1) The equation governing the temperature $u(x, t)$ inside a rod is:

$$\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= \frac{\partial^2 u(x, t)}{\partial x^2} \quad 0 \leq x \leq 1 \\
\frac{\partial u(0, t)}{\partial x} &= ru(0, t) \\
\frac{\partial u(1, t)}{\partial x} &= r(T - u(1, t)) \\
u(x, 0) &= x
\end{align*}$$

a) write and solve the equation for the steady state $v(x)$. 

b) write the equation for the difference $w(x, t) = u(x, t) - v(x)$. 
c) use separation of variable to reduce the problem to a Sturm-Liouville problem. Find the eigenvalues and eigenfunctions. Explain why you can expand in eigenfunctions. Write the general solution for $w(x, t)$ and an expression for the coefficient in term of $w(x, 0)$. 
e) Give an estimate from above and below of the first eigenvalue. How long do you have to wait to be sure that $|w(x, t)| \leq 10^{-3}$. Use only the series truncated at the first term and give the estimate as a function of the first coefficient of $w(x, t)$, i.e. do not try to compute it.
f) **Bonus**: write the solution of the problem. Remember that

\[
\int x \cos(\lambda x)dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x \sin(\lambda x)}{\lambda}
\]

\[
\int x \sin(\lambda x)dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x \cos(\lambda x)}{\lambda}
\]
2) Let \( f(x) \) a continuous and differentiable function defined for all \( x \). Assume that

\[
|f(x)| \leq Ce^{-\lambda|x|}
\]

with \( C \) and \( \lambda \) positive. Finally let

\[
\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} f(x) dx.
\]

Consider now the function

\[
F(x) = \sum_{n=-\infty}^{\infty} f(x + nL)
\]

with \( L > 0 \).

a) Show that \( F(x) \) exists and it is periodic of period \( L \).
b) Let

\[ F(x) = \sum c_n e^{\frac{2\pi i x}{L}}. \]

Find the coefficients \( c_n \). (Hint: write an expression for \( c_n \) as a sum of integrals and then change variable \( y = x + nL \) and ...