1) The equation governing the temperature $u(x, t)$ inside a rod is:

$$
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} & 0 \leq x \leq 1 \\
\frac{\partial u(0, t)}{\partial x} = ru(0, t) \\
\frac{\partial u(1, t)}{\partial x} = r(T - u(1, t)) \\
u(x, 0) = x
\end{cases}
$$

a) write and solve the equation for the steady state $v(x)$.

The equation for the steady state is:

$$
\begin{cases}
\frac{\partial^2 u(x, t)}{\partial x^2} = 0 \\
\frac{\partial v(0)}{\partial x} = rv(0) \\
\frac{\partial v(1)}{\partial x} = r(T - v(1))
\end{cases}
$$

The general solution is still $v(x) = ax + b$. The first b.c. tells me that $a = rb$ while the second tells me that $a = r(T - a - b)$ or, using the other, $a = r(T - a - a/r)$ from which we get

$$a = \frac{rT}{2 + r} \quad b = \frac{T}{2 + r}$$

b) write the equation for the difference $w(x, t) = u(x, t) - v(x)$.

The equation for $w$ is the homogeneous version of that for $u$ so that:

$$
\begin{cases}
\frac{\partial w(x, t)}{\partial t} = \frac{\partial^2 w(x, t)}{\partial x^2} & 0 \leq x \leq 1 \\
\frac{\partial w(0, t)}{\partial x} = rw(0, t) \\
\frac{\partial w(1, t)}{\partial x} = -rw(1, t) \\
u(x, 0) = \left(1 - \frac{rT}{2 + r}\right)x - \frac{T}{2 + r}
\end{cases}
$$
c) use separation of variable to reduce the problem to a Sturm-Liouville problem. Find the eigenvalues and eigenfunctions. Explain why you can expand in eigenfunctions. Write the general solution for \( w(x, t) \) and an expression for the coefficient in term of \( w(x, 0) \).

Writing \( w(x, t) = T(t)s(x) \) we get the equation

\[
\begin{aligned}
\frac{\partial T(t)}{\partial t} &= \mu T(t) \\
\frac{\partial^2 s(x)}{\partial x^2} &= \mu s(x) \\
s'(0) - rs(0) &= 0 \\
s'(1) + rs(1) &= 0
\end{aligned}
\]

Observe that the Theorem on section 2.8 tells you that all \( \mu \) are non negative so that I can write \( \mu = -\lambda^2 \). The general solution of the equation for \( s(x) \) is \( s(x) = a \cos(\lambda x) + b \sin(\lambda x) \) so that \( s'(x) = -a\lambda \sin(\lambda x) + b\lambda \cos(\lambda x) \). The first b.c. tells me \( ra = \lambda b \) and the second tells me

\[
\lambda b \cos(\lambda) + rb \sin(\lambda) = \frac{\lambda^2}{r} b \sin(\lambda) - b \lambda \cos(\lambda x)
\]

that gives

\[
\tan(\lambda) = \frac{2r\lambda}{\lambda^2 - r^2}
\]

Observe that

\[
\lim_{\lambda \to \infty} \frac{2r\lambda}{\lambda^2 - r^2} = 0
\]

so that we have infinitely many solution \( \lambda_n \) and \( \lim_{n \to \infty} \lambda_n = n\pi \). Finally we get

\[
s_n(x) = \lambda_n \cos(\lambda_n x) + r \sin(\lambda_n x)
\]

From the general theory we know that the \( s_n(x) \) are orthogonal because they are the eigenvalue of a regular Sturm-Liouville problem. Setting:

\[
c_n = \int_0^1 s_n^2(x) dx
\]

We have, for every function \( f(x) \), that

\[
f(x) = \sum a_n s_n(x)
\]

where

\[
a_n = \int_0^1 f(x)s_n(x) dx.
\]
So we obtain that the general solution is

\[ u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} s_n(x) \]

and setting

\[ a_n = \frac{1}{c_n} \int_{0}^{1} \left[ \left( 1 - \frac{rT}{2 + r} \right) x - \frac{T}{2 + r} \right] s_n(x) dx \]

we obtain a solution for our problem.
e) Give an estimate from above and below of the first eigenvalue. How long do you have to wait to be sure that $|w(x, t)| \leq 10^{-3}$. Use only the series truncated at the first term but observe that you need an estimate of the first coefficient.

Observe that the function

$$g(\lambda) = \frac{2r\lambda}{\lambda^2 - r^2}$$

is negative for $\lambda \leq r$ and positive after. Moreover $\lim_{\lambda \to r^-} = -\infty$ and $\lim_{\lambda \to r^+} = +\infty$. Finally $g(0) = 0$. This implies that if $0 < r < \pi/2$ then $r < \lambda_1 < \pi/2$, otherwise $\pi/2 < \lambda_1 < \pi$. Writing the truncated solution we have

$$w(x, t) \simeq a_1 e^{-\lambda_1^2 t} s_1(x)$$

Observe that $|s_1(x)| \leq \lambda_1 + r$ so that we have to find $t$ such that

$$|a_1| e^{-\lambda_1^2 t} (\lambda_1 + r) \leq 10^{-3}$$

that is

$$t > \frac{\ln(1000(r + \lambda_1)|a_1|)}{\lambda_1}$$
f) **Bonus:** write the solution of the problem. Remember that

\[
\int x \cos(\lambda x) dx = \frac{\cos(\lambda x)}{\lambda^2} + x \frac{\sin(\lambda x)}{\lambda} \\
\int x \sin(\lambda x) dx = \frac{\sin(\lambda x)}{\lambda^2} - x \frac{\cos(\lambda x)}{\lambda}
\]

We have to compute

\[
\int_0^1 s_n(x) dx = \int_0^1 (\lambda_n \cos(\lambda_n x) + r \sin(\lambda_n x)) dx = \sin(\lambda_n) - r \frac{\cos(\lambda_n) - 1}{\lambda_n} = d_n
\]

and

\[
\int_0^1 x s_n(x) dx = \int_0^1 (\lambda_n x \cos(\lambda_n x) + rx \sin(\lambda_n x)) dx = \\
= \left( \frac{\cos(\lambda_n x)}{\lambda_n} + x \sin(\lambda_n x) + \frac{r \sin(\lambda_n x)}{\lambda_n^2} - \frac{rx \cos(\lambda_n x)}{\lambda_n} \right) \bigg|_0^1 = \\
= \frac{1 - r}{\lambda_n} \cos \lambda_n + \left( 1 + \frac{r}{\lambda_n^2} \right) \sin \lambda_n - \frac{1}{\lambda_n} = e_n
\]

Finally we have

\[
c_n = \int_0^1 \left( \frac{\lambda_n^2 - r^2}{2} \cos(2\lambda_n x) + \frac{\lambda_n^2}{2} + r \lambda_n \sin(2\lambda_n x) \right) dx = \\
= \frac{r^2 \lambda_n^2 - 1}{2\lambda_n} \sin(2\lambda_n) - r (\cos(2\lambda_n) - 1) + \frac{r^2 \lambda_n^2 + 1}{2}
\]

so that

\[
a_n = \left( 1 - \frac{rT}{2 + r} \right) \frac{e_n}{c_n} - \frac{T}{2 + r} \frac{d_n}{c_n}
\]
2) Let $f(x)$ a continuous and differentiable function defined for all $x$. Assume that

$$|f(x)| \leq Ce^{-\lambda|x|}$$

with $C$ and $\lambda$ positive. Finally let

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} f(x) dx.$$  \hspace{1cm} (1)

Consider now the function

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + nL)$$

with $L > 0$.

a) Show that $F(x)$ exists and it is periodic of period $L$.

Observe that

$$F(x + L) = \sum_{n=-\infty}^{\infty} f(x + L + nL) =$$

$$= \sum_{n=-\infty}^{\infty} f(x + (n + 1)L) = \sum_{m=-\infty}^{\infty} f(x + mL) = F(x)$$

so that $F(x)$ is periodic of period $L$. Let now $0 < x < L$. We have

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + nL) \leq C \sum_{n=-\infty}^{\infty} e^{-\lambda|x+nL|} \leq Ce^{\lambda x} \sum_{n=-\infty}^{\infty} e^{-\lambda|n|L} < +\infty$$

where we used that $|x + nL| \geq |nL| - |x|$ so that

$$e^{-\lambda|x+nL|} \leq e^{\lambda x} e^{-\lambda|nL|}.$$
b) Let

\[ F(x) = \sum c_n e^{i \frac{2n\pi}{L} x}. \]

Find the coefficients \( c_n \). (Hint: write an expression for \( c_n \) as a sum of integrals and then change variable \( y = x + nL \) and ...)

\[
\begin{align*}
    c_m &= \frac{1}{L} \int_0^L e^{-i \frac{2m\pi}{L} x} F(x) \, dx = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int_0^L e^{-i \frac{2m\pi}{L} x} f(x + nL) \, dx = \\
    &= \frac{1}{L} \sum_{n=-\infty}^{\infty} \int_{nL}^{(n+1)L} e^{-i \frac{2m\pi}{L} y} f(y) \, dy = \frac{1}{L} \int_{-\infty}^{\infty} e^{-i \frac{2m\pi}{L} y} f(y) \, dy = \\
    &= \hat{f} \left( -\frac{2m\pi}{L} \right)
\end{align*}
\]