)–(4.2). The interval of unstable wave numbers $(0, \alpha_{\max})$ is maximal in the sense that the flow is linearly stable if either $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0,h)$ with (4.5)–(4.6) is nonnegative or $\alpha \geqslant \alpha_{\max}$.*

From a variational consideration, the lowest eigenvalue $-\alpha_{\max}^2$ is characterized as

$$-\alpha_{\max}^2 = \inf_{\substack{\phi \in H^1(0,h) \\ \phi(0)=0}} \frac{\int_0^h (|\phi'|^2 - K(y)|\phi|^2)dy + g_r(U_s)|\phi(h)|^2}{\int_0^h |\phi|^2 dy} \tag{4.8a}$$

in case $U(h) \neq U_s$, and

$$-\alpha_{\max}^2 = \inf_{\substack{\phi \in H^1(0,h) \\ \phi(0)=0=\phi(h)}} \frac{\int_0^h (|\phi'|^2 - K(y)|\phi|^2)dy}{\int_0^h |\phi|^2 dy} \tag{4.8b}$$

in case $U(h) = U_s$.

4.1. **Neutral limiting modes.** The proof of Theorem 4.2 makes use of *neutral limiting modes*, as in the rigid-wall setting [32].

Definition 4.3 (Neutral limiting modes)**.** *A triple (ϕ_s, α_s, c_s) with α_s positive and c_s real is called a neutral limiting mode if it is the limit of a sequence of unstable solutions $\{(\phi_k, \alpha_k, c_k)\}_{k=1}^\infty$ of (4.1)–(4.2) as $k \to \infty$. The convergence of ϕ_k to ϕ_s is in the almost-everywhere sense. For a neutral limiting mode, α_s is called the neutral limiting wave number and c_s is called the neutral limiting wave speed.*

Lemma 4.8 below will establish that neutral limiting wave numbers form the boundary points of the interval of unstable wave numbers, and thus the stability investigation of a shear flow reduces to finding all neutral limiting modes and studying stability properties near them. In general, it is difficult to locate all neutral limiting modes. For flows in class $\mathcal{K}^+$, nonetheless, neutral limiting modes are characterized by the inflection value.

Proposition 4.4 (Characterization of neutral limiting modes). *For $U \in \mathcal{K}^+$ a neutral limiting mode (ϕ_s, α_s, c_s) must solve (4.4)–(4.6) with $c_s = U_s$.*

For the proof of Proposition 4.4, we need several properties of unstable solutions. First, Howard's semicircle theorem holds true in the free-surface setting [48, Theorem 1]. That is, any unstable eigenvalue $c = c_r + ic_i$ ($c_r > 0$) of the Rayleigh equation (4.1)-(4.2) must lie in the semicircle

$$(c_r - \tfrac{1}{2}(U_{\min} + U_{\max}))^2 + c_i^2 \leqslant (\tfrac{1}{2}(U_{\min} - U_{\max}))^2, \tag{4.9}$$

where $U_{\min} = \min_{[0,h]} U(y)$ and $U_{\max} = \max_{[0,h]} U(y)$.

The identities below are useful for future considerations.

Lemma 4.5. *If ϕ is a solution (4.1)–(4.2) with $c = c_r + ic_i$ and $c_i \neq 0$ then for any q real the identities*

(4.10)
$$\int_0^h \left(|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''(U-q)}{|U-c|^2}|\phi|^2 \right) dy = \left(\operatorname{Re} g_r(c) + \frac{c_r - q}{c_i} \operatorname{Im} g_r(c) \right) |\phi(h)|^2,$$

(4.11)
$$\int_0^h \left(|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''(U-q)}{|U-c|^2}|\phi|^2 \right) dy = \left(\operatorname{Re} g_s(c) - \frac{c_r - q}{c_i} \operatorname{Im} g_s(c) \right) |\phi'(h)|^2$$

hold true, where g_r is defined in (4.7) and

$$g_s(c) = \frac{(U(h) - c)^2}{g + U'(h)(U(h) - c)}. \tag{4.12}$$

Proof. Multiplication of (4.1) by ϕ^* and integration by parts using the boundary condition (4.2) yield

$$\int_0^h \left(|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''}{U-c}|\phi|^2 \right) dy = g_r(c)|\phi(h)|^2.$$

Its real and imaginary parts read as

$$\int_0^d \left(|\phi'|^2 + \alpha^2 |\phi|^2 + \frac{U''(U - c_r)}{|U-c|^2} |\phi|^2 \right) dy = \operatorname{Re} g_r(c) |\phi(d)|^2 \tag{4.13}$$

and

$$c_i \int_0^h \frac{U''}{|U-c|^2}|\phi|^2 dy = \operatorname{Im} g_r(c)|\phi(h)|^2, \tag{4.14}$$

respectively. Combining (4.13) and (4.14) then establishes (4.10).

Similarly, combining the real and the imaginary parts of

$$\int_0^h \left(|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''}{U-c}|\phi|^2 \right) dy = g_s(c)|\phi'(h)|^2 \tag{4.15}$$

leads to (4.11). This completes the proof. □

Note that (4.14) is written as

$$c_i \int_0^h \frac{U''}{|U-c|^2}|\phi|^2 dy = \left(\frac{2g(U(h) - c_r)}{|U(h) - c|^4} + \frac{U'(h)}{|U(h) - c|^2} \right) |\phi(h)|^2,$$

and hence, if U'' does not change sign and U is monotone then the flow is linearly stable [48, Section 5]. Proposition 6.4 will prove in the free-surface setting linear stability of general class of shear flows with no inflection points (and U need not be monotone). The proof uses the characterization of neutral limiting modes.

Our next preliminary result is an a priori H^2-estimate of unstable solutions near a neutral limiting mode.

Lemma 4.6. *For $U \in \mathcal{K}^+$ let $\{(\phi_k, \alpha_k, c_k)\}_{k=1}^{\infty}$ be a sequence of unstable solutions to (4.1)–(4.2) such that $\|\phi_k\|_{L^2} = 1$. If $\alpha_k \to \alpha_s > 0$ and $\operatorname{Im} c_k \to 0+$ as $k \to \infty$ then $\|\phi_k\|_{H^2} \leqslant C$, where $C > 0$ is independent of k.*

Proof. By the semicircle theorem (4.9), it follows that $c_k \to c_s$ and $c_s \in [U_{\min}, U_{\max}]$ as $k \to \infty$. The proof is divided into to cases when $U(h) \neq c_s$ and when $U(h) = c_s$.

Case 1: $U(h) \neq c_s$. The proof is similar to that of [32, Lemma 3.7]. It is straightforward to see that

$$|\operatorname{Re} g_r(c_k)| + \left| \frac{\operatorname{Im} g_r(c_k)}{\operatorname{Im} c_k} \right| \leqslant C_0,$$

where $C_0 > 0$ is independent of k. For simplicity, the subscript k is suppressed in the estimate below and C is used to denote generic constants which are independent of k. Let $c = c_r + ic_i$. Let us write (4.10) as

$$\int_0^h \left(|\phi'|^2 + \alpha^2 |\phi|^2 + K(y) \frac{(U - U_s)(U - q)}{|U - c_r|^2 + c_i^2} |\phi|^2 \right) dy = \left(\operatorname{Re} g_r(c) + \frac{c_r - q}{c_i} \operatorname{Im} g_r(c) \right) |\phi(h)|^2 .$$

Note that $g_r(c_s)$ is well defined. Setting $q = U_s - 2\,(U_s - c_r)$ the above identity leads to

$$\begin{aligned}
\int_0^h (|\phi'|^2 + \alpha^2 |\phi|^2) dy &\leqslant \int_0^h K(y) \frac{(U - U_s)^2 + 2(U - U_s)(U_s - c_r)}{|U - c_r|^2 + c_i^2} |\phi|^2 dy + C|\phi(h)|^2 \\
&= \int_0^h K(y) \frac{(U - c_r)^2 - (U_s - c_r)^2}{|U - c_r|^2 + c_i^2} |\phi|^2 dy + C|\phi(h)|^2 \\
&\leqslant \int_0^h K(y) |\phi|^2 dy + C(\varepsilon \|\phi'\|_{L^2}^2 + \frac{1}{\varepsilon} \|\phi\|_{L^2}^2).
\end{aligned}$$

One chooses ε sufficiently small to conclude that $\|\phi\|_{H^1} \leqslant C$.

Next is an H^2-estimate. Multiplication of (4.1) by $(\phi^*)''$ and integration over the interval $[0, h]$ yield

$$\int_0^h (|\phi''|^2 + \alpha^2 |\phi'|^2) dy - \alpha^2 g_r^*(c) |\phi(h)|^2 = \int_0^h (\phi^*)'' \frac{U''}{U - c} \phi \, dy. \tag{4.16}$$

In view of the Rayleigh equation for ϕ^*, the right side is written as

$$\begin{aligned}
\int_0^h (\phi^*)'' \frac{U''}{U - c} \phi \, dy &= \int_0^h \left(\alpha^2 \phi^* + \left(\frac{U''}{U - c} \phi \right)^* \right) \frac{U''}{U - c} \phi \, dy \\
&= \alpha^2 \int_0^h \frac{U''}{U - c} |\phi|^2 dy + \int_0^h \frac{(U'')^2}{|U - c|^2} |\phi|^2 dy.
\end{aligned}$$

The real part of (4.16) then reads as

$$\begin{aligned}
\int_0^h (|\phi''|^2 + \alpha^2 |\phi'|^2) dy &= \alpha^2 \operatorname{Re} g_r(c) |\phi(h)|^2 + \alpha^2 \int_0^h \frac{U''(U - c_r)}{|U - c|^2} |\phi|^2 dy + \int_0^h \frac{(U'')^2}{|U - c|^2} |\phi|^2 dy \\
&= \alpha^2 \left(2 \operatorname{Re} g_r(c) |\phi(h)|^2 - \int_0^h (|\phi'|^2 + \alpha^2 |\phi|^2) dy \right) + \int_0^h \frac{(U'')^2}{|U - c|^2} |\phi|^2 dy.
\end{aligned}$$

The last equality uses (4.13). On the other hand, (4.10) with $q = U_s$ implies

$$\begin{aligned}\int_0^h \frac{(U'')^2}{|U-c|^2}|\phi|^2 dy &\leqslant \|K\|_{L^\infty} \int_0^h \frac{-U''(U-U_s)}{|U-c|^2}|\phi|^2 dy \\ &= \|K\|_{L^\infty}\left(\int_0^h (|\phi'|^2+\alpha^2|\phi|^2)dy - \left(\operatorname{Re} g_r(c) + \frac{c_r-U_s}{c_i}\operatorname{Im} g_r(c)\right)|\phi(h)|^2\right) \\ &\leqslant C\,\|\phi\|_{H^1}^2 .\end{aligned}$$

Since $|\operatorname{Re}\, g_r(c_k)||\phi(h)|^2 \leqslant C_0\|\phi\|_{H^1}^2$, a simple substitution results in

$$\int_0^h (|\phi''|^2+2\alpha^2|\phi'|^2+\alpha^4|\phi|^2)dy = -2\alpha^2 \operatorname{Re} g_r(c)|\phi(h)|^2 + \int_0^h \frac{(U'')^2}{|U-c|^2}|\phi|^2 dy \leqslant C\,\|\phi\|_{H^1}^2 .$$

Therefore, $\|\phi\|_{H^2} \leqslant C$ as desired.

Case 2: $U(h) = c_s$. The proof is identical to that of Case 1 except that one uses $\phi(h) = g_s(c)\phi'(h)$ in place of $\phi'(h) = g_r(c)\phi(h)$. It is straightforward to see that

$$|\operatorname{Re} g_s(c_k)| + \left|\frac{\operatorname{Im} g_s(c_k)}{\operatorname{Im} c_k}\right| \leqslant C_0\, d(c_k, U(h)),$$

where $C_0 > 0$ is independent of k and d is defined as

$$d(c_k, U(h)) = |\operatorname{Re} c_k - U(h)| + (\operatorname{Im} c_k)^2 .$$

Since $U(h) = c_s$ follows that $d(c_k, U(h)) \to 0$ as $k \to \infty$. The same computations as in Case 1 establish

$$\int_0^h (|\phi_k'|^2+\alpha^2|\phi_k|^2)dy \leqslant \int_0^h K(y)|\phi_k|^2 dy + C_0 d(c_k, U(h))\,|\phi_k'(h)|^2$$

and

(4.17)

$$\begin{aligned}\int_0^h (|\phi_k''|^2+2\alpha^2|\phi_k'|^2+\alpha^4|\phi_k|^2)dy &= -2\alpha^2 \operatorname{Re}\, g_s(c)|\phi_k'(h)|^2 + \int_0^h \frac{(U'')^2}{|U-c|^2}|\phi_k|^2 dy \\ &\leqslant C\left(\int_0^h (|\phi_k'|^2+\alpha^2|\phi_k|^2)dy + d(c_k, U(h))\,|\phi_k'(h)|^2\right),\end{aligned}$$

where $C > 0$ is independent of k. Consequently,

$$\|\phi_k\|_{H^2}^2 \leqslant C_1(\|\phi_k\|_{L^2}^2 + d(c_k, U(h))\,|\phi_k'(h)|^2) \leqslant C_2(\|\phi_k\|_{L^2}^2 + d(c_k, U(h))\|\phi_k\|_{H^2}^2),$$

where $C_1, C_2 > 0$ are independent of k. Since $d(c_k, U(h)) \to 0$ as $k \to \infty$ it follows that $\|\phi_k\|_{H^2}^2 \leqslant C$. This completes the proof. □

For $U \in \mathcal{K}^+$, if c is in the range of U then $U(y) = c$ holds at a finite number of points [32, Remark 3.2], denoted by $y_1 < y_2 < \cdots < y_{m_c}$. For convenience, set $y_0 = 0$ and $y_{m_c+1} = h$. We state our last preliminary result.

Lemma 4.7 ([32], Lemma 3.5)**.** *Let ϕ satisfy (4.1) with α positive and c in the range of U and let $U(y) = c$ for $y \in \{y_1, y_2, \ldots, y_{m_c}\}$. If ϕ is sectionally continuous on the open intervals (y_j, y_{j+1}), $j = 0, 1, \ldots, m_c$, then ϕ cannot vanish at both endpoints of any intervals (y_j, y_{j+1}) unless it vanishes identically on that interval.*

Proof of Proposition 4.4. Let (ϕ_s, α_s, c_s) be a neutral limiting mode with $\alpha_s > 0$ and $c_s \in [U_{\min}, U_{\max}]$, and let $\{(\phi_k, \alpha_k, c_k)\}_{k=1}^{\infty}$ be a sequence of unstable solutions of (4.1)–(4.2) such that (ϕ_k, α_k, c_k) converges to (ϕ_s, α_s, c_s) as $k \to \infty$. After normalization, we may assume that $\|\phi_k\|_{L^2} = 1$.

First, the result of Lemma 4.6 says that $\|\phi_k\|_{H^2} \leqslant C$, where $C > 0$ is independent of k. Consequently, ϕ_k converges to ϕ_s weakly in H^2 and strongly in H^1. Then $\|\phi_s\|_{H^2} \leqslant C$ and $\|\phi_s\|_{L^2} = 1$. Let $y_1, y_2, \cdots, y_{m_s}$ be the roots of $U(y) - c_s$ and let S_0 be the complement of the set of points $\{y_1, y_2, \ldots, y_{m_s}\}$ in the interval $[0, h]$. Since ϕ_k converges to ϕ_s uniformly in C^2 on any compact subset of S_0, it follows that ϕ_s'' exists on S_0. Since $(U - c_k)^{-1}$, ϕ_k and their derivatives up to second order are uniformly bounded on any compact subset of S_0, it follows that ϕ_s satisfies

$$\phi_s'' - \alpha_s^2 \phi_s - \frac{U''}{U - c_s}\phi_s = 0 \tag{4.18}$$

almost everywhere on $[0, h]$. Moreover,

$$\phi_s'(h) = \left(\frac{g}{(U(h) - c_s)^2} + \frac{U'(h)}{U(h) - c_s}\right)\phi_s(h) \quad \text{and} \quad \phi_s(0) = 0 \tag{4.19}$$

in case $U(h) \neq c_s$, and

$$\phi_s(h) = 0 \quad \text{and} \quad \phi_s(0) = 0 \tag{4.20}$$

in case $U(h) = c_s$.

Next, we claim that c_s is the inflection value U_s. By Definition 4.1, $U''(y_j) = -K(y_j)(c_s - U_s)$ has the same sign for $j = 1, \cdots, m_s$, say positive. Let

$$E_\delta = \cup_{i=1}^{m_s}\{y \in [0, h] : |y - y_j| < \delta\}.$$

Clearly, $E_\delta^c \subset S_0$ and $U''(y) > 0$ for $y \in E_\delta$ when $\delta > 0$ sufficiently small. The proof is again divided into two cases.

Case 1: $U(h) \neq c_s$. For any q real

$$\int_0^h \left(|\phi_k'|^2 + \alpha_k^2|\phi_k|^2 + \frac{U''(U-q)}{|U - c_k|^2}|\phi_k|^2\right) dy \geqslant \alpha_k^2 + \int_{E_\delta} \frac{U''(U-q)}{|U-c_k|^2}|\phi_k|^2 dy - \sup_{E_\delta^c} \frac{|U''(U-q)|}{|U-c_k|^2}.$$

Since ϕ_s is not identically zero, Lemma 4.7 asserts that $\phi_s(y_j) \neq 0$ for some $y_j \in E_\delta$. If c_s were not an inflection value then near such a y_j it must hold that

$$\int_{E_\delta} \frac{U''(U - U_{\min} + 1)}{|U - c_s|^2}|\phi_s|^2 dy \geqslant \int_{|y - y_j| < \delta} \frac{U''}{|U - c_s|^2}|\phi_s|^2 dy = \infty.$$

Subsequently, by Fatou's Lemma it follows that

$$\liminf_{k\to\infty} \int_0^h \left(|\phi_k'|^2 + \alpha_k^2|\phi_k|^2 + \frac{U''(U - U_{\min} + 1)}{|U - c_k|^2}|\phi_k|^2\right) dy = \infty.$$

On the other hand, (4.10) with $q = U_{\min} - 1$ yields

$$\int_0^h \left(|\phi_k''|^2 + \alpha_k^2|\phi_k|^2 + \frac{U''(U - U_{\min} + 1)}{|U - c_k|^2}|\phi_k|^2\right) dy$$
$$= \left(\operatorname{Re} g_r(c_k) + \frac{\operatorname{Re} c_k - U_{\min} + 1}{\operatorname{Im} c_k}\right)|\phi_k(h)|^2 \leqslant C\|\phi_k\|_{H^2} \leqslant C_1,$$

where $C_1 > 0$ is independent of k. A contradiction proves the claim.

Case 2: $U(h) = c_s$. The proof is identical to that of Case 1 except that we use (4.11) in place of (4.10) and hence is omitted. This completes the proof. □

The following lemma permits us to continue unstable wave numbers until it reaches a neutral limiting wave number, analogous to [32, Theorem 3.9].

Lemma 4.8. *For $U \in \mathcal{K}^+$, the set of unstable wave numbers is open, whose boundary point α satisfies that $-\alpha^2$ is an eigenvalue of the operator $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0, h)$ with the boundary conditions (4.5)–(4.6).*

Proof. An unstable solution (ϕ_0, α_0, c_0) of (4.1)–(4.2) is held fixed, and such $\alpha_0 > 0$ and $\operatorname{Im} c_0 > 0$. For $r_1, r_2 > 0$ let us define

$$I_{r_1} = \{\alpha > 0 : |\alpha - \alpha_0| < r_1\} \quad \text{and} \quad B_{r_2} = \{c \in \mathbb{C}^+ : |c - c_0| < r_2\},$$

where $\mathbb{C}^+$ denotes the set of complex numbers with positive imaginary part. Our goal is to show that for each $\alpha \in I_{r_1}$, there exists $c(\alpha) \in B_{r_2}$ for some $r_2 > 0$ and $\phi_\alpha \in H^2$ such that $(\phi_\alpha, \alpha, c(\alpha))$ satisfies (4.1)–(4.2).

For $\alpha \in (0, \infty)$ and $c \in \mathbb{C}^+$, let $\phi_1(x; \alpha, c)$ and $\phi_2(x; \alpha, c)$ to be the solutions of the differential equation

$$\phi'' - \alpha^2 \phi - \frac{U''}{U - c}\phi = 0 \qquad \text{for} \quad y \in (0, h)$$

normalized at h, that is,

$$\begin{cases} \phi_1(h) = 1, & \phi_2(h) = 0, \\ \phi_1'(h) = 0, & \phi_2'(h) = 1. \end{cases}$$

It is standard that ϕ_1 and ϕ_2 are analytic as a function of c in $\mathbb{C}^+$. Let us consider a function on $(0, \infty) \times \mathbb{C}^+$, defined as

$$\Phi(\alpha, c) = \phi_1(0; \alpha, c) + g_r(c)\phi_2(0; \alpha, c), \tag{4.21}$$

where g_r is defined in (4.7). Clearly, Φ is analytic in c and continuous in α. Note that $\Phi(\alpha, c) = 0$ if and only if the system (4.1)–(4.2) has an unstable solution

$$\phi_\alpha(y) = \phi_1(y; \alpha, c) + g_r(c)\phi_2(y; \alpha, c).$$

Since $\Phi(\alpha_0, c_0) = 0$ and the zeros of an analytic function are isolated, $\Phi(\alpha_0, c) \neq 0$ on the line $\{|c - c_0| = r_2\}$, where $r_2 > 0$ is sufficiently small. By the continuity of $\Phi(\alpha, c)$ in α, subsequently, $\Phi(\alpha, c) \neq 0$ for $\alpha \in I_{r_1}$ and $\{|c - c_0| = r_2\}$, where $r_1, r_2 > 0$ are sufficiently small. Let us consider the function

$$N(\alpha) = \frac{1}{2\pi i} \oint_{|c - c_0| = r_2} \frac{\partial \Phi / \partial c(\alpha, c)}{\Phi(\alpha, c)} dc,$$

where $\alpha \in I_{r_1}$. Observe that $N(\alpha)$ counts the number of zeros of $\Phi(\alpha, c)$ in B_{r_2}. Since $N(\alpha_0) > 0$ and $N(\alpha)$ is continuous as a function of α in I_{r_1} it follows that $N(\alpha) > 0$ for any $\alpha \in I_{r_1}$. This proves that the set of unstable wave numbers is open.

By definition, the boundary points of the set of unstable wave numbers must be neutral limiting wave numbers, say α_s. Proposition 4.4 asserts that $-\alpha_s^2$ must be a negative eigenvalue of the operator $-\frac{d^2}{dy^2} - K(y)$ on $(0, h)$ with (4.5)–(4.6). This completes the proof. □

4.2. **Proof of Theorem 4.2.** The proof of Theorem 4.2 is to examine the bifurcation of unstable modes from each neutral limiting mode. This reduces to the study of the bifurcation of zeros of the algebraic equation $\Phi(\alpha, c) = 0$ from (α_s, c_s) as is pointed out in the proof of Lemma 4.8. However, the differentiability of $\Phi(\alpha, c)$ in c at a neutral limiting mode (α_s, c_s) can only be established in the direction of positive imaginary part, and thus the standard implicit function theorem does not apply. To overcome this difficulty, we construct a contraction mapping in such a half neighborhood and prove the existence of an unstable mode near (α_s, c_s) as is done in [32] in the rigid-wall setting. Our proof, in addition, shows that the existence of an unstable mode can only be established for wave numbers slightly to the left of α_s. Therefore, unstable modes to the left of α_s continue to the zero wave number. Our method is similar to that in the rigid-wall setting in [32, Section 4].

Below we construct unstable solutions for wave numbers slightly to the left of a neutral limiting wave number.

Proposition 4.9. *For $U \in \mathcal{K}^+$, let U_s be the inflection value and $y_1, y_2, \dots, y_{m_s}$ be the inflection points, as such $U(y_j) = U_s$ for $i = 1, 2, \dots, m_s$. If (ϕ_s, α_s, U_s) is a neutral limiting mode with α_s positive, then for $\varepsilon \in (\varepsilon_0, 0)$, where $|\varepsilon_0|$ is sufficiently small, there exist ϕ_ε and $c(\varepsilon)$ such that the nontrivial function ϕ_ε solves*

$$\phi_\varepsilon'' - \left(\alpha_s^2 + \varepsilon\right)\phi_\varepsilon - \frac{U''}{U - U_s - c(\varepsilon)}\phi_\varepsilon = 0 \qquad \textit{for} \quad y \in (0, h), \tag{4.22}$$

$$\phi_\varepsilon'(h) = g_r(U_s + c(\varepsilon))\phi_\varepsilon(h) \quad \textit{and} \quad \phi_\varepsilon(0) = 0 \tag{4.23}$$

with $\operatorname{Im} c(\varepsilon) > 0$. *Moreover,*

$$\lim_{\varepsilon \to 0-} c(\varepsilon) = 0, \tag{4.24}$$

(4.25)

$$\lim_{\varepsilon \to 0-} \frac{dc}{d\varepsilon}(\varepsilon) = \left(\int_0^h \phi_s^2 dy\right)\left(i\pi \sum_{j=1}^{m_s} \frac{K(y_j)}{|U(y_j)|}\phi_s^2(y_j) + p.v.\int_0^h \frac{K}{U - U_s}\phi_s^2 dy + A\right)^{-1};$$

here p.v. means the Cauchy principal part and

$$A = \begin{cases} \frac{2g}{(U(h)-U_s)^3} + \frac{U'(h)}{(U(h)-U_s)^2} & \textit{if} \quad U(h) \neq U_s \\ 0 & \textit{if} \quad U(h) = U_s. \end{cases} \tag{4.26}$$

Proof. As in the proof of Lemma 4.8, for $c \in \mathbb{C}^+$ and $\varepsilon < 0$, let $\phi_1(y; \varepsilon, c)$ and $\phi_2(y; \varepsilon, c)$ be the solutions of

$$\phi'' - (\alpha_s^2 + \varepsilon)\phi - \frac{U''}{U - U_s - c}\phi = 0 \quad \text{for} \quad y \in (0, h) \tag{4.27}$$

normalized at h, that is

$$\begin{cases} \phi_1(h) = 1, & \phi_2(h) = 0, \\ \phi_1'(h) = 0, & \phi_2'(h) = 1. \end{cases}$$

It is standard that ϕ_1 and ϕ_2 are analytic as a function of c in $\mathbb{C}^+$ and that ϕ_1 and ϕ_2 are linearly independent with Wronskian 1. The neutral limiting mode is normalized so that $\phi_s(h) = 1$ and $\phi_s'(h) = g_r(U_s)$. The proof is again divided into two cases.

Case 1: $U(h) \neq U_s$. Let us define

$$\phi_0(y;\varepsilon,c) = \phi_1(y;\varepsilon,c) + g_r(U_s+c)\phi_2(y;\varepsilon,c),$$
$$\Phi(\varepsilon,c) = \phi_1(0;\varepsilon,c) + g_r(U_s+c)\phi_2(0;\varepsilon,c),$$

where g_r is given in (4.7). It is readily seen that ϕ_0 solves (4.27) with $\phi_0(h) = 1$ and $\phi_0'(h) = g_r(U_s+c)$. It is easy to see that $\Phi(\varepsilon,c)$ is analytic in $c \in \mathbb{C}^+$ and differentiable in ε. Note that an unstable solution to (4.22)–(4.23) exists if and only if $\Phi(\varepsilon,c) = 0$ for some $\operatorname{Im} c > 0$. The Green's function of (4.27) is written as

$$G(y,y';\varepsilon,c) = \bar{\phi}_1(y;\varepsilon,c)\phi_2(y';\varepsilon,c) - \phi_2(y;\varepsilon,c)\bar{\phi}_1(y';\varepsilon,c),$$

where $\bar{\phi}_1(y;\varepsilon,c)$ to be the solution of (4.27) with $\bar{\phi}_1(h) = 1$ and $\bar{\phi}_1'(h) = g_r(U_s)$. A similar computation as in [32, pp. 336] for $\phi_j(y;\varepsilon,c)$ $(j=1,2)$ yields that

$$\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon,c) = -\int_0^h G(y,0;\varepsilon,c)\phi_0(y;\varepsilon,c)dy \tag{4.28}$$

and

$$\frac{\partial \Phi}{\partial c}(\varepsilon,c) = \int_0^h G(y,0;\varepsilon,c)\frac{-U''}{(U-U_s-c)^2}\phi_0(y;\varepsilon,c)dy + \frac{d}{dc}g_r(U_s+c)\phi_2(0;\varepsilon,c). \tag{4.29}$$

Let us define the triangle in $\mathbb{C}^+$ as

$$\Delta_{(R,b)} = \{c_r + ic_i : |c_r| < Rc_i\,,\ 0 < c_i < b\}$$

and the Cartesian product in $(0,\infty)\times\mathbb{C}^+$ as

$$E_{(R,b_1,b_2)} = (-b_2,0)\times\Delta_{(R,b_1)},$$

where $R, b_1, b_2 > 0$ are to be determined later.

We claim that:

(a) For fixed R, both $\bar{\phi}_1(\cdot;\varepsilon,c)$ and $\phi_0(\cdot;\varepsilon,c)$ uniformly converge to ϕ_s in $C^1[0,h]$ as $(\varepsilon,c)\to(0,0)$ in $E_{(R,b_1,b_2)}$. That is, for any $\delta > 0$ there exists some $b_0 > 0$ such that whenever $b_1, b_2 < b_0$ and $(\varepsilon,c)\in E_{(R,b_1,b_2)}$ the inequalities

$$\left\|\bar{\phi}_1(\cdot;\varepsilon,c) - \phi_s\right\|_{C^1},\ \|\phi_0(\cdot;\varepsilon,c)-\phi_s\|_{C^1} \leqslant \delta$$

hold.

(b) $\phi_2(\cdot;\varepsilon,c)$ converges uniformly in the sense in (a). Moreover, the limit function $\phi_2(y;0,0)$ has the boundary value $\phi_2(0;0,0) = -\frac{1}{\phi_s'(0)}$ since the Wronskian of $\bar{\phi}_1$ and ϕ_2 and also the Wronskian of ϕ_s and $\phi_2(\cdot;0,0)$ are 1.

Here, we prove the uniform convergence of ϕ_0 only. The proof for $\bar{\phi}_1$ and ϕ_2 follows in the same way. Suppose on the contrary that there would exist a sequence $\{(\varepsilon_k,c_k)\}_{k=1}^{\infty}$ in $E_{(R,b_1,b_2)}$, where $b_1,b_2 < b_0$ such that $(\varepsilon_k,c_k)\to(0,0)$ as $k\to\infty$, yet

$$\|\phi_0(\varepsilon_k,c_k) - \phi_s\|_{C^1} \geqslant \delta_0 > 0.$$

Since $|\operatorname{Re} c_k| < R\operatorname{Im} c_k$ and $\operatorname{Im} c_k < b_0$, it follows immediately that

$$\begin{aligned}\left\|\frac{U''}{U-U_s-c_k}\right\|_{L^\infty} &\leqslant \|K\|_{L^\infty}\left(1 + \left\|\frac{c_k}{U-U_s-c_k}\right\|_{L^\infty}\right)\\ &\leqslant \|K\|_{L^\infty}(1+\sqrt{R^2+1})\end{aligned}$$

and $|g_r(U_s+c_k)| \leqslant C$ independently of k. That is, $\phi_0(\cdot;\varepsilon_k,c_k)$ satisfies the Rayleigh equation (4.27), whose coefficients are uniformly bounded, and the boundary condition $\phi_0(h)=1$, $\phi_0'(h)=g_r(U_s+c)$, whose coefficients are uniformly bounded. Accordingly, $\|\phi_0(\varepsilon_k,c_k)\|_{C^2}$ is uniformly bounded, and by the Ascoli-Arzela theorem $\{\phi_0(\varepsilon_k,c_k)\}$ has a convergent subsequence in C^1. That is, there exists ϕ_∞ such that

$$\|\phi_0(\varepsilon_{k_j},c_{k_j})-\phi_\infty\|_{C^1}\to 0 \qquad \text{as} \quad k_j\to\infty.$$

By continuity, ϕ_∞ satisfies

$$\phi_\infty''-\alpha_s^2\phi_\infty-\frac{U''}{U-U_s}\phi_\infty=0 \qquad \text{for} \quad y\in(0,h),$$

$\phi_\infty(h)=1$ and $\phi_\infty'(h)=g_r(U_s)$. This, however, implies $\phi_\infty=\phi_s$ and $\|\phi_0(\varepsilon_{k_j},c_{k_j})-\phi_s\|_{C^1}\to 0$. A contraction proves the claim.

In the appendix we prove that

$$\frac{\partial\Phi}{\partial\varepsilon}(\varepsilon,c)\to\frac{1}{\phi_s'(0)}\int_0^h\phi_s^2dy, \tag{4.30}$$

$$\frac{\partial\Phi}{\partial c}(\varepsilon,c)\to-\frac{1}{\phi_s'(0)}\left(i\pi\sum_{j=1}^{m_s}\frac{K(y_j)}{|U'(y_j)|}\phi_s^2(y_j)+p.v.\int_0^h\frac{K}{U-U_s}\phi_s^2dy+A\right) \tag{4.31}$$

uniformly as $\varepsilon\to 0-$ and $c\to 0$ in $E_{(R,b_1,b_2)}$, where A is defined by (4.26) and y_j $(j=1,\dots,m_s)$ are the inflection points for U_s. Let us denote

$$\begin{aligned} B&=\frac{1}{\phi_s'(0)}\int_0^h\phi_s^2dy,\\ C&=-\frac{1}{\phi_s'(0)}\left(p.v.\int_0^h\frac{K(y)}{U-U_s}\phi_s^2dy+A\right),\\ D&=-\frac{\pi}{\phi_s'(0)}\sum_{j=1}^{m_s}\frac{K(y_j)}{|U'(y_j)|}\phi_s^2(y_j).\end{aligned} \tag{4.32}$$

Lemma 4.7 asserts that ϕ_s is nonzero at one of the inflection points, say $\phi_s(y_j)\neq 0$. Note that by [32, Remark 4.2], $U'(y_j)\neq 0$ for $j=1,2,\dots,m_s$. Consequently, $D<0$.

The remainder of the proof is identically the same as that of [32, Theorem 4.1] and hence we provide only its sketch. Let us define

$$f(\varepsilon,c)=\Phi(\varepsilon,c)-B\varepsilon-(C+Di)c, \tag{4.33}$$

$$F(\varepsilon,c)=-\frac{B}{C+iD}\varepsilon-\frac{f(\varepsilon,c)}{C+iD}. \tag{4.34}$$

Note that for each $\varepsilon<0$ fixed a zero of an algebraic equation $\Phi(\varepsilon,\cdot)$ corresponds to a fixed point of the mapping $F(\varepsilon,\cdot)$. Our goal is to show that for $\varepsilon\in(\varepsilon_0,0)$, where $|\varepsilon_0|$ is sufficiently small, the mapping $F(\varepsilon,\cdot)$ is contracting on $\Delta_{(R,b)}$ for some R and b.

On account of the uniform convergence of $\partial\Phi/\partial\varepsilon$ and $\partial\Phi/\partial c$ in (4.30) and (4.31), respectively, for any $\delta>0$ there exists $b_0>0$ such that whenever $b_1,b_2<b_0$ the inequality

$$|\partial f/\partial c|,|\partial f/\partial\varepsilon|<\delta \tag{4.35}$$

holds in $E_{(R,b_1,b_2)}$. Subsequently, for any $(\varepsilon, c), (\varepsilon', c') \in E_{(R,b_1,b_2)}$, where $b_1, b_2 < b_0$, it follows that

$$|f(\varepsilon, c) - f(\varepsilon', c')| \leqslant \delta(|\varepsilon - \varepsilon'| + |c - c'|).$$

Since $f(\varepsilon, c), \Phi(\varepsilon, c) \to 0$ as $(\varepsilon, c) \to (0, 0)$, it follows that

$$|f(\varepsilon, c)| \leqslant \delta(|\varepsilon| + |c|) \quad \text{for } (\varepsilon, c) \in E_{(R,b_1,b_2)}. \tag{4.36}$$

Let us choose constants, as such in case $C \neq 0$,

$$R = 4|C/D| \quad \text{and} \quad \delta = \frac{1}{2}\sqrt{C^2 + D^2} \min\left(\frac{|BC|}{Q(C^2 + D^2)}, \frac{-BD}{Q(C^2 + D^2)}, 1\right);$$

in case $C = 0$,

$$R = 1 \quad \text{and} \quad \delta = \frac{1}{2}\sqrt{C^2 + D^2} \min\left(\frac{-BD}{Q(C^2 + D^2)}, 1\right);$$

Here, $Q = -2BD\sqrt{R^2 + 1}(C^2 + D^2)^{-1} + 1$. For such a $\delta > 0$ choose b_0 such that (4.35) and also (4.36) hold true whenever $b_1, b_2 < b_0$. Let

$$b_2 = b_0 \min\left(-\frac{C^2 + D^2}{2BD\sqrt{R^2 + 1}}, 1\right), \quad b_1 = Qb_2.$$

Finally, for $\varepsilon \in (-b_2, 0)$ fixed let

$$b(\varepsilon) = -2DB(C^2 + D^2)^{-1}\varepsilon.$$

It is straightforward to show [32, pp. 339] that for each $\varepsilon \in (-b_2, 0)$ the mapping $F(\varepsilon, \cdot) : \Delta_{(R,b(\varepsilon))} \to \Delta_{(R,b(\varepsilon))}$ is contracting with a contraction ratio not greater than 1/2. That is, for each $\varepsilon \in (-b_2, 0)$ there exists a unique $c(\varepsilon) \in \Delta_{(R,b(\varepsilon))}$ such that $F(\varepsilon, c(\varepsilon)) = c(\varepsilon)$. Since $F(\varepsilon, \cdot)$ is analytic in $\Delta_{(R,b_1)}$ and contracting uniformly for ε, the fixed point $c(\varepsilon)$ is in $\Delta_{(R,b_1)}$ and is differentiable in ε in the interval $(-b_2, 0)$ (see, for instance, [13]). Let $\varepsilon_0 = -b_2$. Since $c(\varepsilon) \in \Delta_{(R,b(\varepsilon))}$ it is immediate that (4.24) holds. Finally, differentiation of $\Phi(\varepsilon, c(\varepsilon)) = 0$ yields

$$c'(\varepsilon) = -\frac{\partial\Phi/\partial\varepsilon}{\partial\Phi/\partial c},$$

which, in view of (4.30) and (4.31), implies (4.25).

Case 2: $U(h) = U_s$. The proof is almost identical to that of Case 1, and thus we only indicate some differences. Let us define

$$\Phi(\varepsilon, c) = g_s(U_s + c)\phi_1(0; \varepsilon, c) + \phi_2(0; \varepsilon, c),$$

where g_s is defined in (4.12). Note that $\phi_s(h) = \phi_s(0) = 0$. Thereby, the Green's function is written as

$$G(y, y'; \varepsilon, c) = \phi_1(y; \varepsilon, c)\phi_2(y'; \varepsilon, c) - \phi_2(y; \varepsilon, c)\phi_1(y'; \varepsilon, c).$$

The same computations as in Case 1 yield that

$$\frac{\partial\Phi}{\partial\varepsilon} \to -\frac{1}{\phi_s'(0)} \int_0^h \phi_s^2 dy$$

and

$$\frac{\partial\Phi}{\partial c}(\varepsilon, c) \to -\frac{1}{\phi_s'(0)} \left(i\pi \sum_{j=1}^{m_s} \frac{K(y_j)}{|U'(y_j)|}\phi_s^2(y_j) + p.v. \int_0^h \frac{K}{U - U_s}\phi_s^2 dy \right)$$

uniformly as $\varepsilon \to 0-$ and $c \to 0$ in $E_{(R,b_1,b_2)}$. This completes the proof. ∎

Proof of Theorem 4.2. Let $-\alpha_N^2 < -\alpha_{N-1}^2 < \cdots < -\alpha_1^2 < 0$ be negative eigenvalues of the operator $-\frac{d^2}{dy^2} - K(y)$ on $(0,h)$ with boundary conditions (4.5)–(4.6. That is, $\alpha_N = \alpha_{\max}$, where $-\alpha_{\max}^2$ is defined either in (4.8a) or (4.8b). We deduce from Lemma 4.8 and Proposition 4.9 that to each $\alpha \in (0, \alpha_N)$ with $\alpha \neq \alpha_j$ $(j = 1, \ldots, N)$ an unstable solution is associated. Our goal is to show the instability at $\alpha = \alpha_j$ for each $j = 1, \ldots, N-1$.

Case 1: $U(h) \neq U_s$. Let $\{(\phi_k, \alpha_k, c_k)\}_{k=1}^{\infty}$ be a sequence of unstable solutions such that $\alpha_k \to \alpha_j+$ as $k \to \infty$. After normalization, we may assume $\phi_k(h) = 1$. Note that ϕ_k satisfies

$$\phi_k'' - \alpha_k^2 \phi_k - \frac{U''}{U - c_k}\phi_k = 0 \qquad \text{for} \quad y \in (0,h) \tag{4.37}$$

and $\phi_k'(h) = g_r(c_k)$, $\phi_k(0) = 0$. Below we will prove that $\operatorname{Im} c_k \geqslant \delta > 0$, where δ is independent of k. Since the coefficients of (4.37) and $g_r(c_k)$ are bounded uniformly for k, the solutions ϕ_k of the above Rayleigh equations are uniformly bounded in C^2, and subsequently, ϕ_k converges in C^2 as $k \to \infty$, say to ϕ_∞. The semicircle theorem (4.9) ensures that $c_k \to c_\infty$ as $k \to \infty$. Note that $\operatorname{Im} c_\infty \geqslant \delta > 0$. By continuity, ϕ_∞ satisfies

$$\phi_\infty'' - \alpha_j^2 \phi_\infty - \frac{U''}{U - c_\infty}\phi_\infty = 0 \qquad \text{for} \quad y \in (0,h),$$

$\phi_\infty(h) = 1$, $\phi_\infty'(h) = g_r(c_\infty)$ and $\phi_\infty(0) = 0$. That is, $(\phi_\infty, \alpha_j, c_\infty)$ is an unstable solution of (4.1)–(4.2).

It remains to show that $\{\operatorname{Im} c_k\}$ has a positive lower bound. Suppose on the contrary that $\operatorname{Im} c_k \to 0$ as $k \to \infty$.

We claim that $\|\phi_k\|_{L^2} \leqslant C$, where $C > 0$ is independent of k. Otherwise, $\|\phi_k\|_{L^2} \to \infty$ as $k \to \infty$. Let $\varphi_k = \phi_k / \|\phi_k\|_{L^2}$ so that $\|\varphi_k\|_{L^2} = 1$. Lemma 4.6 then dictates that $\|\varphi_k\|_{H^2} \leqslant C$ independently of k. Subsequently, Proposition 4.4 ensures that $(\varphi_k, \alpha_k, c_k)$ converges to a neutral limiting mode $(\varphi_s, \alpha_s, U_s)$. By continuity, $\|\varphi_s\|_{L^2} = 1$ and

$$\varphi_s'' - \alpha_j^2 \varphi_s + K(y)\varphi_s = 0 \qquad \text{for} \quad y \in (0,h).$$

On the other hand, $\varphi_s(h) = \varphi_s'(h) = 0$ since $\varphi_k(h) = 1/\|\phi_k\|_{L^2} \to 0$ and $\varphi_k'(h) = g_r(c_k)/\|\phi_k\|_{L^2} \to 0$ as $k \to \infty$. Correspondingly, $\varphi_s \equiv 0$ on $[0,h]$. A contradiction proves the claim.

Since $\|\phi_k\|_{L^2}$ is bounded uniformly for k, Lemma 4.6 and Proposition 4.4 apply and $\|\phi_k\|_{H^2} \leqslant C$, $c_k \to U_s$ and $\phi_k \to \phi_s$ in C^1, where ϕ_s satisfies

$$\phi_s'' - \alpha_j^2 \phi_s - \frac{U''}{U - U_s}\phi_s = 0 \qquad \text{for} \quad y \in (0,h)$$

with $\phi_s(h) = 1$, $\phi_s'(h) = g_r(U_s)$ and $\phi_s(0) = 0$. An integration by parts yields that

$$\begin{aligned} 0 =& \int_0^h \left(\phi_s(\phi_k'' - \alpha_k^2\phi_k - \frac{U''}{U - c_k}\phi_k) - \phi_k(\phi_s'' - \alpha_j^2 \phi_s - \frac{U''}{U - U_s}\phi_s) \right) dy \\ =& (\alpha_j^2 - \alpha_k^2) \int_0^h \phi_s \phi_k \, dy - (c_k - U_s) \int_0^h \frac{U''}{(U - c_k)(U - U_s)} \phi_s \phi_k dy + g_r(c_k) - g_r(U_s). \end{aligned}$$

Let us denote

$$B_k = \int_0^h \phi_s \phi_k dy,$$
$$D_k = -\int_0^h \frac{U''}{(U-c_k)(U-U_s)}\phi_s\phi_k dy + \frac{g_r(c_k)-g_r(U_s)}{c_k-U_s}.$$

It is immediate that $\lim_{k\to\infty} B_k = \int_0^h |\phi_s|^2 dy$. We shall show in the appendix that

$$\lim_{k\to\infty} D_k = A + i\pi \sum_{j=1}^{m_s} \frac{K(y_j)}{|U(y_j)|}\phi_s^2(y_j), \tag{4.38}$$

where A is defined by (4.26) and a_j's ($j = 1, 2, \ldots, m_s$) are inflection points corresponding to the inflection value U_s. Since $\operatorname{Im}(\lim_{k\to\infty} D_k) > 0$ (see the proof of Proposition 4.9) it follows that

$$\operatorname{Im} c_k = (\alpha_k^2 - \alpha_j^2)\operatorname{Im}(B_k/D_k) < 0$$

for k large. A contradiction proves that $\{\operatorname{Im} c_k\}$ has a positive lower bound, uniformly for k.

Case 2: $U(h) = U_s$. We normalize ϕ_k so that $\phi_k'(h) = 1$ and $\phi_k(h) = g_s(c_k)$. The proof is identically the same as that of Case 1 except that

$$D_k = -\int_0^h \frac{U''}{(U-c_k)(U-U_s)}\phi_s\phi_k\, dy + \frac{g_s(c_k)}{c_k-U_s}.$$

We shall show in the appendix that

$$\lim_{k\to\infty} D_k = i\pi \sum_{j=1}^{m_s} \frac{K(y_j)}{|U'(y_j)|}\phi_s^2(y_j) + p.v.\int_0^h \frac{K}{U-U_s}\phi_s^2 dy. \tag{4.39}$$

This proves that there exists an unstable solution for each $\alpha \in (0, \alpha_{\max})$.

It remains to prove linear stability in case either the operator $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0,h)$ with (4.5)–(4.6) is nonnegative or $\alpha \geqslant \alpha_{\max}$. Suppose otherwise, there exists an unstable mode at a wave number $\alpha \geqslant \alpha_{\max}$. By Lemma 4.8, we can continuate this unstable mode for wave numbers larger than α until the growth rate becomes zero. By Lemma 6.3, this continuation must stop at a wave number $\alpha_s > \alpha$, where there is a neutral limiting mode. Then by Proposition 4.4, $-\alpha_s^2$ is a negative eigenvalue of $-\frac{d^2}{dy^2} - K(y)$ with (4.5)-(4.6). But $-\alpha_s^2 < -\alpha_{\max}^2$ which is a contradiction to the fact that $-\alpha_{\max}^2$ is the lowest eigenvalue of $-\frac{d^2}{dy^2} - K(y)$ with (4.5)-(4.6).

This completes the proof. □

Remark 4.10. For $U \in \mathcal{K}^+$, let $-\alpha_d^2$ be the lowest eigenvalue of $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0,h)$ with the Dirichlet boundary conditions $\phi(h) = 0 = \phi(0)$. If $U(h) = U_s$ then $\alpha_{\max} = \alpha_d$. We claim that if $U(h) \neq U_s$ then $\alpha_{\max} > \alpha_d$. To see this, let ϕ_d be the eigenfunction of $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0,h)$ with the Dirichlet boundary conditions corresponding to $-\alpha_d^2$ and let ϕ_m be the eigenfunction of $-\frac{d^2}{dy^2} - K(y)$ with the boundary conditions (4.5)–(4.6) corresponding to $-\alpha_{\max}^2$. By Sturm's theory, we can assume $\phi_d, \phi_m > 0$ on $y \in (0,h)$. An integration by parts yields that

$$0 = \int_0^h \big(\phi_d(\phi_m'' - \alpha_{\max}^2\phi_m + K(y)\phi_m) - \phi_m(\phi_d'' - \alpha_d^2\phi_d + K(y)\phi_d)\big)\,dy$$
$$= -\phi_m(h)\phi_d'(h) + (\alpha_d^2 - \alpha_{\max}^2)\int_0^h \phi_d\phi_m dy.$$

Since $\phi_d'(h) < 0$ the claim follows.

For flows in class $\mathcal{K}^+$, the lowest eigenvalue of (4.4) measures the range of instability, for both the free surface and the rigid wall cases. That means, $(0,\alpha_{\max})$ is the interval of unstable wave numbers in the free-surface setting (Theorem 4.2) and $(0,\alpha_d)$ is the interval of unstable wave numbers in the rigid-wall setting [32, Theorem 1.2]. The fact that $\alpha_{\max} > \alpha_d$ thus indicates that the free surface has a *destabilizing* effect.

4.3. **Monotone unstable shear flows.** In general, for a given shear flow profile with one inflection value, one show the existence of a neutral limiting mode, which is contiguous to exponentially growing modes, and thus the linear instability by numerically computing the negativity of (4.8). For a monotone flow, however, a comparison argument establishes a general statement.

Lemma 4.11. *For any monotone shear flow with exactly one inflection point y_s in the interior, there exists a neutral limiting mode. That is, (4.1)–(4.2) has a nontrivial solution for which $c = U(y_s)$ and $\alpha > 0$.*

Proof. This result is given in Theorem 4 of [48]. Here we present a detailed proof for completeness and also for clarification of some arguments in [48].

W consider an increasing flow $U(y)$ in y only. A decreasing flow in y can be treated in the same way. Let $U_s = U(y_s)$ be the inflection value. Denoted by ϕ_α is the solution of the Rayleigh equation

$$\phi_\alpha'' + \left(\frac{U''}{U_s - U} - \alpha^2\right)\phi_\alpha = 0 \qquad \text{for} \quad y \in (0,h)$$

with $\phi_\alpha(0) = 0$ and $\phi_\alpha'(0) = 1$. As in the proof of Lemma 2.5, integrating the above over $(0,h)$ yields that

$$(U(h) - U_s)\phi_\alpha'(h) - (U(0) - U_s) - \phi_\alpha(h)U'(h) - \alpha^2\int_0^h (U - U_s)\phi_\alpha dy = 0,$$

and thus

$$\frac{\phi_\alpha'(h)}{\phi_\alpha(h)} = \frac{U(0) - U_s}{(U(h) - U_s)\phi_\alpha(h)} + \frac{U'(h)}{U(h) - U_s} + \frac{\alpha^2}{(U(h) - U_s)\phi_\alpha(h)}\int_0^h (U - U_s)\phi_\alpha dy.$$

It is straightforward to see that the boundary condition (4.2) is satisfies if and only if the function

$$f(\alpha) = \frac{U(0) - U_s}{(U(h) - U_s)\phi_\alpha(h)} + \frac{\alpha^2}{(U(h) - U_s)\phi_\alpha(h)}\int_0^h (U - U_s)\phi_\alpha dy - \frac{g}{(U_s - U(h))^2}.$$

vanishes at some $\alpha > 0$.

We claim that $\phi_\alpha(y) > 0$ on $y \in (0,h]$ for any $\alpha \geqslant 0$. Suppose on the contrary that $y_\alpha \in (0,h]$ to be the first zero of ϕ_α other than 0, that is, $\phi_\alpha(y_\alpha) = 0$ and $\phi_\alpha(y) > 0$ for $y \in (0, y_\alpha)$. Then, $y_\alpha > y_s$ must hold. Indeed, if $y_\alpha \leqslant y_s$ were to be true, then Sturm's first comparison theorem would apply to ϕ_α and $U - U_s$ on

$[0, y_\alpha]$ to assert that $U - U_s$ would vanish somewhere in $(0, y_\alpha) \subset (0, y_s)$. This uses that

$$(U - U_s)'' + \frac{U''}{U_s - U}(U - U_s) = 0.$$

A contradiction then asserts that $y_\alpha > y_s$. Correspondingly, ϕ_α and $U - U_s$ have exactly one zero in $[0, y_\alpha]$. On the other hand, by Sturm's second comparison theorem [25], it follows that

$$\frac{\phi'_\alpha(y_\alpha)}{\phi_\alpha(y_\alpha)} \geqslant \frac{U'(y_\alpha)}{U(y_\alpha) - U_s}.$$

This contradicts since $\phi_\alpha(y_\alpha) = 0$ and the left hand side is $-\infty$. Therefore, $\phi_\alpha(y) > 0$ for $y \in (0, h]$ and for any $\alpha \geqslant 0$. In particular, $\phi_\alpha(h) > 0$ for any $\alpha \geqslant 0$. Consequently, f is a continuous function of α and $f(0) < 0$.

It remains to show that $f(\alpha) > 0$ for $\alpha > 0$ big enough. Thereby, by continuity f vanishes at some $\alpha > 0$. Let $\left|\frac{U''}{U_s - U}\right| \leqslant M$ and $\alpha^2 > M$. Le us denote by ϕ_1 and ϕ_2 the solutions of

$$\phi_1'' + (M - \alpha^2)\phi_1 = 0 \quad \text{and} \quad \phi_2'' + (-M - \alpha^2)\phi_2 = 0 \qquad \text{for} \quad y \in (0, h),$$

respectively, with $\phi_i(0) = 0$ and $\phi'_i(0) = 1$. It is straightforward that

$$\phi_1(y) = \frac{1}{\sqrt{\alpha^2 - M}} \sinh \sqrt{\alpha^2 - M} y \quad \text{and} \quad \phi_2(y) = \frac{1}{\sqrt{\alpha^2 + M}} \sinh \sqrt{\alpha^2 + M} y.$$

As in the proof of Lemma 2.7 (a), Sturm's second comparison theorem [25] implies that

$$\frac{1}{\sqrt{\alpha^2 - M}} \sinh \sqrt{\alpha^2 - M} y \leqslant \phi_\alpha(y) \leqslant \frac{1}{\sqrt{\alpha^2 + M}} \sinh \sqrt{\alpha^2 + M} y.$$

This together with the monotone property of U establishes that $f(\alpha) \geqslant C_1\alpha - C_2$ for some constants $C_1, C_2 > 0$. For details we refer to [48, Theorem 4]. Thus, $f(\alpha) > 0$ if $\alpha > 0$ is sufficiently large. This completes the proof . □

Since a monotone flow with one inflection value is in class $\mathcal{K}^+$, the above lemma combined with Theorem 4.2 asserts its instability.

Corollary 4.12. *Any monotone shear flow with exactly one inflection point in the interior is unstable in the free-surface setting for wave numbers in an interval* $(0, \alpha_{\max})$ *with* $\alpha_{\max} > 0$.

Remark 4.13. In the free-surface setting, there are two different kinds of neutral modes, solutions to the Rayleigh system (4.1)–(4.2) with $\operatorname{Im} c = 0$. Neutral limiting modes have their phase speed in the range of the shear profile. In class $\mathcal{K}^+$, moreover, the phase speed of a neutral limiting mode must be the inflection value of the shear profile (Proposition 4.4) and it is contiguous to unstable modes (Proposition 4.9). On the other hand, Lemma 2.5 shows that under the conditions $U''(h) < 0$ and $U(h) > U(y)$ for $h \neq y$ a neutral mode exists with the phase speed $c > \max U$. Such a neutral mode is used in the local bifurcation of nontrivial periodic waves in Theorems 2.2 and 2.6. In view of the semicircle theorem, however, such a neutral mode is not contiguous to unstable modes. This means, neutral modes governing the stability property are different from those governing the bifurcation of nontrivial waves. This is an important difference in the free-surface setting. Since in the rigid wall setting, for any possible neutral modes the phase speed must lie in the range of U, which follows easily from Sturm's first comparison theorem. For class $\mathcal{K}^+$ flows, such neutral modes are contiguous to unstable modes ([32]) and the bifurcation of nontrivial waves from these neutral modes can also be shown ([7]). Thus, in the rigid-wall setting, the same neutral modes govern both stability and bifurcation.

Remark 4.14. In [15], the $\mathcal{J}$-formal stability was introduced via a quadratic form, which is related to the local bifurcation of nontrivial waves in the transformed variables (see also Introduction), and it was concluded that this formal stability of the trivial solutions switches exactly at the bifurcation point. Below we discuss two examples for which the linear stability property does not change along the line of trivial solutions passing the bifurcation point, which indicates that the $\mathcal{J}$-formal stability in [15] is unrelated to linear stability of the physical water wave problem. However, the $\mathcal{J}$-formal stability results [15] does give more information about the structure of the periodic water wave branch.

Let us consider a monotone increasing flow $U(y)$ on $y \in [0,h]$ with one inflection point $y_s \in (0,h)$, for example, $U(y) = a\sin b(y-h/2)$ on $y \in [0,h]$ for which $y_s = h/2$. By lemma 4.11 and Corollary 4.12, such a shear flow is equipped with a neutral limiting mode with $c = U(y_s)$ and $\alpha = \alpha_{\max} > 0$, and it is linearly unstable for any wave number $\alpha \in (0, \alpha_{\max})$. In addition, by lemma 2.5 and Theorem 2.6 for any wave number $\alpha \in (0, \alpha_{\max})$ this shear flow has a neutral mode with $c(\alpha) > U(h) = \max U$. Moreover, such a neutral mode is a nontrivial solution to the bifurcation equation (2.9)–(2.10), and thus there exists a local curve of bifurcation of nontrivial waves with a wave speed $c(\alpha)$ and period $2\pi/\alpha$. Let p_0 and γ be the flux and vorticity relation determined by $U(y)$, $c(\alpha)$ and h via (2.11). Consider the trivial solutions with shear flows $U(y;\mu)$ defined in Lemma 2.1, with above p_0, γ and the parameter μ. The bifurcation point $U(y;\mu_0) = U(y) - c(\alpha)$ corresponds to $\mu_0 = (U(h) - c(\alpha))^2$. The instability of $U(y;\mu_0)$ at the wave number α continuates to shear flows $U(y;\mu)$ with μ near μ_0, which can be shown by a similar argument as in the proof of Lemma 4.8. So at the bifurcation point μ_0, there is *NO* switch of stability of trivial solutions.

Let us consider $U \in C^2([0,h])$ satisfying that $U'(y) > 0$ and $U''(y) < 0$ in $y \in [0,h]$. For such a shear flow Lemma 2.5 and Theorem 2.6 applies as well and there exists a local curve of bifurcation of nontrivial waves for any wave number α which travel at the speed $c(\alpha) > \max U$, where $c(\alpha)$ is chosen so that the bifurcation equation (2.9)–(2.10) is solvable. For this bifurcation flow $U(y) - c(\alpha)$, the vorticity relation γ determined via (2.11) is monotone since U'' does not change sign. Consequently, any shear flows $U(y;\mu)$ defined in Lemma 2.1 with the same γ has no inflection points, as also remarked at the end of Section 3. Therefore, by Theorem 6.4 all trivial solutions corresponding to these shear flows $U(y;\mu)$ are stable. This again shows that the bifurcation of nontrivial periodic waves does not involve the switch of stability of trivial solutions.

5. Linear instability of periodic water waves with free surface

We now turn to investigating the linear instability of periodic traveling waves near an unstable background shear flow. Suppose that a shear flow $(U(y),0)$ with $U \in C^{2+\beta}([0,h_0])$ and $U \in \mathcal{K}^+$ has an unstable wave number $\alpha > 0$, that is, for such a wave number $\alpha > 0$ the Rayleigh system (4.1)–(4.2) has a nontrivial solution with $\operatorname{Im} c > 0$. Suppose moreover that for the unstable wave number

$\alpha > 0$ the bifurcation equation (2.9)–(2.10) is solvable with some $c(\alpha) > \max U$. Then Remark 2.4 and Theorem 2.2 apply to state that there exists a one-parameter curve of small-amplitude traveling-wave solutions $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ satisfying (2.2) of period $2\pi/\alpha$ and the wave speed $c(\alpha)$, where $\epsilon \geqslant 0$ is the amplitude parameter. A natural question is: are these small-amplitude nontrivial periodic waves generated over the unstable shear flow also unstable? The answer is YES under some technical assumptions, which is the subject of the forthcoming investigation.

5.1. **The main theorem and examples.** We prove the linear instability of the steady periodic water-waves $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ by finding a growing-mode solution to the linearized water-wave problem. As in Section 3, let

$$\mathcal{D}_\epsilon = \{(x,y) : 0 < x < 2\pi/\alpha,\ 0 < y < \eta_\epsilon(x)\} \quad \text{and} \quad \mathcal{S}_\epsilon = \{(x, \eta_\epsilon(x)) : 0 < x < 2\pi/\alpha\}$$

denote, respectively, the fluid domain of the steady wave $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ of one period and the steady surface. The growing-mode problem (3.8) of the linearized periodic water-wave problem around $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ reduces to

$$\Delta\psi + \gamma'(\psi_\epsilon)\psi - \gamma'(\psi_\epsilon)\int_{-\infty}^{0} \lambda e^{\lambda s}\psi(X_\epsilon(s), Y_\epsilon(s))ds = 0 \quad \text{in } \mathcal{D}_\epsilon; \tag{5.1a}$$

$$\lambda\eta(x) + \frac{d}{dx}\big(\psi_{\epsilon y}(x, \eta_\epsilon(x))\eta(x)\big) = -\frac{d}{dx}\psi(x, \eta_\epsilon(x)); \tag{5.1b}$$

$$P(x, \eta_\epsilon(x)) + P_{\epsilon y}(x)\eta(x) = 0; \tag{5.1c}$$

$$\lambda\psi_n(x) + \frac{d}{dx}\big(\psi_{\epsilon y}(x, \eta_\epsilon(x))\psi_n(x)\big) = -\frac{d}{dx}P(x, \eta_\epsilon(x)) - \Omega\frac{d}{dx}\psi(x, \eta_\epsilon(x)); \tag{5.1d}$$

$$\psi(x, 0) = 0. \tag{5.1e}$$

Here and in sequel, let us abuse notation and denote that $P_{\epsilon y}(x) = P_{\epsilon y}(x, \eta_\epsilon(x))$, that is, $P_{\epsilon y}(x,y)$ restricted on the steady wave-profile $y = \eta_\epsilon(x)$. By Theorem 2.2, $P_{\epsilon y}(x) = P_{\epsilon y}(x, \eta_\epsilon(x)) = -g + O(\epsilon)$. Recall that

$$\psi_n(x) = \partial_y\psi(x, \eta_\epsilon(x)) - \eta_{\epsilon x}(x)\partial_x\psi(x, \eta_\epsilon(x))$$

is the derivative of $\psi(x, \eta_\epsilon(x))$ in the direction normal to the free surface $(x, \eta_\epsilon(x))$ and that $\Omega = \gamma(0)$ is the vorticity of the steady flow of $\psi_\epsilon(x,y)$ on the steady wave-profile $y = \eta_\epsilon(x)$. Note that Ω is a constant independent of ϵ.

Theorem 5.1 (Linear instability of small-amplitude periodic water-waves)**.** *Let $U \in C^{2+\beta}([0, h_0])$, $\beta \in (0,1)$, be in class $\mathcal{K}^+$. Suppose that $U(h_0) \neq U_s$, where U_s is the inflection value of U, and that $\alpha_{\max}$ defined by (4.8a) is positive, as such Theorem 4.2 applies to find the interval of unstable wave numbers $(0, \alpha_{\max})$. Suppose moreover that for some $\alpha \in (\alpha_{\max}/2, \alpha_{\max})$ there exists $c(\alpha) > \max U$ such that the bifurcation equation (2.9)– (2.10) has a nontrivial solution. Let us denote by $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ the family of nontrivial waves with the period $2\pi/\alpha$ and the wave speed $c(\alpha)$ bifurcating from the trivial solution $\eta_0(x) \equiv h_0$ and $(\psi_{0y}(y), -\psi_{0x}(y)) = U(y) - c(\alpha), 0)$, where $\epsilon \geqslant 0$ is the small-amplitude parameter. Provided that*

$$g + U'(h_0)(U(h_0) - U_s) > 0, \tag{5.2}$$

then for each $\epsilon > 0$ sufficiently small, there exists an exponentially growing solution $(e^{\lambda t}\eta(x), e^{\lambda t}\psi(x,y))$ of the linearized system (3.5), where $\operatorname{Re}\lambda > 0$, with the regularity property

$$(\eta(x), \psi(x,y)) \in C^{2+\beta}([0, 2\pi/\alpha]) \times C^{2+\beta}(\bar{\mathcal{D}}_\epsilon).$$

Remark 5.2 (Examples). As is discussed in Remark 4.14, any increasing flow shear flow $U \in C^{2+\beta}([0, h_0])$, $\beta \in (0, 1)$, with exactly one inflection point in $y \in (0, h_0)$ satisfies $\alpha_{\max} > 0$. Moreover, Lemma 2.5 applies and small-amplitude periodic waves bifurcate at any wave number $\alpha > 0$. Since (5.2) holds true, therefore, by Theorem 5.1 small-amplitude periodic waves bifurcating from such a shear flow at any wave number $\alpha \in (\alpha_{\max}/2, \alpha_{\max})$ are unstable. An example includes

$$U(y) = a \sin b(y - h_0/2) \qquad \text{for} \quad y \in [0, h_0], \tag{5.3}$$

where $h_0, b > 0$ satisfy $h_0 b \leqslant \pi$ and $a > 0$ is arbitrary.

(1) The shear flow in (5.3) is unstable under periodic perturbations of a wave number $\alpha \in (0, \alpha_{\max})$, where $\alpha_{\max} > 0$. Note that in the rigid-wall setting [32], the same shear flow is stable under perturbations of any wave number. This indicates that free surface has a destabilizing effect. This serves as an example of Remark 4.10 since $\alpha_{\max} > 0$ and $\alpha_d = 0$.

(2) The amplitude a and the depth h_0 in (5.3) may be chosen arbitrarily small, and the shear flow as well as the nontrivial periodic waves near the shear flow are unstable for any wave number $\alpha \in (0, \alpha_{\max})$, which conflicts with the result in [15] that small-amplitude rotational periodic water-waves are $\mathcal{J}$-formally stable if the vorticity strength and the depth are sufficiently small. Thus, as also commented in Remark (4.14), the $\mathcal{J}$-formal stability in [15] is not directly related to the linear stability of water waves. Indeed, while $\partial\mathcal{J}(\eta, \psi) = 0$ gives the equations for steady steady waves, the linearized water-wave problem is not in the form

$$\partial_t(\eta, \psi) = (\partial^2\mathcal{J})(\eta, \psi),$$

which is implicitly required in [15] in order to apply the Crandall-Rabanowitz theory [16] of exchange of stability.

(3) Our example (5.3) also indicates that adding an arbitrarily small vorticity to the irrotational water wave system of an arbitrary depth may induce instability. That means, although small irrotational periodic waves are found to be stable under perturbations of the same period [39], [45], they are not structurally stable; Vorticity has a subtle influence on the stability of water waves.

The proof of Theorem 5.1 uses a perturbation argument. At $\epsilon = 0$ the trivial solution $(\eta_0(x), \psi_0(x, y))$ corresponds to the shear flow $(U(y) - c(\alpha), 0)$ under the flat surface $\{y = h_0\}$. For the simplicity of notations, in the remainder of this section, we write $U(y)$ for $U(y) - c(\alpha)$, as is done in Section 4. Thereby, $U(y) < 0$. Since α is an unstable wave number of $U(y)$, there exist an unstable solution ϕ_α to the Rayleigh system (4.1)–(4.2) and an unstable phase speed c_α. That is, $\phi_\alpha \not\equiv 0$, $\operatorname{Im} c_\alpha > 0$ and

$$\begin{aligned} &\phi_\alpha'' - \alpha^2\phi_\alpha + \frac{U''}{U - c_\alpha}\phi_\alpha = 0 \qquad \text{for} \quad y \in (0, h_0), \\ &\phi_\alpha'(h_0) = \left(\frac{g}{(U(h_0) - c_\alpha)^2} + \frac{U'(h_0)}{U(h_0) - c_\alpha}\right)\phi_\alpha(h_0), \\ &\phi_\alpha(0) = 0. \end{aligned} \tag{5.4}$$

This corresponds to a growing mode solution satisfying (5.1) at $\epsilon = 0$, where $\lambda_0 = -i\alpha c_\alpha$ has a positive real part. Our goal is to show that for $\epsilon > 0$ sufficiently small, there is found λ_ϵ near λ_0 such that the growing-mode problem (5.1) at $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ is solvable. First, the system (5.1) is reduced to an operator equation defined in a function space independent of ϵ. Then, by showing the continuity of this operator with respect to the small-amplitude parameter ϵ, the continuation of the unstable mode follows from the eigenvalue perturbation theory of operators.

5.2. **Reduction to an operator equation.** The purpose of this subsection is to reduce the growing mode system (5.1) to an operator equation on $L^2_{\text{per}}(\mathcal{S}_\epsilon)$. Here and elsewhere the subscript *per* denotes the periodicity in the x-variable. The idea is to express $\eta(x)$ on $\mathcal{S}_\epsilon$ and $\psi(x,y)$ in $\mathcal{D}_\epsilon$ (and hence $P(x,\eta_\epsilon(x))$) in terms of $\psi(x,\eta_\epsilon(x))$.

Our first task is to relate $\eta(x)$ with $\psi(x,\eta_\epsilon(x))$.

Lemma 5.3. *For $|\lambda-\lambda_0| \leqslant (\operatorname{Re}\lambda_0)/2$, where $\lambda_0 = -i\alpha c_\alpha$, let us define the operator $\mathcal{C}^\lambda : L^2_{per}(\mathcal{S}_\epsilon) \to L^2_{per}(\mathcal{S}_\epsilon)$ by*

$$\begin{aligned}\mathcal{C}^\lambda\phi(x) = &-\frac{1}{\psi_{\epsilon y}(x)}\phi(x) + \frac{1}{\psi_{\epsilon y}(x)e^{\lambda a(x)}}\int_0^x \lambda e^{\lambda a(x')}\psi_{\epsilon y}^{-1}(x')\phi(x')dx' \\ &- \frac{\lambda}{\psi_{\epsilon y}(x)e^{\lambda a(x)}\left(1-e^{\lambda a(2\pi/\alpha)}\right)}\int_0^{2\pi/\alpha} e^{\lambda a(x')}\psi_{\epsilon y}^{-1}(x')\phi(x')dx',\end{aligned} \tag{5.5}$$

where $a(x) = \int_0^x \psi_{\epsilon y}^{-1}(x',\eta_\epsilon(x'))dx'$. For simiplicity, here and in the sequel we identify $\psi_{ey}(x)$ with $\psi_{ey}(x,\eta_\epsilon(x))$ and $\phi(x)$ with $\phi(x,\eta_\epsilon(x))$, etc. Then,

(a) The operator $\mathcal{C}^\lambda$ is analytic in λ for $|\lambda-\lambda_0| \leqslant (\operatorname{Re}\lambda_0)/2$, and the estimate

$$\left\|\mathcal{C}^\lambda\right\|_{L^2_{\text{per}}(\mathcal{S}_\epsilon)\to L^2_{\text{per}}(\mathcal{S}_\epsilon)} \leqslant K$$

holds, where $K>0$ is independent of λ and ϵ.

(b) For any $\phi \in L^2_{\text{per}}(\mathcal{S}_\epsilon)$, the function $\varphi = \mathcal{C}^\lambda\phi$ is the unique $L^2_{\text{per}}(\mathcal{S}_\epsilon)$- weak solution of the first-order ordinary differential equation

$$\lambda\varphi + \frac{d}{dx}\left(\psi_{\epsilon y}(x)\varphi\right) = -\frac{d}{dx}\phi. \tag{5.6}$$

If, in addition, $\phi \in C^1_{\text{per}}(\mathcal{S}_\epsilon)$ then $\varphi \in C^1_{\text{per}}(\mathcal{S}_\epsilon)$ is the unique classical solution of (5.6).

Proof. Assertions of (a) follows immediately since $\psi_{\epsilon y}(x) < 0$ and thus $a(x) < 0$ and since $\operatorname{Re}\lambda \geqslant \operatorname{Re}\lambda_0$.

(b) First, we consider the case $\phi \in C^1_{\text{per}}(\mathcal{S}_\epsilon)$ to motivate the definition of $\mathcal{C}^\lambda$.

Let us write (5.6) as the first-order ordinary differential equation

$$\frac{d}{dx}\varphi + \frac{1}{\psi_{\epsilon y}}\left(\lambda + \frac{d}{dx}\psi_{\epsilon y}\right)\varphi = -\frac{1}{\psi_{\epsilon y}}\frac{d}{dx}\phi,$$

which has a unique $2\pi/\alpha$-periodic solution

$$\begin{aligned}\varphi(x) = -\frac{1}{\psi_{\epsilon y}(x)e^{\lambda a(x)}}\Big(&\int_0^x e^{\lambda a(x')}\frac{d}{dx}\phi(x')dx' \\ &- \frac{1}{1-e^{\lambda a(2\pi/\alpha)}}\int_0^{2\pi/\alpha} e^{\lambda a(x')}\frac{d}{dx}\phi(x')dx'\Big).\end{aligned}$$

An integration by parts of the above formula yields that $\varphi(x) = \mathcal{C}^\lambda\phi$ is as defined in (5.5).

In case $\phi \in L^2_{\rm per}(\mathcal{S}_\epsilon)$, the integral representation (5.5) makes sense and solves (5.6) in the weak sense. Indeed, an integration by parts shows that φ defined by (5.5) satisfies the weak form of equation (5.6)

$$\int_0^{2\pi/\alpha} \left(\lambda\varphi(x)h(x) - \psi_{\epsilon y}(x)\varphi(x)\frac{d}{dx}h(x) - \phi(x)\frac{d}{dx}h(x) \right) dx = 0$$

for any $2\pi/\alpha$-periodic function $h \in H^1_{\rm per}([0,2\pi/\alpha])$. In order to show the uniqueness, suppose that $\tilde\varphi \in L^2_{\rm per}(\mathcal{S}_\epsilon)$ is another weak solution of (5.6). Let $\varphi_1 = \varphi - \tilde\varphi$. Then, $\varphi_1 \in L^2_{\rm per}(\mathcal{S}_\epsilon)$ is a weak solution of the homogeneous differential equation

$$\lambda\varphi_1 + \frac{d}{dx}\left(\psi_{\epsilon y}(x)\varphi_1\right) = 0.$$

It is readily seen that $\int_0^{2\pi/\alpha}\varphi_1(x)dx = 0$. Note that $h_1(x) = \int_0^x \varphi_1(x')dx'$ defines $2\pi/\alpha$-periodic function in $H^1_{\rm per}([0,2\pi/\alpha])$. Multiplication of the above homogeneous equation by $(\lambda h_1)^*$ and an integration by parts then yields that

$$\begin{aligned} 0 &= \operatorname{Re}\int_0^{2\pi/\alpha}\left(|\lambda|^2\varphi_1(x)h_1^*(x) - \psi_{\epsilon y}(x)\lambda^*\varphi_1(x)\left(\frac{d}{dx}h_1(x)\right)^*\right)dx \\ &= |\lambda|^2\int_0^{2\pi/\alpha}\frac{d}{dx}\left(\tfrac12 |h_1|^2\right)dx - \operatorname{Re}\lambda\int_0^{2\pi/\alpha}\psi_{\epsilon y}(x)\,|\varphi_1|^2\,dx \\ &= -\operatorname{Re}\lambda\int_0^{2\pi/\alpha}\psi_{\epsilon y}(x)\,|\varphi_1|^2\,dx. \end{aligned}$$

Here and elsewhere, the asterisk denotes the complex conjugation. Since $\operatorname{Re}\lambda > 0$ and $\psi_{\epsilon y}(x) < 0$, it follows that $\varphi_1 \equiv 0$, and in turn, $\varphi \equiv \tilde\varphi$. This completes the proof. □

Note that the range of $\mathcal{C}^\lambda$ is of mean-zero. That is, $\oint_{\mathcal{S}_\epsilon}(\mathcal{C}^\lambda\phi)(x)dx = 0$. Indeed, integrating (5.6) with $\varphi = \mathcal{C}^\lambda\phi$ proves the assertion. Formally, the operator $\mathcal{C}^\lambda$ is written as

$$\mathcal{C}^\lambda\phi(x) = -\left(\lambda + \frac{d}{dx}\left(\psi_{\epsilon y}(x)\phi(x)\right)\right)^{-1}\frac{d}{dx}\phi(x).$$

Let us denote $f(x) = \psi(x,\eta_\epsilon(x))$. With the use of $\mathcal{C}^\lambda$ then the boundary conditions (5.1b)–(5.1d) are written in terms of f as

$$\begin{gathered} \eta(x) = \mathcal{C}^\lambda f(x), \\ P(x,\eta_\epsilon(x)) = -P_{\epsilon y}(x)\mathcal{C}^\lambda f(x), \\ \psi_n(x) = -\mathcal{C}^\lambda(P_{\epsilon y}(x)\mathcal{C}^\lambda + \Omega\, id)f(x), \end{gathered}$$

where $id : L^2(\mathcal{S}_\epsilon) \to L^2(\mathcal{S}_\epsilon)$ is the identity operator.

Our next task is to relate $\psi(x,y)$ in $\mathcal{D}_\epsilon$ with $f(x) = \psi(x,\eta_\epsilon(x))$. Given $b \in L^2_{\rm per}(\mathcal{S}_\epsilon)$ let $\psi_b \in H^1(\mathcal{D}_\epsilon)$ be a weak solution of the elliptic partial differential equation

(5.7a) $$\Delta\psi + \gamma'(\psi_\epsilon)\psi - \gamma'(\psi_\epsilon)\int_{-\infty}^0 \lambda e^{\lambda s}\psi(X_\epsilon(s),Y_\epsilon(s))ds = 0 \quad \text{in } \mathcal{D}_\epsilon$$

(5.7b) $$\psi_n(x) := \partial_y\psi(x,\eta_\epsilon(x)) - \eta_{\epsilon x}(x)\partial_x\psi(x,\eta_\epsilon(x)) = b(x) \quad \text{on } \mathcal{S}_\epsilon,$$

(5.7c) $$\psi(x,0) = 0$$

such that ψ_b is $2\pi/\alpha$-periodic in the x-variable. Lemma 5.6 below proves that the boundary value problem (5.7) is uniquely solvable and $\psi_b \in H^1(\mathcal{D}_\epsilon)$ provided that $|\lambda - \lambda_0| \leqslant (\mathrm{Re}\,\lambda_0)/2$ and $(\eta_\epsilon(x), \psi_\epsilon(x,y))$ is near the trivial solution with the flat surface $y = h_0$ and the unstable shear flow $(U(y), 0)$ given in Theorem 5.1. This, together with the trace theorem, allows us to define an operator $\mathcal{T}_\epsilon : L^2_{\mathrm{per}}(\mathcal{S}_\epsilon) \to L^2_{\mathrm{per}}(\mathcal{S}_\epsilon)$ by

$$\mathcal{T}_\epsilon b(x) = \psi_b(x, \eta_\epsilon(x)), \tag{5.8}$$

which is the unique solution ψ_b of (5.7) restricted on the steady surface $\mathcal{S}_\epsilon$.

Be definition, it follows that

$$f(x) = \psi(x, \eta_\epsilon(x)) = \mathcal{T}_\epsilon \psi_n(x).$$

This, together with the boundary conditions written in terms of f as above yields that

$$f = -\mathcal{T}_\epsilon \mathcal{C}^\lambda (P_{\epsilon y}(x)\mathcal{C}^\lambda + \Omega\, id) f. \tag{5.9}$$

The growing-mode problem (5.1) is thus reduced to find a nontrivial solution $f(x) = \psi(x, \eta_\epsilon(x)) \in L^2_{\mathrm{per}}(\mathcal{S}_\epsilon)$ of the equation (5.9), or equivalently, to show that the operator

$$id + \mathcal{T}_\epsilon \mathcal{C}^\lambda (P_{\epsilon y}(x)\mathcal{C}^\lambda + \Omega\, id)$$

has a nontrivial kernel for some $\lambda \in \mathbb{C}$ with Re $\lambda > 0$.

The remainder of this subsection concerns with the unique solvability of (5.7). Our first task is to compare (5.7) at $\epsilon = 0$ with the Rayleigh system, which is useful in later consideration.

Lemma 5.4. *For $U \in \mathcal{K}^+$ for $y \in [0, h_0]$ and $U(h_0) \neq U_s$, let $-\alpha^2_{\max}$ be the lowest eigenvalue of $-\frac{d^2}{dy^2} - K(y)$ for $y \in (0, h_0)$ subject to the boundary conditions*

$$\phi'(h_0) = \left(\frac{g}{(U(h_0) - U_s)^2} + \frac{U'(h_0)}{U(h_0) - U_s} \right) \phi(h_0) \quad \text{and} \quad \phi(0) = 0, \tag{5.10}$$

where K is defined in (4.3). Let $-\alpha_n^2$ be the lowest eigenvalue of $-\frac{d^2}{dy^2} - K(y)$ for $y \in (0, h_0)$ subject to the boundary conditions

$$\phi'(h_0) = 0 \quad \text{and} \quad \phi(0) = 0. \tag{5.11}$$

If $g + U'(h_0)(U(h_0) - U_s) > 0$, then $\alpha_{\max} > \alpha_n$.

Proof. The argument is nearly identical to that in Remark 4.10. Let us denote by ϕ_m the eigenfunction of $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0, h_0)$ with (5.10) corresponding to the eigenvalue $-\alpha^2_{\max}$ and by ϕ_n the eigenfunction of $-\frac{d^2}{dy^2} - K(y)$ on $y \in (0, h_0)$ with (5.11) corresponding to the eigenvalue $-\alpha_n^2$. By the standard theory of Sturm-Liouville systems we may assume $\phi_m > 0$ and $\phi_n > 0$ on $y \in (0, h_0)$. An integration by parts yields that

$$\begin{aligned} 0 =& \int_0^{h_0} \Big(\phi_n(\phi_m'' - \alpha^2_{\max}\phi_m + K(y)\phi_m) - \phi_m(\phi_n'' - \alpha_n^2\phi_n + K(y)\phi_n) \Big) dy \\ =& (\alpha_n^2 - \alpha^2_{\max}) \int_0^{h_0} \phi_n\phi_m dy + \frac{g + U'(h_0)(U(h_0) - U_s)}{(U(h_0) - U_s)^2} \phi_m(h_0)\phi_n(h_0). \end{aligned}$$

The assumption $g + U'(h_0)(U(h_0) - U_s) > 0$ then proves the assertion. □

Our next task is the unique solvability of the homogeneous problem of (5.7).

Lemma 5.5. *Assume that* $U(h_0) \neq U_s$ *and* $g + U'(h_0)(U(h_0) - U_s) > 0$. *For* $\epsilon > 0$ *sufficiently small and* $|\lambda - \lambda_0| \leqslant (\operatorname{Re}\lambda_0)/2$, *the following elliptic partial differential equation*

$$\Delta\psi + \gamma'(\psi_\epsilon)\psi - \gamma'(\psi_\epsilon)\int_{-\infty}^{0} \lambda e^{\lambda s}\psi(X_\epsilon(s), Y_\epsilon(s))ds = 0 \quad \text{in } \mathcal{D}_\epsilon \tag{5.12a}$$

subject to

$$\psi(x,0) = 0 \tag{5.12b}$$

and the Neumann boundary condition

$$\psi_n = 0 \qquad \text{on} \quad \mathcal{S}_\epsilon \tag{5.12c}$$

admits only the trivial solution $\psi \equiv 0$.

A main difficulty in the proof of Lemma 5.5 is that the domain $\mathcal{D}_\epsilon$ depends on the small amplitude parameter $\epsilon > 0$ whereas the statement of Lemma 5.5 calls for an estimate of solutions of the system (5.12) uniform for $\epsilon > 0$. In order to compare (5.12) for different values of ϵ, we employ the *action-angle variables*, which map the domain $\mathcal{D}_\epsilon$ into a common domain independent of ϵ. For any $(x,y) \in \mathcal{D}_\epsilon$, let us denote by $\{(x',y') : \psi_\epsilon(x',y') = \psi_\epsilon(x,y) = p\}$ the streamline containing (x,y), by σ the arc-length variable on the streamline $\{\psi_\epsilon(x',y') = p\}$, and by $\sigma(x,y)$ the value of σ corresponding to the point (x,y) along the streamline. Let us define the normalized action-angle variables as

(5.13)

$$I = \frac{\alpha}{2\pi}\frac{h_0}{h_\epsilon}\iint_{\{\psi_\epsilon(x',y')<p\}} dy'dx', \quad \text{and} \quad \theta = \upsilon_\epsilon(I)\int_0^{\sigma(x,y)} \frac{1}{|\nabla\psi_\epsilon|}\Big|_{\{\psi_\epsilon(x'y')=p\}} d\sigma',$$

where h_0 and h_ϵ are the mean water depth at the parameter values 0 and ϵ, respectively, and

$$\upsilon_\epsilon(I) = \frac{2\pi}{\alpha}\left(\oint_{\{\psi_\epsilon(x',y')=p\}} \frac{1}{|\nabla\psi_\epsilon|}\right)^{-1}.$$

The action variable I represents the (normalized) area in the phase space under the streamline $\{(x',y') : \psi_\epsilon(x',y') = \psi_\epsilon(x,y) = p\}$ and the angle variable θ represents the position along the streamline of $\psi_\epsilon(x,y)$. The assumption of no stagnation, i.e. $\psi_{\epsilon y}(x,y) < 0$ throughout $\mathcal{D}_\epsilon$, implies that all stream lines are non-closed. For, otherwise, the horizontal velocity $\psi_{\epsilon y}$ must change signs on a closed streamline. Moreover, for $\epsilon > 0$ sufficiently small, all streamlines are close to those of the trivial flow, that is, they almost horizontal. Therefore, the action-angle variable (θ, I) is defined globally in $\mathcal{D}_\epsilon$. The mean-zero property (2.8) of the wave profile $\eta_\epsilon(x)$ implies that the area of the (steady) fluid region $\mathcal{D}_\epsilon$ is $(2\pi/\alpha)h_\epsilon$. Accordingly, by the definition in (5.13) it follows that $0 < I < h_0$ and $0 < \theta < 2\pi/\alpha$, independently of ϵ.

Let us define the mapping by the action-angle variables by $\mathcal{A}_\epsilon(x,y) = (\theta, I)$. From the above discussions follows that $\mathcal{A}_\epsilon$ is bijective and maps $\mathcal{D}_\epsilon$ to

$$D = \{(\theta, I) : 0 < \theta < 2\pi/\alpha,\ 0 < I < h_0\}.$$

At $\epsilon = 0$, the action-angle mapping reduces to the identity mapping on $\mathcal{D}_0 = D$. For $\epsilon > 0$, the mapping has a scaling effect. More precisely,

$$\iint_D f(\mathcal{A}_\epsilon^{-1}(\theta, I))d\theta dI = \frac{h_0}{h_\epsilon}\iint_{\mathcal{D}_\epsilon} f(x,y)dydx$$

for any function f defined in $\mathcal{D}_\epsilon$.

Another motivation to employ the action-angle variables comes from that they simplify the equation on the particle trajectory. In the action-angle variables (θ, I), the characteristic equation (3.7) becomes [3, Section 50]

$$\begin{cases} \dot\theta = -\upsilon_\epsilon(I) \\ \dot I = 0, \end{cases}$$

where the dot above a variable denotes the differentiation in the σ-variable. This observation will be useful in future considerations.

Proof of Lemma 5.5. Suppose on the contrary that there would exist sequences $\epsilon_k \to 0+$, $\lambda_k \to \lambda_0$ as $k \to \infty$ and $\psi_k \in H^1(\mathcal{D}_{\epsilon_k})$ such that $\psi_k \not\equiv 0$ is a solution of (5.12) with $\epsilon = \epsilon_k$. After normalization, $\|\psi_k\|_{L^2(\mathcal{D}_{\epsilon_k})} = 1$. We claim that

$$\|\psi_k\|_{H^2(\mathcal{D}_{\epsilon_k})} \leqslant C, \tag{5.14}$$

where $C > 0$ is independent of k. Indeed, by Minkovski's inequality it follows that

$$\begin{aligned} &\left\|\gamma'(\psi_{\epsilon_k})\psi_k - \gamma'(\psi_{\epsilon_k}) \int_{-\infty}^0 \lambda e^{\lambda s}\psi_k(X_{\epsilon_k}(s), Y_{\epsilon_k}(s))ds\right\|_{L^2(\mathcal{D}_{\epsilon_k})} \\ &\leqslant \|\gamma'(\psi_{\epsilon_k})\|_{L^\infty} \left(\|\psi_k\|_{L^2(\mathcal{D}_{\epsilon_k})} + \int_{-\infty}^0 |\lambda|\, e^{\operatorname{Re}\lambda s}\, \|\psi_k(X_{\epsilon_k}(s), Y_{\epsilon_k}(s))\|_{L^2(\mathcal{D}_{\epsilon_k})}\, ds\right) \\ &= \|\gamma'(\psi_{\epsilon_k})\|_{L^\infty} \|\psi_k\|_{L^2} \left(1 + \frac{|\lambda|}{\operatorname{Re}\lambda}\right) \leqslant \|\gamma'(\psi_{\epsilon_k})\|_{L^\infty} \left(1 + \frac{|\lambda_0| + \delta}{\delta}\right). \end{aligned} \tag{5.15}$$

This uses the fact that the mapping $(x, y) \to (X_{\epsilon_k}(s), Y_{\epsilon_k}(s))$ is measure preserving and that $|\lambda - \lambda_0| < \operatorname{Re}\lambda_0/2$. The standard elliptic regularity theory [2] for a Neumann problem adapted for (5.12) then proves the estimate (5.14). This proves the claim.

In order to study the convergence of $\{\psi_k\}$, we perform the mapping by the action-angle variables (5.13) and write $\mathcal{A}_{\epsilon_k}(x, y) = (\theta, I)$. Note that the image of $\mathcal{D}_{\epsilon_k}$ under the mapping $\mathcal{A}_{\epsilon_k}$ is

$$D = \{(\theta, I) : 0 < \theta < 2\pi/\alpha\,,\ 0 < I < h_0\},$$

which is independent of k. Let us denote $A\psi_k(\theta, I) = \psi_k(\mathcal{A}_{\epsilon_k}^{-1}(\theta, I))$. It is immediate to see that $A\psi_k \in H^2(D)$.

Since $h_\epsilon = h_0 + O(\epsilon)$ it follows that

$$\|A\psi_k\|_{L^2(D)} = (h_0/h_\epsilon)\|\psi_k\|_{L^2(\mathcal{D}_{\epsilon_k})} = 1 + O(\epsilon).$$

This, together with (5.14), implies that

$$\begin{aligned} A\psi_k &\to \psi_\infty \quad \text{weakly in } H^2(D) \quad \text{as } k \to \infty, \\ A\psi_k &\to \psi_\infty \quad \text{strongly in } L^2(D) \quad \text{as } k \to \infty \end{aligned}$$

for some ψ_∞. By continuity, $\|\psi_\infty\|_{L^2(D_0)} = 1$. Our goal is to show that $\psi_\infty \equiv 0$ and thus prove the assertion by contradiction.

At the limit as $\epsilon_k \to 0$, the limiting mapping $\mathcal{A}_0$ is the identity mapping on $D = \mathcal{D}_0$ and the limit function ψ_∞ satisfies

$$\begin{gathered}\Delta\psi_\infty + \gamma'(\psi_0)\psi_\infty - \gamma'(\psi_0)\int_{-\infty}^0 \lambda_0 e^{\lambda_0 s}\psi_\infty(X_0(s),Y_0(s))ds = 0 \qquad \text{in} \quad D;\\ \partial_y\psi_\infty(x,h_0) = 0;\\ \psi_\infty(x,0) = 0.\end{gathered} \tag{5.16}$$

Here, $\lambda_0 = -i\alpha c_\alpha$ and $(X_0(s),Y_0(s)) = (x+U(y)s,y)$.

Since the above equation and the boundary conditions are separable in the x and y variables, ψ_∞ can be written as

$$\psi_\infty(x,y) = \sum_{l=0}^{\infty} e^{il\alpha x}\phi_l(y).$$

In case $l=0$, the boundary value problem (5.16) reduces to

$$\begin{cases}\phi_0'' = 0 & \text{for} \quad y\in(0,h_0)\\ \phi_0'(h_0) = 0, \quad \phi_0(0) = 0,\end{cases}$$

and thus, $\phi_0 \equiv 0$.

Next, consider the solution ϕ_1 of (5.16) when $l=1$:

$$\begin{cases}\phi_1'' - \alpha^2\phi_1 - \frac{U''}{U-c_\alpha}\phi_1 = 0 & \text{for} \quad y\in(0,h_0)\\ \phi_1'(h_0) = 0, \quad \phi_1(0) = 0.\end{cases}$$

This uses that $\gamma'(\psi_0) = -U''/U$. Recall that to the unstable wave number α and the unstable wave speed c_α is associated an unstable solution ϕ_α satisfying (5.4). As is done in the proof of Lemma 5.4, an integration by parts yields that

$$\begin{aligned}0 &= \int_0^{h_0}\left(\phi_1\left(\phi_\alpha'' - \alpha^2\phi_\alpha - \frac{U''}{U-c_\alpha}\phi_\alpha\right) - \phi_\alpha\left(\phi_1'' - \alpha^2\phi_1 - \frac{U''}{U-c_\alpha}\phi_1\right)\right)dy\\ &= \left(\frac{g}{(U(h_0)-c_\alpha)^2} + \frac{U'(h_0)}{U(h_0)-c_\alpha}\right)\phi_\alpha(h_0)\phi_1(h_0).\end{aligned}$$

Since $\phi_\alpha(h_0) \neq 0$, this implies $\phi_1(h_0) = 0$ and, in turn, $\phi_1 \equiv 0$. In addition, $\phi_1'(h_0) = 0$.

For $l \geqslant 2$, the solution ϕ_l of (5.16) ought to satisfy

$$\begin{cases}\phi_l'' - l^2\alpha^2\phi_l - \frac{U''}{U-c_\alpha/l}\phi_l = 0 & \text{for} \quad y\in(0,h_0)\\ \phi_l'(h_0) = 0, \quad \phi_l(0) = 0.\end{cases}$$

An integration by parts yields that

$$\int_0^{h_0}\Big(|\phi_l'|^2 + l^2\alpha^2|\phi_l|^2 + \frac{U''}{U-c_\alpha/l}|\phi_l|^2\Big)dy = 0,$$

and subsequently, for any q real follows that (see the proof of Lemma 4.6)

$$\int_0^{h_0}\Big(|\phi_l'|^2 + l^2\alpha^2|\phi_l|^2 + \frac{U''(U-q)}{|U-c_\alpha/l|^2}|\phi_l|^2\Big)dy = 0.$$

The same argument as in Lemma 4.6 applies to assert that

$$\int_0^{h_0}(|\phi_l'|^2 + l^2\alpha^2|\phi_l|^2)dy \leqslant \int_0^{h_0} K(y)|\phi_l|^2 dy.$$

Recall that $K(y) = -U''(y)/(U(y) - U_s) > 0$.

Let $-\alpha_n^2$ be as in Lemma 5.4 the lowest eigenvalue of $\frac{d^2}{dy^2} - K(y)$ on $y \in (0, h_0)$ with the boundary conditions $\phi'(h_0) = 0$ and $\phi(0) = 0$. By Lemma 5.4, $\alpha_{\max} > \alpha_n$. On the other hand, the variational characterization of $-\alpha_n^2$ asserts that

$$\int_0^h (|\phi_l'|^2 - K(y)|\phi_l|^2)dy \geqslant -\alpha_n^2 \int_0^h |\phi_l|^2 dy.$$

Accordingly,

$$\int_0^{h_0} (l^2\alpha^2 - \alpha_n^2)|\phi_l|^2 dy \leqslant 0$$

must hold. Since $\alpha > \alpha_{\max}/2$ and $l \geqslant 2$, it follows that $l^2\alpha^2 - \alpha_n > 2\alpha_{\max}^2 - \alpha_n^2 > 0$. Consequently, $\phi_l \equiv 0$.

Therefore, $\psi_\infty \equiv 0$, which contradicts since $\|\phi_\infty\|_{L^2} = 1$. This completes the proof. □

Lemma 5.6. *Under the assumption of Lemma 5.5, for any $b \in L^2(\mathcal{S}_\epsilon)$, there exists a unique solution ψ_b to (5.7). Moreover, the estimate*

$$\|\psi_b\|_{H^1(\mathcal{D}_\epsilon)} \leqslant C\|b\|_{L^2(\mathcal{S}_\epsilon)} \tag{5.17}$$

holds, where $C > 0$ is independent of ϵ, b.

Proof. The proof uses the theory of Fredholm alternative as adapted to usual elliptical problems [22, Section 6.2]. Let us introduce the Hilbert space

$$H(\mathcal{D}_\epsilon) = \{\psi \in H^1(\mathcal{D}_\epsilon) : \psi(x, 0) = 0\} \tag{5.18}$$

and a bilinear form $B_z : H \times H \to \mathbb{R}$, defined as

$$B_z[\phi, \psi] = \iint_{\mathcal{D}_\epsilon} \left(\nabla\phi \cdot \nabla\psi^* + \phi(K^\lambda\psi)^*\right) dydx + z\,(\phi, \psi)\,.$$

Here,

$$K^\lambda\psi = -\gamma(\psi_\epsilon)\psi + \gamma'(\psi_\epsilon)\int_{-\infty}^0 \lambda e^{\lambda s}\psi(X_\epsilon(s), Y_\epsilon(s))ds,$$

$z \in \mathbb{R}$ and $(\cdot, \cdot)$ denotes the $L^2(\mathcal{D}_\epsilon)$ inner product. By the estimate (5.15) it follows that

$$|B_z[\phi, \psi]| \leqslant (C(\gamma, \delta) + z)\|\phi\|_{H^1}\|\psi\|_{H^1} \tag{5.19}$$

and

$$B_z[\phi, \phi] \geqslant c_0\|\phi\|_{H^1}^2 \tag{5.20}$$

for $z > 2C(\gamma, \delta)$, where

$$C(\gamma, \delta) = \|\gamma'(\psi_\epsilon)\|_{L^\infty}\left(1 + \frac{|\lambda_0| + \delta}{\delta}\right)$$

and $c_0 = \min(1, C(\gamma, \delta)) > 0$. Then, the Lax-Milgram theorem applies and there exists a bounded operator $L_z : H^* \to H$ such that $B_z[L_z f, \phi] = \langle f, \phi\rangle$ for any $f \in H^*$ and $\phi \in H$, where H^* denotes the dual space of H and $\langle\cdot, \cdot\rangle$ is the duality pairing. For $b \in L^2(\mathcal{S}_\epsilon)$, the trace theorem [22, Section 5.5] permits us to define $b^* \in H^*$ by

$$< b^*, \psi >= \oint_{\mathcal{S}_\epsilon} b(x)\psi^*(x, \eta_\epsilon(x))dx \qquad \text{for} \quad \psi \in H.$$

Note that $\psi_b \in H$ is a weak solution of (5.7) if and only if

$$B_z(\psi_b, \phi) = < z\psi_b + b^*, \phi > \qquad \text{for all} \quad \phi \in H.$$

That is to say, $\psi_b = L_z(z\psi_b + b^*)$, or equivalently

$$(id - zL_z)\psi_b = L_z b^*. \tag{5.21}$$

The operator $L_z : L^2(\mathcal{D}_\epsilon) \to L^2(\mathcal{D}_\epsilon)$ is compact. Indeed,

$$\|L_z\phi\|_{H^1(\mathcal{D}_\epsilon)} \leqslant \|L_z\|_{H^* \to H} \|\phi\|_{H^*} \leqslant C \|\phi\|_{L^2(\mathcal{D}_\epsilon)}$$

for any $\phi \in L^2(\mathcal{D}_\epsilon)$. Moreover, the result of Lemma 5.5 states that $\ker(I - zL_z) = \{0\}$. Thus, by the Fredholm alternative theory for compact operators, the equation (5.21) is uniquely solvable for any $b \in L^2(\mathcal{S}_\epsilon)$ and

$$\psi_b = (id - zL_z)^{-1} L_z b^*. \tag{5.22}$$

Next is the proof of (5.17). From (5.22) it follows that

$$\|\psi_b\|_{L^2(\mathcal{D}_\epsilon)} \leqslant \left\|(id - zL_z)^{-1}\right\|_{L^2 \to L^2} \|L_z\|_{H^* \to H} \|b^*\|_{H^*} \leqslant C\|b\|_{L^2(\mathcal{S}_\epsilon)},$$

where $C > 0$ is independent of b and ϵ. Then, by (5.21) it follows that

$$\begin{aligned} \|\psi_b\|_{H^1(\mathcal{D}_\epsilon)} &= \|L_z(z\psi_b) + L_z b^*\|_H \leqslant C(\|\psi_b\|_{L^2(\mathcal{D}_\epsilon)} + \|b\|_{L^2(\mathcal{S}_\epsilon)}) \\ &\leqslant C'\|b\|_{L^2(\mathcal{S}_\epsilon)}, \end{aligned}$$

where $C, C' > 0$ are independent of b and ϵ. This completes the proof. □

Similar considerations to the above proves the unique solvability of the inhomogeneous problem.

Corollary 5.7. *Under the assumption of Lemma 5.5, for any $f \in H^*(\mathcal{D}_\epsilon)$ there exists an unique weak solution $\psi_f \in H^1(\mathcal{D}_\epsilon)$ to the elliptic problem*

$$\begin{gathered} \Delta\psi + \gamma'(\psi_\epsilon)\psi - \gamma'(\psi_\epsilon)\int_{-\infty}^0 \lambda e^{\lambda s}\psi(X_\epsilon(s), Y_\epsilon(s))ds = f \quad \text{in } \mathcal{D}_\epsilon, \\ \psi_n = 0 \qquad \text{on} \quad \mathcal{S}_\epsilon, \\ \psi(x, 0) = 0. \end{gathered}$$

The estimate

$$\|\psi_f\|_{H^1(\mathcal{D}_\epsilon)} \leqslant C\|f\|_{H^*(\mathcal{D}_\epsilon)} \tag{5.23}$$

holds, where $C > 0$ is independent of f and ϵ. Moreover, if $f \in L^2(\mathcal{D}_\epsilon)$ then an improved estimate

$$\|\psi_f\|_{H^2(\mathcal{D}_\epsilon)} \leqslant C\|f\|_{L^2(\mathcal{D}_\epsilon)} \tag{5.24}$$

holds, where $C > 0$ is independent of f and ϵ.

Proof. As in the proof of Lemma 5.6, the function ψ_f is a solution to the above boundary value problem if and only if

$$(id - zL_z)\psi_f = L_z f.$$

The unique solvability and the H^1-estimate (5.23) are identically the same as those in Lemma 5.6. When $f \in L^2(\mathcal{D}_\epsilon)$ the elliptic regularity theory [2] for Neumann boundary condition implies (5.24). □

For any $b \in L^2_{\mathrm{per}}(\mathcal{S}_\epsilon)$ let us define the operator

$$\mathcal{T}_\epsilon b = \psi_b|_{\mathcal{S}_\epsilon} = \psi_b\left(x, \eta_\epsilon(x)\right), \tag{5.25}$$

where ψ_b is the unique solution of (5.7) in Lemma 5.6 with periodicity. Then, the elliptic estimate (5.17) and the trace theorem [22, Section 5.5] implies that

$$\|\mathcal{T}_\epsilon b\|_{H^{1/2}(\mathcal{S}_\epsilon)} \leqslant C\|\psi_b\|_{H^1(\mathcal{D}_\epsilon)} \leqslant C'\|b\|_{L^2}; \tag{5.26}$$

Therefore, $\mathcal{T}_\epsilon : L^2_{\mathrm{per}}(\mathcal{S}_\epsilon) \to L^2_{\mathrm{per}}(\mathcal{S}_\epsilon)$ is a compact operator.

5.3. **Proof of Theorem 5.1.** This subsection is devoted to the proof of Theorem 5.1 pertaining to the linear instability of small-amplitude periodic traveling waves over an unstable shear flow.

For $\epsilon \geqslant 0$ sufficiently small and $|\lambda - \lambda_0| \leqslant (\mathrm{Re}\,\lambda_0)/2$, let us denote

$$\mathcal{F}(\lambda, \epsilon) = \mathcal{T}\left(\lambda, \epsilon\right) \mathcal{C}\left(\lambda, \epsilon\right) \left(P_{\epsilon y}\mathcal{C}\left(\lambda, \epsilon\right) + \Omega\, id\right) : L^2_{\mathrm{per}}(\mathcal{S}_\epsilon) \to L^2_{\mathrm{per}}(\mathcal{S}_\epsilon), \tag{5.27}$$

where $\mathcal{C}(\lambda, \epsilon) = \mathcal{C}^\lambda$ and $\mathcal{T}(\lambda, \epsilon) = \mathcal{T}_\epsilon$ are defined in (5.5) and (5.25), respectively. In the light of the discussion in the previous subsection, it suffices to show that for each small parameter $\epsilon \geqslant 0$ there exists $\lambda(\epsilon)$ with $|\lambda(\epsilon) - \lambda_0| \leqslant (\mathrm{Re}\,\lambda_0)/2$ such that the operator $id + \mathcal{F}(\lambda(\epsilon), \epsilon)$ has a nontrivial null space. The result of Theorem 4.2 states that there exists $\lambda_0 = -i\alpha c_\alpha$ with $\mathrm{Im}\, c_\alpha > 0$ such that $id + \mathcal{F}(\lambda_0, 0)$ has a nontrivial null space. The proof for $\epsilon > 0$ uses a perturbation argument, based on the following lemma due to Steinberg [41].

Lemma 5.8. *Let $F(\lambda, \epsilon)$ be a family of compact operators on a Banach space, analytic in λ in a region Λ in the complex plane and jointly continuous in (λ, ϵ) for each $(\lambda, \epsilon) \in \Lambda \times \mathbb{R}$. Suppose that $id - F(\lambda_0, \epsilon)$ is invertible for some $\lambda_0 \in \Lambda$ and all $\epsilon \in \mathbb{R}$. Then, $R(\lambda, \epsilon) = (id - F(\lambda, \epsilon))^{-1}$ is meromorphic in Λ for each $\epsilon \in \mathbb{R}$ and jointly continuous at (z_0, ϵ_0) if λ_0 is not a pole of $R(\lambda, \epsilon)$; its poles depend continuously on ϵ and can appear or disappear only at the boundary of Λ.*

In order to apply Lemma 5.8 to our situation, we need to transform the operator (5.27) to one on a function space independent of the parameter ϵ. This calls for the employment of the action-angle mapping $\mathcal{A}_\epsilon$, as is done in the proof of Lemma 5.5:

$$\mathcal{A}_\epsilon : \mathcal{D}_\epsilon \to D \quad \text{and} \quad \mathcal{A}_\epsilon(x, y) = (\theta, I),$$

where the action-angle variables (θ, I) are defined in (5.13) and

$$D = \{(\theta, I) : 0 < \theta < 2\pi/\alpha\,,\, 0 < I < h_0\}.$$

Note that $\mathcal{A}_\epsilon$ maps $\mathcal{S}_\epsilon$ bijectively to $\{(\theta, h_0) : 0 < \theta < 2\pi/\alpha\}$. The latter may be identified with $(2\pi/\alpha, h_0)$. This naturally induces an homeomorphism

$$\mathcal{B}_\epsilon : L^2_{\mathrm{per}}(\mathcal{S}_\epsilon) \to L^2_{\mathrm{per}}([0, 2\pi/\alpha])$$

by

$$(\mathcal{B}_\epsilon f)(\theta) = f(\mathcal{A}_\epsilon^{-1}\left(\theta, h_0\right)).$$

Let us denote the following operators from $L^2_{\mathrm{per}}\left([0, 2\pi/\alpha]\right)$ to itself:

$$\tilde{\mathcal{T}}\left(\lambda, \epsilon\right) = \mathcal{B}_\epsilon \mathcal{T}\left((\lambda, \epsilon)\right)(\mathcal{B}_\epsilon)^{-1},$$
$$\tilde{\mathcal{C}}\left(\lambda, \epsilon\right) = \mathcal{B}_\epsilon \mathcal{C}\left(\lambda, \epsilon\right)(\mathcal{B}_\epsilon)^{-1},$$
$$\tilde{\mathcal{P}}_\epsilon = \mathcal{B}_\epsilon P_{\epsilon y}(\mathcal{B}_\epsilon)^{-1},$$

and

$$\tilde{\mathcal{F}}(\lambda, \epsilon) = \mathcal{B}_\epsilon \mathcal{F}(\lambda, \epsilon)(\mathcal{B}_\epsilon)^{-1} = \tilde{\mathcal{T}}\left(\lambda, \epsilon\right)\tilde{\mathcal{C}}(\lambda, \epsilon)(\tilde{\mathcal{P}}_\epsilon \tilde{\mathcal{C}}\left(\lambda, \epsilon\right) + \Omega id). \tag{5.28}$$

Since $\mathcal{T}(\lambda,\epsilon)$ is compact and $\mathcal{T}(\lambda,\epsilon)$ and $\mathcal{C}(\lambda,\epsilon)$ are analytic in λ with $|\lambda-\lambda_0|\leqslant(\operatorname{Re}\lambda_0)/2$, the operator $\mathcal{F}(\lambda,\epsilon)$ is compact and analytic in λ. Subsequently, $\tilde{\mathcal{F}}(\lambda,\epsilon)$ is compact and analytic in λ. Clearly, $P_{\epsilon y}(x)$, $\mathcal{C}(\lambda,\epsilon)$ and $\mathcal{B}_\epsilon$ are continuous in ϵ, and in turn, $\tilde{\mathcal{C}}(\lambda,\epsilon)$ and $\tilde{\mathcal{P}}_\epsilon$ are continuous in ϵ. Thus, it remains to show the continuity of $\tilde{\mathcal{T}}(\lambda,\epsilon)$ in ϵ to obtain the the continuity of $\tilde{\mathcal{F}}(\lambda,\epsilon)$ in ϵ.

Lemma 5.9. *For $\epsilon\geqslant 0$ sufficiently small and $|\lambda-\lambda_0|\leqslant(\operatorname{Re}\lambda_0)/2$, the operator $\tilde{\mathcal{T}}(\lambda,\epsilon)$ satisfies the estimate*

$$\|\tilde{\mathcal{T}}(\lambda,\epsilon_1)-\tilde{\mathcal{T}}(\lambda,\epsilon_2)\|_{L^2_{per}([0,2\pi/\alpha])\to L^2_{per}([0,2\pi/\alpha])}\leqslant C|\epsilon_1-\epsilon_2|, \tag{5.29}$$

where $C>0$ is independent of λ and ϵ.

Proof. For a given $b\in L^2_{\text{per}}([0,2\pi/\alpha])$, let us denote $b_\epsilon=\mathcal{B}_\epsilon^{-1}b\in L^2(\mathcal{S}_\epsilon)$. By definition

$$\tilde{\mathcal{T}}\,(\lambda,\epsilon)\,b=\mathcal{B}_\epsilon\mathcal{T}_\epsilon b_\epsilon=\mathcal{B}_\epsilon(\psi_{b,\epsilon}|_{\mathcal{S}_\epsilon}),$$

where $\psi_{b,\epsilon}\in H^1\,(\mathcal{D}_\epsilon)$ is the unique weak solution of

$$\Delta\psi_{b,\epsilon}+\gamma'(\psi_\epsilon)\psi_{b,\epsilon}-\gamma'(\psi_\epsilon)\int_{-\infty}^0\lambda e^{\lambda s}\psi_{b,\epsilon}(X_\epsilon(s),Y_\epsilon(s))ds=0\quad\text{in }\mathcal{D}_\epsilon; \tag{5.30a}$$

$$(\psi_{b,\epsilon})_n=b_\epsilon\quad\text{on }\mathcal{S}_\epsilon; \tag{5.30b}$$

$$\psi_{b,\epsilon}(x,0)=0. \tag{5.30c}$$

Our goal is to estimate the L^2-operator norm of $\tilde{\mathcal{T}}\,(\lambda,\epsilon_1)-\tilde{\mathcal{T}}\,(\lambda,\epsilon_2)$ in terms of $|\epsilon_1-\epsilon_2|$. Since the domain of the boundary value problem (5.30) depends on ϵ, we use the action-angle mapping $\mathcal{A}_{\epsilon_j}$ $(j=1,2)$ to transform functions and the Laplacian operator in $\mathcal{D}_{\epsilon_j}$ $(j=1,2)$ to those in the fixed domain D. For the simplicity of notation, we write $\mathcal{A}_{\epsilon_j}$ $(j=1,2)$ to denote the induced transformations for functions and operators. Let $\psi_j=\mathcal{A}_{\epsilon_j}(\psi_{b,\epsilon_j})$, which are $H^1(D)$-functions, and let

$$\Delta_j=\mathcal{A}_{\epsilon_j}(\Delta),\qquad\gamma_j(I)=\mathcal{A}_{\epsilon_j}(\gamma'(\psi_{\epsilon_j})),$$

where $j=1,2$. By definition, $\mathcal{A}_{\epsilon_j}(b_{\epsilon_j})=\mathcal{B}_{\epsilon_j}(b_{\epsilon_j})=b$. Note that the characteristic equation (3.7) in the action-angle variables (θ,I) becomes

$$\begin{cases}\dot{\theta}=\upsilon_j(I)\\ \dot{I}=0\end{cases}$$

for $j=1,2$, where $\upsilon_j(I)=\upsilon_{\epsilon_j}(I)$. That means, the trajectory $(X_{\epsilon_j}(s),Y_{\epsilon_j}(s))$ in the phase space transforms under the mapping $\mathcal{A}_{\epsilon_j}$ into $(\theta+\upsilon_j(I)s,I)$. Since ψ_j $(j=1,2)$ are $2\pi/\alpha$-periodic in θ, it is equipped with the Fourier expansion

$$\psi_j(\theta,I)=\sum_l e^{il\theta}\psi_{j,l}(I).$$

Under the action-angle mapping $\mathcal{A}_{\epsilon_j}$ $(j=1,2)$ the left side of (5.30) becomes

$$\begin{aligned}
&\mathcal{A}_{\epsilon_j}\left(\gamma'(\psi_\epsilon)\psi_{b,\epsilon}-\gamma'(\psi_\epsilon)\int_{-\infty}^{0}\lambda e^{\lambda s}\psi_{b,\epsilon}(X_\epsilon(s),Y_\epsilon(s))ds\right)\\
&=\gamma_j(I)\left(\sum_l e^{il\theta}\psi_{j,l}(I)-\int_{-\infty}^{0}\lambda e^{\lambda s}\sum_l e^{il(\theta+\upsilon_j(I)s)}\psi_{j,l}(I)ds\right) \qquad (5.31)\\
&=\gamma_j(I)\sum_l \frac{il\upsilon_j(I)}{\lambda+il\upsilon_j(I)}\psi_{j,l}(I)e^{il\theta},
\end{aligned}$$

and thus the system (5.30) becomes

$$\begin{aligned}
\Delta_j\psi_j+\gamma_j(I)\sum_l \frac{il\upsilon_j(I)}{\lambda+il\upsilon_j(I)}\psi_{j,l}(I)e^{il\theta}&=0 \quad \text{in } D,\\
\partial_I\psi_j(\theta,h_0)&=b(\theta),\\
\psi_j(\theta,0)&=0.
\end{aligned}$$

Accordingly, the difference $\psi_1-\psi_2$ is a weak solution of the partial differential equation

(5.32)

$$\begin{aligned}
&\Delta_1(\psi_1-\psi_2)+(\Delta_1-\Delta_2)\psi_2+\gamma_1\sum_l\frac{il\upsilon_1}{\lambda+il\upsilon_1}(\psi_{1,l}(I)-\psi_{2,l}(I))e^{il\theta}\\
&\quad+\gamma_1\sum_l\left(\frac{il\upsilon_1}{\lambda+il\upsilon_1}-\frac{il\upsilon_2}{\lambda+il\upsilon_2}\right)\psi_{2,l}(I)e^{il\theta}+(\gamma_1-\gamma_2)\sum_l\frac{il\upsilon_2}{\lambda+il\upsilon_2}\psi_{2,l}(I)e^{il\theta}=0
\end{aligned}$$

with the boundary conditions

$$\partial_I(\psi_1-\psi_2)(\theta,h_0)=0, \qquad (5.33a)$$

$$(\psi_1-\psi_2)(\theta,0)=0. \qquad (5.33b)$$

Let us write (5.32) as

$$\Delta_1(\psi_1-\psi_2)+\gamma_1\sum_l\frac{il\upsilon_1(I)}{\lambda-ik\upsilon_1(I)}(\psi_{1,l}(I)-\psi_{2,l}(I))e^{il\theta}=f, \qquad (5.34)$$

where $f=f_1+f_2+f_3$ and

$$\begin{aligned}
f_1&=-(\Delta_1-\Delta_2)\psi_2,\\
f_2&=-\gamma_1\sum_l\frac{i\lambda l(\upsilon_1-\upsilon_2)}{(\lambda+il\upsilon_1)(\lambda+il\upsilon_2)}\psi_{2,l}(I)e^{il\theta},\\
f_3&=-(\gamma_1-\gamma_2)\sum_l\frac{il\upsilon_2}{\lambda+il\upsilon_2}\psi_{2,l}(I)e^{il\theta}.
\end{aligned}$$

We estimate f_1, f_2, f_3 separately. To simplify notations, $C>0$ in the estimates below denotes a generic constant independent of ϵ and λ.

First, we claim that $f_1\in H^*(D)$ with the estimates

$$\|f_1\|_{H^*(D_0)}\leqslant C|\epsilon_1-\epsilon_2|\|b\|_{L^2([0,2\pi/\alpha])}, \qquad (5.35)$$

where $H^*(D)$ is the dual space of

$$H(D)=\{\psi\in H^1(D):\psi(\theta,I)=\psi(\theta+2\pi/\alpha,I),\ \psi(\theta,0)=0\}.$$

Let us write

$$\Delta_j = a^j_{II}\partial_{II} + a^j_{I\theta}\partial_{I\theta} + a^j_{\theta\theta}\partial_{\theta\theta} + b^j_I\partial_I + b^j_\theta\partial_\theta \qquad \text{for} \quad j = 1,2,$$

and the difference of the coefficients as

$$\bar{a}_{II} = a^1_{II} - a^2_{II}, \quad \bar{a}_{I\theta} = a^1_{I\theta} - a^2_{I\theta}, \quad \bar{a}_{\theta\theta} = a^1_{\theta\theta} - a^2_{\theta\theta},$$
$$\bar{b}_I = b^1_I - b^2_I, \quad \text{and} \quad \bar{b}_\theta = b^1_\theta - b^2_\theta.$$

Then formally, for any $\phi \in H(D) \cap C^2(\bar{D})$ it follows that

$$\begin{aligned}(5.36)\qquad &\int_{D_0} f_1\phi \, dId\theta \\ =\; &\int_{D_0} -\phi\left(\bar{a}_{II}\partial_{II} + \bar{a}_{I\theta}\partial_{I\theta} + \bar{a}_{\theta\theta}\partial_{\theta\theta} + \bar{b}_\theta\partial_\theta + \bar{b}_I\partial_I\right)\psi_2 dId\theta \\ =\; &\int_{D_0} \left[\partial_I\psi_2\partial_I\left(\bar{a}_{II}\phi\right) + \partial_I\psi_2\partial_\theta\left(\bar{a}_{I\theta}\phi\right) + \partial_\theta\psi_2\partial_\theta\left(\bar{a}_{\theta\theta}\phi\right)\right] dId\theta \\ &- \int_{D_0} \phi\left(\bar{b}_I\partial_I\psi_2 + \bar{b}_\theta\partial_\theta\psi_2\right) dId\theta - \int_{\{I=h_0\}} \bar{a}_{II}\phi b\left(\theta\right) d\theta,\end{aligned}$$

This uses that ψ_2 and ϕ are periodic in the θ-variable and that

$$\partial_I\psi_2(\theta, h_0) = b(\theta), \qquad \phi(\theta, 0) = 0.$$

Note that the elliptic estimate (5.17) and the equivalence of norms under the transformation $\mathcal{A}^{-1}_{\epsilon_2}$ assert that

$$\begin{aligned}\|\psi_2\|_{H^1(D)} &\leqslant C\left\|\mathcal{A}^{-1}_{\epsilon_2}\psi_2\right\|_{H^1(\mathcal{D}_{\epsilon_2})} = C\left\|\psi_{b,\epsilon_2}\right\|_{H^1(\mathcal{D}_{\epsilon_2})} \\ &\leqslant C\left\|b_{\epsilon_2}\right\|_{L^2(\mathcal{S}_{\epsilon_2})} \leqslant C\left\|b\right\|_{L^2([0,2\pi/\alpha])}.\end{aligned}$$

Since

$$\left|\bar{a}_{II}\right|_{C^1} + \left|\bar{a}_{I\theta}\right|_{C^1} + \left|\bar{a}_{\theta\theta}\right|_{C^1} + \left|\bar{b}_\theta\right|_{C^1} + \left|\bar{b}_I\right|_{C^1} = O(|\epsilon_1 - \epsilon_2|),$$

by using the trace theorem it follow from (5.36) the estimate

$$\begin{aligned}\left|\int_D f_1\phi \, dId\theta\right| &\leqslant C\left|\epsilon_1 - \epsilon_2\right|\left(\|\psi_2\|_{H^1(D)} + \|b\|_{L^2([0,2\pi/\alpha])}\right)\|\phi\|_{H^1(D)} \\ &\leqslant C\left|\epsilon_1 - \epsilon_2\right|\|b\|_{L^2([0,2\pi/\alpha])}\|\phi\|_{H^1(D)}.\end{aligned}$$

This proves the estimate (5.35).

We claim that if b is smooth then the formal manipulations in (5.36) are valid and $\psi_2 \in C^2(\bar{D})$. Note that Theorem 2.2 ensures that the steady state $(\eta_{\epsilon_2}(x), \psi_{\epsilon_2}(x,y))$ is in $C^{3+\beta}$ class, where $\beta \in (0,1)$. Since b is smooth it follows that $b_{\epsilon_2} = \mathcal{B}^{-1}_{\epsilon_2}b$ is at least in $H^2(\mathcal{S}_{\epsilon_2})$. Then, the similar argument as in the regularity proof of Theorem 5.1 asserts that $\psi_{b,\epsilon_2} \in H^{7/2}(\mathcal{D}_{\epsilon_2}) \subset C^2(\bar{\mathcal{D}}_{\epsilon_2})$. Since the definition of the action-angle variables guarantees that the mapping $\mathcal{A}_{\epsilon_2}$ is at least of C^2, subsequently, $\psi_2 = \mathcal{A}_{\epsilon_2}\psi_{b,\epsilon_2} \in C^2(\bar{D})$. This proves the claim. If $b \in L^2$, an approximation of b by smooth functions asserts (5.35). This proves the claim.

Next, since

$$\left|\frac{1}{\lambda + il\upsilon_j}\right| = \frac{1}{(|\operatorname{Re}\lambda|^2 + |\operatorname{Im}\lambda - l\upsilon_j|^2)^{1/2}} \leqslant \frac{1}{|\operatorname{Re}\lambda|} \leqslant \frac{2}{\operatorname{Re}\lambda_0},$$

by the estimate (5.17) it follows that

$$(5.37)\qquad \|f_2\|_{L^2(D)} \leqslant C|\epsilon_1 - \epsilon_2|\|\psi_2\|_{L^2(D)} \leqslant C|\epsilon_1 - \epsilon_2|\|b\|_{L^2([0,2\pi/\alpha])}.$$

Similarly,

$$\|f_3\|_{L^2(D_0)} \leqslant C|\epsilon_1 - \epsilon_2|\|b\|_{L^2([0,2\pi/\alpha])}. \tag{5.38}$$

Combining the estimates (5.35), (5.37) and (5.38) asserts that $f \in H^*(D)$ and

$$\|f\|_{H^*(D)} \leqslant C|\epsilon_1 - \epsilon_2|\|b\|_{L^2([0,2\pi/\alpha])}.$$

Let $\psi = \psi_1 - \psi_2 \in H^1(D)$ and $\phi = \mathcal{A}_{\epsilon_1}^{-1}\psi \in H^1(\mathcal{D}_{\epsilon_1})$. It remains to transform back to the physical space of the boundary value problem for ψ and to compute the operator norm of $\tilde{\mathcal{T}}(\lambda, \epsilon_1) - \tilde{\mathcal{T}}(\lambda, \epsilon_2)$. Under the transformation $\mathcal{A}_{\epsilon_1}^{-1}$, the equations (5.34), (5.33a), (5.33b) become

$$\begin{aligned} \Delta\phi + \gamma'(\psi_{\epsilon_1})\phi - \gamma'(\psi_{\epsilon_1}) \int_{-\infty}^{0} \lambda e^{\lambda s}\phi(X_{\epsilon_1}(s), Y_{\epsilon_1}(s))ds = \mathcal{A}_{\epsilon_1}^{-1} f \quad \text{in } \mathcal{D}_{\epsilon_1}; \\ (\phi^{\epsilon_1})_n = 0 \quad \text{on } \mathcal{S}_{\epsilon_1}; \\ \phi^{\epsilon_1}(x,0) = 0, \end{aligned}$$

Then, $\mathcal{A}_{\epsilon_1}^{-1} f \in H^*(\mathcal{D}_{\epsilon_1})$ and

$$\left\|\mathcal{A}_{\epsilon_1}^{-1} f\right\|_{H^*(\mathcal{D}_{\epsilon_1})} \leqslant C\left\|f\right\|_{H^*(D)} \leqslant C|\epsilon_1 - \epsilon_2|\|b\|_{L^2([0,2\pi/\alpha])}.$$

Corollary 5.7 thus applies to assert that

$$\begin{aligned} \|\psi\|_{H^1(D)} \leqslant \|\phi\|_{H^1(\mathcal{D}_{\epsilon_1})} &\leqslant C\left\|\mathcal{A}_{\epsilon_1}^{-1} f\right\|_{H^*(\mathcal{D}_{\epsilon_1})} \\ &\leqslant C|\epsilon_1 - \epsilon_2|\|b\|_{L^2([0,2\pi/\alpha])}. \end{aligned}$$

Finally, by the trace theorem it follows

$$\begin{aligned} \|\tilde{\mathcal{T}}(\lambda, \epsilon_1)\, b - \tilde{\mathcal{T}}(\lambda, \epsilon_2)\, b\|_{L^2_{\text{per}}([0,2\pi/\alpha])} &= \|(\psi_1 - \psi_2)(\theta, h_0)\|_{L^2_{\text{per}}([0,2\pi/\alpha])} \\ &\leqslant C\|\psi\|_{H^1(D)} \leqslant C|\epsilon_1 - \epsilon_2|\|b\|_{L^2_{\text{per}}([0,2\pi/\alpha])}. \end{aligned}$$

This completes the proof. □

We are now in a position to prove our main theorem.

Proof of Theorem 5.1. For $|\lambda - \lambda_0| \leqslant (\operatorname{Re}\lambda_0)/2$, where $\lambda_0 = -i\alpha c_\alpha$, and $\epsilon \geqslant 0$ small, consider the family of operators $\tilde{\mathcal{F}}(\lambda, \epsilon)$ on $L^2_{\text{per}}([0, 2\pi/\alpha])$, defined in (5.28). The discussions following (5.28) and Lemma 5.9 assert that $\tilde{\mathcal{F}}(\lambda, \epsilon)$ is compact, analytic in λ and continuous in ϵ.

By assumption, $\operatorname{Im} c_\alpha > 0$ and c_α is an unstable eigenvalue of the Rayleigh system (4.1)-(4.2) which corresponds to $\epsilon = 0$. In other words, λ_0 is a pole of $(id + \mathcal{F}(\lambda, 0))^{-1}$. Subsequently, it is a pole of $(id + \tilde{\mathcal{F}}(\lambda, 0))^{-1}$. Since λ_0 is an isolated pole, we may choose $\delta > 0$ small enough so that the operator $id + \tilde{\mathcal{F}}(\lambda, 0)$ is invertible on $|\lambda - \lambda_0| = \delta$. By the continuity of $\tilde{\mathcal{F}}(\lambda, \epsilon)$ in ϵ, the following estimate

$$\|\tilde{\mathcal{F}}(\lambda, \epsilon) - \tilde{\mathcal{F}}(\lambda, 0)\|_{L^2_{\text{per}}([0,2\pi/\alpha]) \to L^2_{\text{per}}([0,2\pi/\alpha])} \leqslant C\epsilon$$

holds. Hence, $id + \tilde{\mathcal{F}}(\lambda, \epsilon)$ is invertible on $|\lambda - \lambda_0| = \delta$ and $\epsilon \geqslant 0$ sufficiently small. Lemma 5.8 then applies the poles of $(id + \tilde{\mathcal{F}}(\lambda, \epsilon))^{-1}$ are continuous in ϵ and can only appear or disappear in the boundary of $\{\lambda : |\lambda - \lambda_0 < \delta\}$. Therefore, at each $\epsilon \geqslant 0$, there is found a pole $\lambda(\epsilon)$ of $id + \tilde{\mathcal{F}}(\lambda, \epsilon)$ in $|\lambda(\epsilon) - \lambda_0| < \delta$. Then, $\operatorname{Re}\, \lambda(\epsilon) > 0$ and there exists a nonzero function $\tilde{f} \in L^2_{\text{per}}([0, 2\pi/\alpha])$ such that

$$(id + \tilde{\mathcal{F}}(\lambda, \epsilon))\tilde{f} = 0.$$

Our next task is to construct an exponentially growing solution to the linearized system (3.5) associated to $\tilde{f}$. Let $f = (\mathcal{B}_\epsilon)^{-1}\tilde{f}$ be the function in $L^2_{\text{per}}(\mathcal{S}_\epsilon)$ which transforms to $\tilde{f}$ under the action-angle mapping. By the construction of $\tilde{\mathcal{F}}$, it follows that

$$(id + \mathcal{F}(\lambda, \epsilon))f = 0 \tag{5.39}$$

Given $f \in L^2(\mathcal{S}_\epsilon)$ let $\psi \in H^1(\mathcal{D}_\epsilon)$ be the unique solution of (5.7) with $\lambda = \lambda(\epsilon)$ and

$$\psi_n(x) = -\mathcal{C}^\lambda(P_{\epsilon y}(x)\mathcal{C}^\lambda + \Omega\ id)f(x),$$

and let $\eta \in L^2(\mathcal{S}_\epsilon)$ be defined as $\eta(x) = \mathcal{C}^\lambda f(x)$. Let

$$\psi(x, \eta_\epsilon(x)) = -\mathcal{T}_\epsilon(\mathcal{C}^\lambda(P_{\epsilon y}(x)\mathcal{C}^\lambda + \Omega\ id))f(x).$$

This yields

$$\psi_n(x) = -\mathcal{C}^\lambda\left(P(x, \eta_\epsilon(x)) + \Omega\psi(x, \eta_\epsilon(x))\right), \tag{5.40}$$

$$\eta(x) = \mathcal{C}^\lambda(\psi(x, \eta_\epsilon(x)). \tag{5.41}$$

We shall show that $(e^{\lambda(\epsilon)t}\eta(x), e^{\lambda(\epsilon)t}\psi(x,y))$ satisfies the linearized system (3.5). Since $\psi(x,0) = 0$, the bottom boundary condition (3.5e) is satisfied. It is readily seen the dynamic boundary condition (3.5c) is satisfied. In view of Lemma 5.3, the equations (5.41), (5.40) imply that η and $\psi_n(x)$ satisfy (5.1b) and (5.1d), respectively, in the weak sense. In turn, the boundary conditions on the top surface (3.5b) and (3.5d) are satisfied in the weak sense. Since ψ solves (5.7a), the vorticity is given as

$$\omega = -\Delta\psi = \gamma'(\psi_\epsilon)\psi - \gamma'(\psi_\epsilon)\int_{-\infty}^0 \lambda e^{\lambda s}\psi(X_\epsilon(s), Y_\epsilon(s))ds. \tag{5.42}$$

As shown in [35], above equation implies that the vorticity ω satisfies the the vorticity equation (3.6) in the weak sense. In other words, ψ satisfies (3.5a) in the weak sense. In summary, $(e^{\lambda(\epsilon)t}\eta(x), e^{\lambda(\epsilon)t}\psi(x,y))$ is a weak solution of the linearized system (3.5).

Our last step of the proof is to study the regularity of the growing-mode $(e^{\lambda(\epsilon)t}\eta(x), e^{\lambda(\epsilon)t}\psi(x,y))$ and thus to show the solvability in the strong sense. First, we claim that $\psi \in H^2(\mathcal{D}_\epsilon)$. (Recall that the result of Lemma 5.6 is $\psi \in H^1(\mathcal{D}_\epsilon)$.) Indeed, the trace theorem asserts that $\psi \in H^1(\mathcal{D}_\epsilon)$ implies $\psi(x, \eta_\epsilon(x)) \in H^{1/2}(\mathcal{S}_\epsilon)$. Since the operator $\mathcal{C}^\lambda$ defined by (5.5) is regularity preserving, by (5.40) it follows that $\psi_n(x) \in H^{1/2}(\mathcal{S}_\epsilon)$. Recall from the statement of Theorem 2.2 that η_ϵ is of $C^{3+\beta}(\mathbb{R})$, where $\beta \in (0,1)$. That is $\mathcal{D}_\epsilon$ is of $C^{3+\beta}$. Since $\omega = -\Delta\psi \in L^2(\mathcal{D}_\epsilon)$ from the regularity theorem ([2]) of elliptic boundary problems it follows that $\psi \in H^2(\mathcal{D}_\epsilon)$. Then, by the trace theorem and (5.40), respectively, follows that $\psi(x, \eta_\epsilon(x)) \in H^{3/2}(\mathcal{S}_\epsilon)$ and $\psi_n(x) \in H^{3/2}(\mathcal{S}_\epsilon)$.

In order to obtain a higher regularity of ψ, we need to show that $\omega \in H^1(\mathcal{D}_\epsilon)$. The argument presented below is a simpler version of that in [33]. Taking the gradient of (5.42) yields that

$$\begin{aligned}\nabla\omega =& \nabla(\gamma'(\psi_\epsilon))\psi + \gamma'(\psi_\epsilon)\nabla\psi - \nabla(\gamma'(\psi_\epsilon))\int_{-\infty}^0 \lambda e^{\lambda s}\psi(X_\epsilon(s), Y_\epsilon(s))ds \\ &- \gamma'(\psi_\epsilon)\int_{-\infty}^0 \lambda e^{\lambda s}\nabla\psi(X_\epsilon(s), Y_\epsilon(s))\frac{\partial(X_\epsilon(s), Y_\epsilon(s))}{\partial(x,y)}ds.\end{aligned} \tag{5.43}$$

Note that the particle trajectory is written in the action-angle variables $(\theta, I) = \mathcal{A}_\epsilon(x,y)$ as

$$(X_\epsilon(s;x,y), Y_\epsilon(s;x,y)) = \mathcal{A}_\epsilon^{-1}((\theta + \upsilon_\epsilon(I)s, I))$$

This relies on that the action-angle mapping $\mathcal{A}_\epsilon$ is globally defined, a consequence of the assumption of no stagnation. With the use of the above description of the trajectory the estimate of the Jacobi matrix

$$\left|\frac{\partial\left(X_\epsilon(s;x,y), Y_\epsilon(s;x,y)\right)}{\partial\left(x,y\right)}\right| \leqslant C_1\left|s\right| + C_2 \tag{5.44}$$

follows, where $C_1, C_2 > 0$ are independent of s.

It is straightforward to see that calculations as in the proof of (5.15), L^2-norm of the first three terms of (5.43) is bounded by the H^1 norm of ψ. The last term in (5.43) is treated as

$$\begin{aligned}&\left\|\gamma'(\psi_\epsilon)\int_{-\infty}^{0}\lambda e^{\lambda s}\nabla\psi(X_\epsilon(s),Y_\epsilon(s))\frac{\partial(X_\epsilon(s),Y_\epsilon(s))}{\partial(x,y)}ds\right\|_{L^2}\\ &\quad\leqslant ||\gamma'(\psi_\epsilon)||_{L^\infty}\int_{-\infty}^{0}|\lambda|e^{\operatorname{Re}\lambda s}(C_1\left|s\right|+C_2)||\nabla\psi(X_\epsilon(s),Y_\epsilon(s))||_{L^2(\mathcal{D}_\epsilon)}ds\\ &\quad\leqslant C\left\|\psi\right\|_{H^1(\mathcal{D}_\epsilon)}.\end{aligned}$$

This uses (5.44), $\operatorname{Re}\lambda \geqslant \delta > 0$ and that the mapping $(x,y) \mapsto (X_\epsilon(s), Y_\epsilon(s))$ is measure-preserving. Therefore,

$$\left\|\nabla\omega\right\|_{L^2(\mathcal{D}_\epsilon)} \leqslant C\left\|\psi\right\|_{H^1(\mathcal{D}_\epsilon)}.$$

In turn, $\omega \in H^1(\mathcal{D}_\epsilon)$. Since $\psi_n(x) \in H^{3/2}(\mathcal{S}_\epsilon)$, by the elliptical regularity theorem [2] it follows that that $\psi \in H^3\left(\mathcal{D}_\epsilon\right)$. In view of the trace theorem this implies $\psi_n \in H^{5/2}(\mathcal{S}_\epsilon)$.

We repeat the process again. Taking the gradient of (5.43) and using the linear stretching property (5.3) of the trajectory, it follows that $\omega \in H^2\left(\mathcal{D}_\epsilon\right)$. The elliptic regularity applies to assert that $\psi \in H^4(\mathcal{D}_\epsilon) \subset C^{2+\beta}(\bar{\mathcal{D}}_\epsilon)$, where $\beta \in (0,1)$. By the trace theorem then it follows that $\psi(x, \eta_\epsilon(x)) \in H^{7/2}(\mathcal{S}_\epsilon)$. On account of (5.41) this implies that $\eta \in H^{7/2}\left(\mathcal{S}_\epsilon\right) \subset C^{2+\beta}([0, 2\pi/\alpha])$. Therefore, $(e^{\lambda(\epsilon)t}\eta(x), e^{\lambda(\epsilon)t}\psi(x,y))$ is a classical solution of (3.5). This completes the proof. □

6. Instability of general shear flows

Linear instability of free-surface shear flows is of independent interests. This section extends our instability result in Theorem 4.2 to a more general class of shear flows. The following class of flows was introduced in [32] and [35] in the rigid-wall setting.

Definition 6.1. *A function $U \in C^2([0,h])$ is said to be in the class $\mathcal{F}$ if U'' takes the same sign at all points such that $U(y) = c$, where c in the range of U but not an inflection value of U.*

Examples of the class-$\mathcal{F}$ flows include monotone flows and symmetric flows with a monotone half. If $U''(y) = f(U(y))k(y)$ for f continuous and $k(y) > 0$ then U is in class $\mathcal{F}$. All flows in class $\mathcal{K}^+$ are in class $\mathcal{F}$.

The lemma below shows that for a flow in class $\mathcal{F}$ a neutral limiting wave speed must be an inflection value. The main difference of the proof from that in class $\mathcal{K}^+$

(Proposition 4.4) lies in the lack of uniform H^2-bounds of unstable solutions near a neutral limiting mode.

Lemma 6.2. *For $U \in \mathcal{F}$, let $\{(\phi_k, \alpha_k, c_k)\}_{k=1}^{\infty}$ with $\operatorname{Im} c_k > 0$ be a sequence of unstable solutions which satisfy (4.1)–(4.2). If (α_k, c_k) converges to (α_s, c_s) as $k \to \infty$ with $\alpha_s > 0$ and c_s in the range of U then c_s must be an inflection value of U.*

Proof. Suppose on the contrary that c_s is not an inflection value. Let $y_1, y_2, \dots, y_m$ be in the pre-image of c_s so that $U(y_j) = c_s$, and let S_0 be the complement of the set of points $\{y_1, y_2, \dots, y_m\}$ in the interval $[0, h]$. Since c_s is not an inflection value, Definition 6.1 asserts that $U''(y_j)$ takes the same sign for $j = 1, 2, \dots, m$, say positive. As in the proof of Proposition 4.4, let $E_\delta = \{y \in [0, h] : |y - y_j| < \delta$ for some j, where $j = 1, 2, \cdots, m\}$. It is readily seen that $E_\delta^c \subset S_0$. Note that $U''(y) > 0$ for $y \in E_\delta$ if $\delta > 0$ small enough. After normalization, $\|\phi_k\|_{L^2} = 1$. The result of Lemma 3.6 in [32] is that ϕ_k converges uniformly to ϕ_s on any compact subset of S_0. Moreover, ϕ_s'' exists on S_0 and ϕ_s satisfies

$$\phi_s'' - \alpha_s^2 \phi_s - \frac{U''}{U - c_s}\phi_s = 0 \qquad \text{for} \quad y \in (0, h).$$

Our first task is to show that ϕ_s is not identically zero. Suppose otherwise. The proof is again divided into two cases.

Case 1: $U(h) \neq c_s$. In this case, $[h - \delta_1, h] \subset S_0$ for some $\delta_1 > 0$. As is done in the Proposition 4.4, for any q real follows that

$$\begin{aligned}
&\int_0^h \left(|\phi_k'|^2 + \alpha_k^2 |\phi_k|^2 + \frac{U''(U-q)}{|U - c_k|^2} |\phi_k|^2 \right) dy \\
&\geqslant \int_0^h |\phi_k'|^2 \, dy + \alpha_k^2 + \int_{E_\delta^c} \frac{U''(U-q)}{|U - c_k|^2} |\phi_k|^2 dy + \int_{E_\delta} \frac{U''(U-q)}{|U - c_k|^2} |\phi_k|^2 dy \\
&\geqslant \int_0^h |\phi_k'|^2 \, dy + \alpha_k^2 - \sup_{E_\delta^c} \frac{|U''(U-q)|}{|U - c_k|^2} \int_{E_\delta^c} |\phi_k|^2 dy.
\end{aligned}$$

On the other hand, (4.10) with $q = U_{\min} - 1$ yields that

$$\begin{aligned}
&\int_0^h \left(|\phi_k'|^2 + \alpha_k^2 |\phi_k|^2 + \frac{U''(U - U_{\min} + 1)}{|U - c_k|^2} |\phi_k|^2 \right) dy \\
&= \left(\operatorname{Re} g_r(c_k) + (\operatorname{Re} c_k - U_{\min} + 1) \frac{\operatorname{Im} g_r(c_k)}{\operatorname{Im} c_k} \right) |\phi_k(h)|^2 \\
&\leqslant C |\phi_k(h)|^2 \leqslant C_1 \left(\varepsilon \int_{h-\delta_1}^h |\phi_k'^2 dy + \frac{1}{\varepsilon} \int_{h-\delta_1}^h |\phi_k|^2 dy \right).
\end{aligned}$$

If ε is chosen to be small then the above two inequalities lead to

$$0 \geqslant \alpha_k^2 - \sup_{E_\delta^c} \frac{|U''(U - U_{\min} + 1)|}{|U - c_k|^2} \int_{E_\delta^c} |\phi_k|^2 dy - C_\varepsilon \int_{h-\delta_1}^h |\phi_k|^2 dy.$$

Since ϕ_k converges to $\phi_s \equiv 0$ uniformly on E_δ^c and $[h - \delta_1, h]$, this implies $0 \geqslant \alpha_s^2/2$. A contradiction proves that ϕ_s is not identically zero.

Case 2: $U(h) = c_s$. Since

$$\operatorname{Im} c_k \int_0^h \frac{U''}{|U - c_k|^2} |\phi_k|^2 dy = -\operatorname{Im} g_s(c_k) \left| \phi_k' \, (h) \right|^2,$$

the identity (4.17) yields that

$$\begin{aligned}
&\int_0^h (|\phi_k''|^2 + 2\alpha_k^2 |\phi_k'|^2 + \alpha_k^4|\phi_k|^2)dy \\
&= -2\alpha_k^2 \operatorname{Re} g_s(c_k)\,|\phi_k'(h)|^2 + \int_0^h \frac{(U'')^2}{|U-c_k|^2}|\phi_k|^2dy \\
&= \left(-2\alpha_k^2 \operatorname{Re} g_s(c_k) - \frac{\operatorname{Im} g_s(c_k)}{\operatorname{Im} c_k}\right)|\phi_k'(h)|^2 \\
&\leqslant Cd(c_k, U(h))\,|\phi_k'(h)|^2 \leqslant Cd(c_k, U(h))\|\phi_k\|_{H^2}^2.
\end{aligned}$$

Here, $d(c_k, U(h)) = |\operatorname{Re} c_k - U(h)| + (\operatorname{Im} c_k)^2$. Since $d(c_k, U(h)) \to 0$ as $k \to \infty$, this implies that $0 \geqslant \alpha_s^4/4$. A contradiction proves that ϕ_s is not identically zero. Subsequently, Lemma 4.7 asserts that $\phi_s(y_j) \neq 0$ for some y_j.

It remains to prove that c_s is the inflection value of U. In Case 1 when $U(h) \neq c_s$, it is straightforward to see that

$$\int_{E_\delta} \frac{U''(U-U_{\min}+1)}{|U-c_s|^2}|\phi_s|^2dy \geqslant \int_{|y-y_j|<\delta} \frac{U''}{|U-c_s|^2}|\phi_s|^2dy = \infty,$$

where $\phi_s(y_j) \neq 0$. As in the proof of Proposition 4.4, Fatou's lemma then states that

$$\liminf_{k\to\infty} \int_{E_\delta} \frac{U''(U-U_{\min}+1)}{|U-c_k|^2}|\phi_k|^2dy = \infty.$$

Subsequently, (4.10) applies and

$$\begin{aligned}
0 &= \int_0^h \left(|\phi_k'|^2 + \alpha_k^2|\phi_k|^2 + \frac{U''(U-U_{\min}+1)}{|U-c|^2}|\phi_k|^2\right)dy \\
&\quad - \left(\operatorname{Re} g_r(c_k) + (\operatorname{Re} c_k - U_{\min}+1)\frac{\operatorname{Im} g_r(c_k)}{\operatorname{Im} c_k}\right)|\phi_k(h)|^2 \\
&\geqslant \int_{E_\delta} \frac{U''(U-U_{\min}+1)}{|U-c_k|^2}|\phi_k|^2dy - \sup_{E_\delta^c}\frac{|U''(U-U_{\min}+1)|}{|U-c_k|^2} - C_\varepsilon > 0
\end{aligned}$$

for k large. A contradiction asserts that c_s is an inflection value. A similar consideration in Case 2 when $U(h) = c_s$ states that

$$\liminf_{k\to\infty} \int_{E_\delta} \frac{U''(U''_{\max}+1-U'')}{|U-c_k|^2}|\phi_k|^2dy = \infty,$$

where $U''_{\max} = \max_{[0,h]} U''(y)$. On the other hand, (4.17) implies

$$0 \geqslant \int_{E_\delta} \frac{U''(U''_{\max}+1-U'')}{|U-c_k|^2}|\phi_k|^2dy - \sup_{E_\delta^c}\frac{|U''(U''_{\max}+1-U'')|}{|U-c_k|^2} > 0$$

for k large. A contradiction asserts that c_s is an inflection value. This completes the proof. □

The proof of above lemma indicates that a flow in class $\mathcal{F}$ is linearly stable when the wave number is large.

Lemma 6.3. *For $U \in \mathcal{F}$ there exists $\alpha_{\max} > 0$ such that for each $\alpha \geqslant \alpha_{\max}$ corresponds no unstable solution to (4.1)–(4.2).*

Proof. Suppose otherwise. Then, there would exist a sequence of unstable solutions $\{(\phi_k, \alpha_k, c_k)\}_{k=1}^{\infty}$ of (4.1)–(4.2) such that $\alpha_k \to \infty$ as $k \to \infty$. After normalization, let $\|\phi_k\|_{L^2} = 1$. Our goal is to show that $\lim_{k\to\infty} \operatorname{Im} c_k = 0$. If $\operatorname{Im} c_k \geqslant \delta > 0$ for some δ then $1/|U - c_k|$ and $|g_r(c_k)|$ are uniformly bounded. Accordingly, with $q = \operatorname{Re} c_k$ in (4.10) yields

$$\begin{aligned}
0 &= \int_0^h \left(|\phi_k'|^2 + \alpha_k^2 |\phi_k|^2 + \frac{U''(U - \operatorname{Re} c_k)}{|U - c_k|^2} |\phi_k|^2 \right) dy - \operatorname{Re} g_r(c_k) |\phi_k(h)|^2 \\
&\geqslant \int_0^h |\phi_k'|^2 \, dy + \alpha_k^2 - \sup_{[0,h]} \frac{|U''(U - \operatorname{Re} c_k)|}{|U - c_k|^2} - C \left(\varepsilon \int_0^h |\phi_k'|^2 \, dy + \frac{1}{\varepsilon} \int_0^\varepsilon |\phi_k|^2 dy \right) \\
&\geqslant \alpha_k^2 - \sup_{[0,h]} \frac{|U''(U - \operatorname{Re} c_k)|}{|U - c_k|^2} - C/\varepsilon > 0
\end{aligned}$$

for k sufficiently large. A contradiction proves that $c_k \to c_s \in [U_{\min}, U_{\max}]$ as $k \to \infty$. The remainder of the

proof is nearly identical to that of Lemma 6.2 and hence is omitted. ☐

The following theorem gives a necessary condition for free surface instability that the flow profile should have an inflection point, which generalize the classical result of Lord Rayleigh [42] in the rigid wall case.

Theorem 6.4. *A flow $U \in \mathcal{F}$ without an inflection point throughout $[0, h]$ is linearly stable.*

Proof. Suppose otherwise; Then, there would exist an unstable solution (ϕ, α, c) to (4.1)–(4.2) with $\alpha > 0$ and $\operatorname{Im} c > 0$. Lemma 4.8 allows us to continue this unstable mode for wave numbers to the right of α until the growth rate becomes zero. Note that a flow without an inflection point is trivially in class F. So by Lemma 6.3, this continuation must end at a finite wave number $\alpha_{\max}$ and a neutral limiting mode therein. On the other hand, Lemma 6.2 asserts that the neutral limiting wave speed c_s corresponding to this neutral limiting mode must be an inflection value. A contradiction proves the assertion. ☐

Remark 6.5. Our proof of the above no-inflection stability theorem is very different from the rigid wall case. In the rigid-wall setting, where $\phi(h) = 0$, the identity (4.14) reduces to

$$c_i \int_0^h \frac{U''}{|U - c|^2} |\phi|^2 dy = 0,$$

which immediately shows that if U is unstable ($c_i > 0$) then $U''(y) = 0$ at some point $y \in (0, h)$. The same argument was adapted in [48, Section 5] for the free-surface setting, however, it does not give linear stability for general flows with no inflection points. More specifically, in the free-surface setting, (4.14) becomes

$$c_i \int_0^h \frac{U''}{|U - c|^2} |\phi|^2 \, dy = \left(\frac{2g(U(h) - c_r)}{|U(h) - c|^4} + \frac{U'(h)}{|U(h) - c|^2} \right) |\phi(h)|^2 ,$$

which only implies linear stability ([48, Section 5]) for special flows satisfying $U''(y) < 0, U'(y) \geqslant 0$ or $U''(y) > 0, U'(y) \leqslant 0$. In the proof of Theorem 6.4, we use the characterization of neutral limiting modes and remove above additional assumptions.

Let us now consider a shear flow $U \in \mathcal{F}$ with multiple inflection values $U_1, U_2, \ldots, U_n$. Lemma 6.2 states that a neutral limiting wave speed c_s must be one of the inflection values $U_1, U_2, \ldots, U_n$, say $c_s = U_j$. Repeating the argument in the proof of Lemma 4.6 around inflection points corresponding to the inflection value U_j establishes an uniform H^2-bound for a sequence of unstable solutions near the neutral limiting mode whose wave speed is near U_j. The proof is very similar to that in the rigid-wall setting [35, Lemma 2.6] and hence it is omitted. Thus, neutral limiting modes in class $\mathcal{F}$ are characterized by inflection values.

Proposition 6.6. *If $U \in \mathcal{F}$ has inflection values $U_1, U_2, \ldots, U_n$ then for a neutral limiting mode (ϕ_s, α_s, c_s) with $\alpha_s > 0$ the neutral limiting wave speed must be one of the inflection values, that is, $c_s = U_j$ for some j. Moreover, ϕ_s must solve*

$$\phi_s'' - \alpha_s^2 \phi_s + K_j(y)\phi_s = 0 \quad \textit{for } y \in (0,h),$$

where $K_j(y) = -\dfrac{U''(y)}{U(y) - U_j}$, and boundary conditions

$$\begin{cases} \phi_s'(h) = g_r(U_j), & \phi_s(0) = 0 \qquad \textit{if} \quad U(h) \neq U_j \\ \phi_s(h) = 0, & \phi_s(0) = 0 \qquad \textit{if} \quad U(h) = U_j. \end{cases} \tag{6.1}$$

One may exploit the instability analysis of Theorem 4.9 for a flow in class $\mathcal{F}$ with possibly multiple inflection values). The main difference of the analysis in class $\mathcal{F}$ from that in class $\mathcal{K}^+$ is that unstable wave numbers in class $\mathcal{F}$ may bifurcate to the left and to the right of a neutral limiting wave number, whereas unstable wave numbers in class $\mathcal{K}^+$ bifurcate only to the left of a neutral limiting wave number. In the rigid-wall setting, with an extension of the proof of [32, Theorem 1.1] Zhiwu Lin [35, Theorem 2.7] analyzed this more complicated structure of the set of unstable wave numbers. The remainder of this section establishes an analogous result in the free-surface setting.

In order to study the structure of unstable wave numbers in class $\mathcal{F}$ with possibly multiple inflection values, we need several notations to describe. A flow $U \in \mathcal{F}$ is said to be in class $\mathcal{F}^+$ if each $K_j(y) = -U''(y)/(U(y) - U_j)$ is nonzero, where U_j for $j = 1, \cdots, n$ are inflection values of U. It is readily seen that for such a flow K_j takes the same sign at all inflection points of U_j. A neutral limiting mode (ϕ_j, α_j, U_j) is said to be positive if the sign of K_j is positive at inflection points of U_j, and negative if the sign of K_j is negative. Proposition 6.6 asserts that $-\alpha_s^2$ is a negative eigenvalue of $-\frac{d^2}{dy^2} - K_j(y)$ on $y \in (0,h)$ with boundary conditions (6.1). We employ the argument in the proof of Theorem 4.9 to conclude that an unstable solution exists near a positive (negative) neutral limiting mode if and only if the perturbed wave number is slightly to the left (right) of the neutral limiting wave number. Thus, the structure of the set of unstable wave numbers with multiple inflection values is more intricate. We remark that a class-$\mathcal{K}^+$ flow has a unique positive neutral limiting mode and hence unstable solutions bifurcate to the left of a neutral limiting wave number.

Let us list all neutral limiting wave numbers in the increasing order. If the sequence contains more than one successive negative neutral limiting wave numbers, then we pick the smallest (and discard others). If the sequence contains more than one successive positive neutral limiting wave numbers, then we pick the largest (and discard others). If the smallest member in this sequence is a positive neutral limiting wave number, then we add zero into the sequence. Thus, we obtain a new sequence of neutral limiting wave numbers. Let us denote the resulting sequence by $\alpha_0^- < a_0^+ < \cdots < \alpha_N^- < \alpha_N^+$, where, α_0^- (might be 0),..., α_N^- are negative neutral limiting wave numbers and $\alpha_0^+, \ldots, \alpha_N^+$ are positive neutral limiting wave numbers. The largest member of the sequence must be a positive neutral wave number since no unstable modes exist to its right.

Theorem 6.7. *For $U \in \mathcal{F}^+$ with inflection values $U_1, U_2, \ldots, U_n$, let $\alpha_0^- < a_0^+ < \cdots < \alpha_N^- < \alpha_N^+$ be defined as above. For each $\alpha \in \cup_{j=0}^N (\alpha_j^-, \alpha_j^+)$, there exists an unstable solution of (4.1)–(4.2. Moreover, the flow is linear stable if either $\alpha \geqslant \alpha_N^+$ or all operators $-\frac{d^2}{dy^2} - K_j(y)$ $(j = 1, 2, \ldots, n)$ on $y \in (0, h)$ with (6.1 are nonnegative.*

Theorem 6.7 indicates that there possibly exists a gap in $(0, \alpha_N^+)$ of stable wave numbers. Indeed, in the rigid-wall setting, a numerical computation [6] demonstrates that for a certain shear-flow profile the onset of the unstable wave numbers is away from zero, that is $\alpha_0^- > 0$.

Appendix A. Proofs of (4.30), (4.31), (4.38) and (4.39)

Our first task is to show that the limit (4.30) holds as $\varepsilon \to 0-$ uniformly in $E_{(R,b_1,b_2)}$.

In the proof of Theorem 4.9, we have already established that both $\bar{\phi}_1(y; \varepsilon, c)$ and $\phi_0(y; \varepsilon, c)$ uniformly converge to ϕ_s in C^1, as $(\varepsilon, c) \to (0, 0)$ in $E_{(R,b_1,b_2)}$. Moreover, $\phi_2(0; \varepsilon, c) \to -\frac{1}{\phi_s'(0)}$ uniformly as $(\varepsilon, c) \to (0, 0)$ in $E_{(R,b_1,b_2)}$. So the function

$$\begin{aligned} &G(y, 0; \varepsilon, c)\phi_0(y; \varepsilon, c) \\ &= \left(\bar{\phi}_1(y; \varepsilon, c)\, \phi_2(0; \varepsilon, c) - \phi_2(y; \varepsilon, c)\, \bar{\phi}_1(0; \varepsilon, c)\right) \phi_0(y; \varepsilon, c) \end{aligned}$$

converges uniformly to $-\phi_s^2(y)/\phi_s'(0)$ in $C^1[0, d]$,as $(\varepsilon, c) \to (0, 0)$ in $E_{(R,b_1,b_2)}$.

The use of (4.28) proves the uniform convergence of (4.30).

The proof of (4.31) uses the following lemma.

Lemma A.1 ([32], Lemma 7.3)**.** *Assume that a sequence of differentiable functions $\{\Gamma_k\}_{k=1}^\infty$ converges to Γ_∞ in C^1 and that $\{c_k\}_{k=1}^\infty$ converges to zero, where $\operatorname{Im} c_k > 0$ and $|\operatorname{Re} c_k| \leqslant R \operatorname{Im} c_k$ for some $R > 0$. Then,*

(A.1)
$$\lim_{k\to\infty} - \int_0^h \frac{U''}{(U - U_s - c_k)^2} \Gamma_k dy = p.v. \int_0^h \frac{K(y)}{U - U_s} \Gamma_\infty dy + i\pi \sum_{i=1}^{m_s} \frac{K(a_j)}{|U'(a_j)|} \phi_s(a_j),$$

provided that $U'(y) \neq 0$ at each a_j. Here $a_1, \ldots, a_{m_s}$ are roots of $U - U_s$.

We now prove (4.31). That is, We shall show that (4.31) holds uniformly in $E_{(R,b_1,b_2)}$. Suppose for some $\delta > 0$ and a sequence $\{(\varepsilon_k, c_k)\}_{k=1}^\infty$ in $E_{(R,b_1,b_2)}$with $\max(b_1^k, b_2^k)$ tending to zero

$$\left|\frac{\partial \Phi}{\partial c}(\varepsilon_k, c_k) - (C + iD)\right| > \delta$$

holds, where C and D are defined in (4.32). Let us write

$$\frac{\partial \Phi}{\partial c}(\varepsilon_k, c_k) = -\int_0^h \frac{U''}{(U - U_s - c_k)^2} \Gamma_k dy + \frac{d}{dc} g_r(U_s + c)\phi_2(0; \varepsilon_k, c_k) = I + II,$$

where
$$\Gamma_k(y) = G(y,0;\varepsilon_k,c_k)\phi_0(y;\varepsilon_k,c_k) \to -\frac{1}{\phi_s'(0)}\phi_s^2 \quad \text{in } C^1.$$
By Lemma A.1 follows that
$$\lim_{k\to\infty} I = -\frac{1}{\phi_s'(0)}\left(p.v.\int_0^h \frac{K(y)}{U-U_s}\phi_s^2 dy + i\pi\sum_{k=1}^{m_s}\frac{K(a_j)}{|U'(a_j)|}\phi_s^2(a_j)\right).$$
A straightforward calculation yields that
$$\lim_{k\to\infty} II = -\frac{d}{dc}g_r(U_s+c)\frac{1}{\phi_s'(0)} = -\left(\frac{2g}{(U(h)-U_s)^3}+\frac{U'(h)}{(U(h)-U_s)^2}\right)\frac{1}{\phi_s'(0)} = -\frac{A}{\phi_s'(0)},$$
where A is given in (4.26). Therefore,
$$\lim_{k\to\infty}\frac{\partial\Phi}{\partial c}(\varepsilon_k,c_k) = C+iD.$$
A contradiction then proves the uniform convergence.

The proofs of (4.38) and (4.39) use the following lemma.

Lemma A.2 ([32], Lemma 7.1)**.** *Assume that $\{\psi_k\}_{k=1}^\infty$ converges to ψ_∞ in $C^1([0,h])$ and that $\{c_k\}_{k=1}^\infty$ with $\operatorname{Im} c_k > 0$ converges to zero. Let us denote $W_k(y) = U(y)-U_s-\operatorname{Re} c_k$. Then, the limits*

$$\lim_{k\to\infty}\int_0^h \frac{W_k}{W_k^2+\operatorname{Im} c_k^2}\psi_k dy = p.v.\int_0^h\frac{\psi_\infty}{U-U_s}dy, \tag{A.2}$$
$$\lim_{k\to\infty}\int_0^h \frac{\operatorname{Im} c_k}{W_k^2+\operatorname{Im} c_k^2}\psi_k dy = \pi\sum_{j=1}^{m_s}\frac{\psi_\infty(a_j)}{|U(a_j)|} \tag{A.3}$$

hold provided that $U'(y)\neq 0$ at each a_j. Here, $a_1,\ldots,a_{m_s}$ satisfy $U(a_j)=U_s$.

We now prove (4.38). Since $K(y)\phi_s\phi_k \to K(y)\phi_s^2$ in $C^1([0,h])$, by (A.2) and (A.3) it follows that
$$\begin{aligned}
&-\int_0^h \frac{U''}{(U-c_k)(U-U_s)}\phi_s\phi_k dy\\
&= \int_0^h \frac{W_k}{W_k^2+\operatorname{Im} c_k^2}K(y)\phi_s\phi_k dy + i\int_0^h\frac{\operatorname{Im} c_k}{W_k^2+\operatorname{Im} c_k^2}K(y)\phi_s\phi_k dy\\
&\to p.v.\int_0^h\frac{K}{(U-U_s)}\phi_s^2 dy + i\pi\sum_{j=1}^{m_s}\frac{K(a_j)}{|U(a_j)|}\phi_s^2(a_j)
\end{aligned}\tag{A.4}$$
as $k\to\infty$. Since $c_k\to U_s$ as $k\to\infty$ in the proof of Theorem 4.2, it follows that
$$\lim_{k\to\infty}\frac{g_r(c_k)-g_r(U_s)}{c_k-U_s} = g_r'(U_s) = \frac{2g}{(U(h)-U_s)^3}+\frac{U'(h)}{(U(h)-U_s)^2}. \tag{A.5}$$
Addition of (A.4) and (A.5) proves (4.38). In case $U(h)=U_s$ the same computations as above prove (4.39). This uses that
$$\lim_{k\to\infty}\frac{g_s(c_k)}{c_k-U_s} = -\lim_{k\to\infty}\frac{U(h)-c_k}{g+U'(h)(U(h)-c_k)} = 0.$$

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