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Let $A$ and $B$ be two matrices such that $AB \neq BA$. Consider the functions:

\[ E_0(t) = e^{tA} \]
\[ E_1(t) = \int_0^t e^{(t-t_1)A} Be^{t_1A} dt_1 = \int_0^t e^{(t-t_1)A} BE_0(t_1) dt_1 \]

(a) (5 points) Show that
\[ \frac{d}{dt} E_1(t) = AE_1(t) + BE_0(t) \]

(b) (5 points) Define recursively
\[ E_{n+1}(t) = \int_0^t e^{(t-t_1)A} BE_n(t_1) dt_1 = \int_0^t dt_1 e^{(t-t_1)A} B \int_0^{t_1} dt_2 e^{(t_1-t_2)A} B \ldots \int_0^{t_{n-1}} dt_n e^{(t_{n-1}-t_n)A} Be^{t_nA} \]
and show that
\[ \frac{d}{dt} E_n(t) = AE_n(t) + BE_{n-1}(t) \]

(c) (10 points) Use the above relation to show that
\[ e^{(A+B)t} = \sum_{n=0}^{\infty} E_n(t) \]

(d) (10 points) Use the above to show that
\[ \frac{d}{dh} e^{A+hB} \bigg|_{h=0} = \int_0^1 e^{(1-s)A} Be^sA ds \]

(e) (10 points) Compute
\[ \frac{d^2}{dh^2} e^{A+hB} \bigg|_{h=0} \]
Question 2 ................................................................. 30 point

Consider the system of equation

\[
\begin{align*}
\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2)^\alpha \\
\dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2)^\alpha
\end{align*}
\]

with \( \alpha > -1 \).

(a) (10 points) For which values of \( \alpha \) does the system admit a unique solution for every initial condition?

(b) (10 points) For the values of \( \alpha \) for which the solution is not unique, give an initial value and (at least) two solutions starting from it.

(c) (10 points) For which values of \( \alpha \) does the system admit a global solution for every initial condition?
Consider the differential equations:

\[ \dot{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X \]  

and

\[ \dot{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y \]

Write the conjugacy \( Y = \phi(X) \) that conjugate eq.(3) and eq.(4).
Consider the differential equation

\[ \dot{X} = \begin{pmatrix} a & 1 + b \\ 1 - b & a \end{pmatrix} X \]

In the $a - b$ plane, describe the regions where the origin is a sink, a source, a saddle or a center. For each of these regions, give the general solution of the equation. Describe what happens on the boundary of these regions.