Exercise 1
Consider the differential equation
\[ \dot{x} = f(x, t) \]  
with initial condition \( x(t_0) = x_0 \). Assume that \( f \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n) \). Given \( h > 0 \) we call \( x^h(t) \) (Euler approximation) the function defined by
\[
\begin{align*}
  x^h(nh + t) &= x^h(nh) + f(x^h(nh), nh)t \quad \text{for } n \geq 0 \text{ and } 0 \leq t \leq h \\
  x^h(nh + t) &= x^h(nh) + f(x^h(nh), nh)t \quad \text{for } n \leq 0 \text{ and } -h \leq t \leq 0
\end{align*}
\]
Prove existence and uniqueness of the solution of eq.(1) using the Euler approximations. Show how it happens that, if the function \( f \) is not Lipschitz, the solution may fail to be unique.

Exercise 2
Let \( x(t) \) be a solution of
\[ \dot{x} = f(x) \]  
with \( x(0) = x_0 \) and \( x(1) = x_1 \). Call \( \gamma \) the trajectory \( \{x(t), \ t \in [0,1]\} \). Assume that \( f \in C^2(\mathbb{R}^n, \mathbb{R}^n) \). Let \( h_0(x) \) and \( h_1(x) \) to smooth function from \( \mathbb{R}^n \) in \( \mathbb{R} \) such that
\[ h_0(x_0) = 0 \quad h_1(x_1) = 0 \]  
Under which condition the equations:
\[ h_0(x) = 0 \quad h_1(x) = 0 \]  
define two \((n-1)\)-cells \( S_0 \) and \( S_1 \) transverse to \( \gamma \)?
Under these condition, show that there is a differentiable function \( F \) from a small neighbor of \( x_0 \) on \( S_0 \) to a small neighbor of \( x_1 \) in \( S_1 \) such that \( F(x) \) is on the trajectory of eq.(3) starting from \( x \). Compute
\[ \frac{\partial F}{\partial x}(x) \]

Exercise 3
Consider the differential equation
\[
\begin{aligned}
\dot{x} &= -y + \epsilon f_x(x, y) \\
\dot{y} &= x + \epsilon f_y(x, y)
\end{aligned}
\] (7)

where \( f = (f_x, f_y) \) is a smooth function from \( \mathbb{R}^2 \) in \( \mathbb{R}^2 \) and \( \epsilon \) is a small parameter. Call \( \phi(\xi, t) \) the solution of eq.(7) starting at \( \xi \) at time 0. Let \( \xi = (x, 0), \ x > 0, \) be a point on the positive \( x \) axis. Show that if \( \epsilon \) is small enough, there is a time \( t_\epsilon(x) \) close to \( 2\pi \) such that \( \phi((x, 0), t(x)) \) is again on the positive \( x \) axis.

Call \( F_\epsilon(x) \) the map define by \( F_\epsilon(x) = \phi_x((x, 0), t(x)) \) where \( \phi(\xi, t) = (\phi_x(\xi, t), \phi_y(\xi, t)) \). Show that, for \( \epsilon \) small enough, \( F_\epsilon \) is a smooth map from a neighbor of \( x \) in \( \mathbb{R} \) to a neighbor of \( F_\epsilon(x) \) in \( \mathbb{R} \). Compute

\[
\partial_\epsilon F_\epsilon(x) = \frac{\partial F_\epsilon}{\partial \epsilon}(x)
\] (8)

by treating \( \epsilon \) as a parameter. Show that if there are \( x_1 \) and \( x_2, \ x_1 < x_2, \) such that \( \partial_\epsilon F_\epsilon(x_1) > 0 > \partial_\epsilon F_\epsilon(x_2) \) then there is a periodic orbit starting from some point \( (\bar{x}, 0) \) with \( x_1 \leq \bar{x} \leq x_2 \).