No books or notes allowed. No laptop, graphic calculator or wireless devices allowed. Write clearly. Show your work and justify your answers. Remember to add your name to every page.
1. Evaluate the following definite integrals.

(a) (13 points)
\[ \int_{0}^{1} \frac{(x + 2)^2 - 4}{x} \, dx \]

**Solution:**
\[ \int_{0}^{1} \frac{(x + 2)^2 - 4}{x} \, dx = \int_{0}^{1} \frac{x^2 + 4x}{x} \, dx = \int_{0}^{1} (x + 4) \, dx = \frac{1}{2} + 4 \]

(b) (13 points)
\[ \int_{0}^{1} (x + 1)^2 \sqrt{3 + (x + 1)^3} \, dx \]

**Solution:** Calling \( u = 3 + (x + 1)^3 \) we get \( du = 3(x + 1)^2 \, dx \). Thus
\[ \int_{0}^{1} (x + 1)^2 \sqrt{3 + (x + 1)^3} \, dx = \frac{1}{3} \int_{4}^{11} \sqrt{u} \, du = \frac{2}{9} u^{3/2} \bigg|_{4}^{11} = \frac{2}{9} (11\sqrt{11} - 8) \]
2. Evaluate the following indefinite integrals.

(a) (13 points)
\[
\int \sin(x) \cos(x) \left( \cos^2(x) - \sin^2(x) \right) \, dx
\]

**Solution:**
\[
\int \sin(x) \cos(x) \left( \cos^2(x) - \sin^2(x) \right) \, dx = \frac{1}{2} \int \sin(2x) \cos(2x) \, dx = \frac{1}{8} \sin^2(2x) + C
\]
Alternatively call \( u = \sin(x) \cos(x) \) so that \( du = \left( \cos^2(x) - \sin^2(x) \right) \, dx \) and
\[
\int \sin(x) \cos(x) \left( \cos^2(x) - \sin^2(x) \right) \, dx = \int u \, du = \frac{1}{2} (\sin(x) \cos(x))^2 + C
\]
Observe that
\[
(\sin(x) \cos(x))^2 = \frac{1}{4} \sin^2(2x) = \frac{1}{4} - \frac{1}{4} \cos^2(2x) = \frac{1}{4} - \frac{1}{4} (\cos^2(x) - \sin^2(x))^2
\]
Since the term 1/4 can be included in the constant \( C \) this gives several different ways to write the result.

(b) (12 points)
\[
\int \sin(x) \cos(x) \, dx
\]

**Solution:** Call \( u = \sin(x) \) so that \( du = \cos(x) \, dx \) and
\[
\int \sin(x) \cos(x) \, dx = \int u \, du = \frac{1}{2} \sin^2(x) + C
\]
Equivalently call \( u = \cos(x) \) so that \( du = -\sin(x) \, dx \) and
\[
\int \sin(x) \cos(x) \, dx = -\int u \, du = -\frac{1}{2} \cos^2(x) + C
\]
3. Let $F(x)$ be the function defined by:

$$F(x) = \int_{\frac{x}{2}}^{x} \frac{\sin(t)}{t} \, dt.$$ 

for $x > 0$.

(a) (12 points) Find where $F$ is increasing/decreasing.

**Solution:** We need to find where $F'(x) \leq 0$ or $\geq 0$. We have that

$$F'(x) = \frac{\sin(x)}{x} \quad (1)$$

Since $1/x > 0$ if $x > 0$ we have that $F'(x) \leq 0$ if $x \in [\pi, 2\pi] \cup [3\pi, 4\pi] \cup [5\pi, 6\pi] \cup \ldots$ and $F'(x) \geq 0$ if $x \in (0, \pi] \cup [2\pi, 3\pi] \cup [4\pi, 5\pi] \cup \ldots$ since $F(x)$ is defined for $x > 0$.

Thus we have

- $F(x)$ decreases on $[\pi, 2\pi], [3\pi, 4\pi], [5\pi, 6\pi], \ldots$
- $F(x)$ increases on $(0, \pi], [2\pi, 3\pi], [4\pi, 5\pi], \ldots$

(b) (12 points) Is it possible that $F(\pi) > 1$?

**Solution:** No, it is not possible. Indeed se have that, for $\pi/2 \leq t \leq \pi$:

$$0 \leq \frac{\sin(t)}{t} \leq \frac{1}{t} \leq \frac{2}{\pi}$$

so that

$$F(\pi) = \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(t)}{t} \, dt \leq \left(\pi - \frac{\pi}{2}\right) \frac{\sin(t)}{t} \leq \frac{\pi}{2} \frac{2}{\pi} = 1.$$ 

Thus we have $F(\pi) \leq 1$. 

Name: _______________________________
4. Let \( A \) be the region of the plane bounded by the curves

\[
f(x) = (x - 2)^2 - 1
\]

and

\[
g(x) = -f(x) = -(x - 2)^2 + 1
\]

for \( 1 \leq x \leq 3 \).

(a) (12 points) Compute the area of \( A \).

\[\text{Solution:}\] The area of \( A \) is given by:

\[
\text{Area}(A) = 2 \int_1^3 [-(x - 2)^2 + 1] dx = 2 \int_1^3 (-x^2 + 4x - 3) dx = 2 \left( -\frac{x^3}{3} \right)_1^3 + 4 \left( \frac{x^2}{2} \right)_1^3 - 3 \left( x \right)_1^3 = \frac{8}{3}
\]

(b) (13 points) Let \( S \) be the solid obtained by rotating \( A \) around the \( y \) axis. Compute the volume of \( S \). You may use any of the methods you studied.

\[\text{Solution:}\] The easiest way is to use Pappus theorem. The centroid of \( A \) is at \( \bar{x} = 2 \) and \( \bar{y} = 0 \) since \( A \) is symmetric with respect to the lines \( y = 0 \) and \( x = 2 \).

Thus the volume is

\[
\text{Vol}(R) = 2\pi \bar{x} \text{Area}(A) = \frac{32\pi}{3}
\]

Alternatively you can use the shells method and get

\[
\text{Vol}(R) = 4\pi \int_1^3 x[-(x - 2)^2 + 1] dx = 2 \int_1^3 (-x^3 + 4x^2 - 3x) dx = 4\pi \left( -\frac{x^4}{4} \right)_1^3 + 4 \left( \frac{x^3}{3} \right)_1^3 - 3 \left( x \right)_1^3 = \frac{32\pi}{3}
\]