a) \[ A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \]

\[ \lambda^2 - 4 \lambda + 3 = 0 \quad \lambda_+ = 2 \pm 1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \]

\[ V_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad V_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ X(t) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} \]

b) \[ A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \]

\[ \lambda^2 - 2 \lambda = 0 \quad \lambda_+ = \lambda_- = 0 \]

\[ V_+ = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad V_- = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \]

\[ X(t) = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} \]
\[ A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \]

\[ \lambda^2 - \lambda - 2 = 0 \]

\[ \lambda_+ = \frac{1 \pm \sqrt{9}}{2} = 2 \]

\[ V_+ = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad V_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ X(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \]

\[ B = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix} \]

\[ \lambda^2 + 2\lambda - 9\lambda = 0 \]

\[ \lambda_+ = -1 \pm \sqrt{10} \]
(a) \rightarrow 4
(b) \rightarrow 2
(c) \rightarrow 1
(d) \rightarrow 3

6) \dot{x} + b \dot{y} + kx = 0

X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \ddot{x} = y, \quad \ddot{y} = -by - kx

X = AX

A = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}

Real eigenvalue if \((\text{Tr}A)^2 - 4 \det A > 0\)

\(b^2 - 4k^2 > 0\)

If \(b > 0\) and \(k > 0\) Then mean \(b > 2k\)
\[ \lambda_+ = \frac{-b \pm \sqrt{b^2 - 4k^2}}{2} \]

\[ V_+ = \begin{pmatrix} 2 \\ b \pm \sqrt{b^2 + 4k^2} \end{pmatrix} \]

\[ X(t) = \alpha e^{\lambda_+ t} V_+ + \beta e^{\lambda_- t} V_- \]

Observe that \( \lambda_+ < 0 \) and \( \lambda_- < 0 \) so that \( X(t) \to 0 \) as \( t \to \infty \).

2) \[ A = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \]

Clearly for \( a = 1 \) \( A \) has 1 as repeated eigenvalue. In general near \( a = 1 \)

\[ \lambda - (a+1)\lambda + a = 0 \]

\[ \lambda_+ = \frac{(a+1)^2 - (a-1)^2}{2} \]

\[ \lambda_+ = a \quad \lambda_- = 1 \]

\[ V_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_- = \begin{pmatrix} 1 \\ 1-a \end{pmatrix} \]

Thus \( \lim_{a \to 1} V_- = V_+ \). The two eigenvectors
get closer and closer when \( \alpha \to 1 \)

12) Observe that

\[
\lambda + \mu = \text{Tr } A \quad \lambda \cdot \mu = \det A
\]

Thus

\[
\begin{pmatrix}
\alpha_{11} - \mu & \alpha_{12} \\
\alpha_{21} & \alpha_{22} - \mu
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} - \lambda \\
\alpha_{21}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha_{11} - \mu \left( \alpha_{11} - \lambda \right) + \alpha_{12} \alpha_{21} \\
\alpha_{21} \left( \alpha_{11} - \lambda \right) + \alpha_{21} \left( \alpha_{22} - \mu \right)
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha_{11} - \alpha_{11} \text{Tr } A + \det A + \alpha_{12} \alpha_{21} \\
\alpha_{21} \left( \alpha_{11} + \alpha_{22} \right) - \alpha_{21} \text{Tr } A
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

and similarly for the second line of

\( A - \lambda I \). So calling \( V = AE_1 = \begin{pmatrix} \alpha_{11} - \lambda \\ \alpha_{21} \end{pmatrix} \)

we have \( \left( A - \mu I \right) V = 0 \)

and similarly for \( AE_2 \). If they are
non-zero. They are eigenvectors of $\mu$.

14. Let $V_\lambda$ and $V_\mu$ be eigenvectors.

If $V_\lambda$ and $V_\mu$ are not linearly independent, then for $\alpha, \beta \neq 0$ such that

$$\alpha V_\lambda + \beta V_\mu = 0$$

it follows that

$$A(\alpha V_\lambda + \beta V_\mu) = 0$$

$$= \alpha \lambda V_\lambda + \beta \mu V_\mu$$

Let

$$B = \begin{pmatrix} \alpha & \beta \\ \lambda & \mu \end{pmatrix}$$

and

$$W = \begin{pmatrix} V_\lambda \\ V_\mu \end{pmatrix}$$

we have

$$BW = 0$$

so that $\det B = \alpha \beta (\lambda - \mu) = 0$ that

is impossible.
(a) $\lambda^2 - \lambda + 4 = 0 \quad \lambda = \frac{1 \pm i\sqrt{15}}{2}$

If $X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $X = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

The phase portrait is 2.

(b) $\lambda^2 + \lambda + 4 = 0 \quad \lambda = -\frac{1 \pm i\sqrt{15}}{2}$

$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\dot{X} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

The phase portrait is 5.

(c) $\lambda^2 - \lambda + 4 = 0 \quad \lambda = \frac{1 \pm i\sqrt{15}}{2}$

Phase portrait 6

(d) $\lambda^2 + 1 = 0 \quad \lambda = \pm i$

$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\dot{X} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Phase portrait is 1
(e) $\lambda = \pm i$ phase portrait is 2

(f) $\lambda = -1 \pm i\sqrt{15}$ phase portrait is 4
\[
\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[\lambda = \frac{1}{2}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\]

\[\lambda = -1, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\]

\[T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\]

\[X^*(t) = \mathbf{a} e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{b} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\]

\[Y(t) = \mathbf{a} e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{b} e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\]
\[ A_k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \lambda^2 - \lambda + 1 = 0 \]

\[ \lambda_1 = \frac{1 \pm \sqrt{5}}{2}, \quad \lambda_2 = \frac{-2}{1 \pm \sqrt{5}} \]

\[ T = \begin{pmatrix} -2 & -2 \\ 1 + \sqrt{5} & 1 - \sqrt{5} \end{pmatrix} \]

\[ X(t) = a \begin{pmatrix} -2 \\ 1 - \sqrt{5} \end{pmatrix} e^{(1 + \sqrt{5})t} + b \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix} e^{(1 - \sqrt{5})t} \]

\[ Y(t) = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{(1 - \sqrt{5})t} + b \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{(1 + \sqrt{5})t} \]
\[
A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
\lambda^2 - \lambda + 1 = 0 \quad \lambda = \frac{-1 \pm i \sqrt{3}}{2}
\]

\[
V_+ = \begin{pmatrix} -2 \\ 1 + i \sqrt{3} \end{pmatrix} = U i W
\]

\[
U = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}
\]

Thus

\[
T = \begin{pmatrix} -2 & 0 \\ 1 & \sqrt{3} \end{pmatrix}
\]

and

\[
T^{-1} A T = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}
\]

\[
Y(t) = a e^{\frac{t}{2}} \begin{pmatrix} \cos \frac{\sqrt{3}}{2} t \\ \sin \frac{\sqrt{3}}{2} t \end{pmatrix} + b e^{\frac{t}{2}} \begin{pmatrix} \sin \frac{\sqrt{3}}{2} t \\ \cos \frac{\sqrt{3}}{2} t \end{pmatrix}
\]

\[
X(t) = e^{\frac{t}{2}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} a \cos \frac{\sqrt{3}}{2} t + b \sin \frac{\sqrt{3}}{2} t \\ a \sin \frac{\sqrt{3}}{2} t - b \cos \frac{\sqrt{3}}{2} t \end{pmatrix} +
\]

\[
e^{\frac{t}{2}} \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \begin{pmatrix} a \sin \frac{\sqrt{3}}{2} t - b \cos \frac{\sqrt{3}}{2} t \\ a \cos \frac{\sqrt{3}}{2} t + b \sin \frac{\sqrt{3}}{2} t \end{pmatrix} =
\]

\[
e^{\frac{t}{2}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{\sin \frac{\sqrt{3}}{2} t + \theta} + e^{\frac{t}{2}} \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} e^{\cos \frac{\sqrt{3}}{2} t + \theta}
\]

where \((a, b) = e^{(\cos \theta, \sin \theta)}\)
$\begin{pmatrix}
1 & 1 \\
-1 & 2
\end{pmatrix}$

$\lambda = 2$ double

Only one eigenvector

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ Then

$A v_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = 2 v_1 + 2 v_2$

Thus $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ satisfy

$A v_2 = 2 v_1 + v_2$

$T = (v_1, v_2) = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix}$

$T^T A T = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$Y(t) = a e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + b e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

$X(t) = e^{2t} (a + b) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + be^{2t} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$
\[
V = \begin{pmatrix}
1 & 1 \\
-1 & -3
\end{pmatrix}
\]

\[
\lambda_{\pm} = -1 \pm \sqrt{3} \quad V_{\pm} = \begin{pmatrix} 1 \\ -2 \pm \sqrt{3} \end{pmatrix}
\]

\[
T = \begin{pmatrix}
1 & 1 \\
-2 + \sqrt{3} & -2 - \sqrt{3}
\end{pmatrix}
\]

\[
T^{-1} A T = \begin{pmatrix}
-1 + \sqrt{3} & 0 \\
0 & -1 - \sqrt{3}
\end{pmatrix}
\]

\[
Y(t) = a e^{(-1 + \sqrt{3})t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b e^{(-1 - \sqrt{3})t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
X(t) = a e^{(-1 + \sqrt{3})t} \begin{pmatrix} 1 \\ -2 + \sqrt{3} \end{pmatrix} + b e^{(-1 - \sqrt{3})t} \begin{pmatrix} 1 \\ -2 - \sqrt{3} \end{pmatrix}
\]
$$(v_1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

\[ \lambda_\pm = \pm \sqrt{2} \quad V_\pm = \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} \quad V = \begin{pmatrix} \sqrt{2} & -1 \\ -1 & 0 \end{pmatrix} \]

\[ T = \begin{pmatrix} 1 & \sqrt{2} - 1 \\ \sqrt{2} - 1 & -1 \end{pmatrix} \]

\[ T^{-1} A T = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \]

\[ Y(t) = a e^{\sqrt{2} t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b e^{-\sqrt{2} t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ X(t) = a e^{\sqrt{2} t} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} + b e^{-\sqrt{2} t} \begin{pmatrix} \sqrt{2} - 1 \\ -1 \end{pmatrix} \]
\[(5)\]
\[
A = \begin{pmatrix}
\alpha & 1 \\
2\alpha & 2
\end{pmatrix}
\]

\[
\lambda_+ = (\alpha + 2), \quad \lambda_- = 0
\]

\[
V_+ = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad V_- = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}
\]

\[\alpha > 2\]

\[\alpha \leq -2\]
Observe that the point rotates counterclockwise if the angle \( \theta \) between \( \vec{x} \) and \( \dot{\vec{x}} \) is in \( [0, \pi] \).

\[
\sin \theta = \frac{\vec{x} \cdot \dot{\vec{x}}}{\|\vec{x}\| \|\dot{\vec{x}}\|}
\]

This means that \( \sin \theta > 0 \). Since it is enough to ask that \( \vec{x} \cdot \dot{\vec{x}} > 0 \).

Choosing \( \vec{x} = (1, 0) \) we get \( a_{e_1} > 0 \).
Exercise 2.12: Tell us that if \( \lambda \) is an eigenvalue, then either the columns of \( A - \lambda I \) are eigenvectors. This means that

\[
(A - \mu I)(A - \lambda I) E_i = 0
\]

for \( i = 1 \) or \( 2 \). This means

\[
(A - \mu I)(A - \lambda I) = 0
\]

The proof works even if \( \lambda \) and \( \mu \) are complex or \( \lambda = \mu \). But then

\[
\lambda + \mu = -\alpha
\]

\[
\lambda \mu = \beta
\]

so that

\[
A^2 - (\lambda + \mu) A + \lambda \mu I = 0
\]

\[
= A^2 + \alpha A - \beta I.
\]