8. Let $G$ be a region and let $f$ and $g$ be analytic functions on $G$ such that $f(z)g(z) = 0$ for all $z$ in $G$. Show that either $f \equiv 0$ or $g \equiv 0$.

9. Let $U: \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all $z$ in $\mathbb{C}$; prove that $U$ is constant.

10. Show that if $f$ and $g$ are analytic functions on a region $G$ such that $fg$ is also analytic on $G$, then $\overline{f} \overline{g}$ is analytic on $G$. Does the converse hold?
integer \( k \) there is a point \( u \in \{ \gamma \} \) with \( n(\gamma, u) = k \).

3. Let \( p(z) \) be a polynomial of degree \( n \) and let \( R > 0 \) be sufficiently large so that \( p \) never vanishes in \( \{ z : |z| \geq R \} \). If \( \gamma(t) = Re^{it}, 0 \leq t \leq 2\pi \), show that \( \int_{\gamma} \frac{p'(z)}{p(z)} \, dz = 2\pi i n \).
Exercises

1. Suppose \( f : G \to \mathbb{C} \) is analytic and define \( \varphi : G \times G \to \mathbb{C} \) by \( \varphi(z, w) = \left( f(z) - f(w) \right) (z - w)^{-1} \) if \( z \neq w \) and \( \varphi(z, z) = f'(z) \). Prove that \( \varphi \) is continuous and for each fixed \( w \), \( z \to \varphi(z, w) \) is analytic.

2. Give the details of the proof of Theorem 5.6.

3. Let \( B_\pm = \overline{B}(\pm 1; \frac{1}{2}) \), \( G = B(0; 3) - (B_+ \cup B_-) \). Let \( \gamma_1, \gamma_2, \gamma_3 \) be curves whose traces are \( |z - 1| = 1 \), \( |z + 1| = 1 \), and \( |z| = 2 \), respectively. Give \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) orientations such that \( n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0 \) for all \( w \) in \( \mathbb{C} - G \).

4. Show that the Integral Formula follows from Cauchy's Theorem.

5. Let \( \gamma \) be a closed rectifiable curve in \( \mathbb{C} \) and \( a \notin \{ \gamma \} \). Show that for \( n \geq 2 \)
   \[ \int_\gamma (z - a)^{-n} \, dz = 0. \]

6. Let \( f \) be analytic on \( D = B(0; 1) \) and suppose \( |f(z)| \leq 1 \) for \( |z| < 1 \). Show \( |f'(0)| \leq 1 \).

7. Let \( \gamma(t) = 1 + e^{it} \) for \( 0 \leq t \leq 2\pi \). Find \( \int_{\gamma} \left( \frac{z}{z - 1} \right)^n \, dz \) for all positive integers \( n \).

8. Let \( G \) be a region and suppose \( f_n : G \to \mathbb{C} \) is analytic for each \( n \geq 1 \). Suppose that \( \{ f_n \} \) converges uniformly to a function \( f : G \to \mathbb{C} \) Show that \( f \) is analytic.

9. Show that if \( f : \mathbb{C} \to \mathbb{C} \) is a continuous function such that \( f \) is analytic off \([ -1, 1 ]\) then \( f \) is an entire function.

10. Use Cauchy's Integral Formula to prove the Cayley–Hamilton Theorem: If \( A \) is an \( n \times n \) matrix over \( \mathbb{C} \) and \( f(z) = \text{det}(z - A) \) is the characteristic polynomial of \( A \) then \( f(A) = 0 \). (This exercise was taken from a paper by C. A. McCarthy, Amer. Math. Monthly, 82 (1975), 390–391).
1. Let $G$ be a region and let $\sigma_1, \sigma_2 : [0, 1] \to G$ be the constant curves $\sigma_1(t) \equiv a$, $\sigma_2(t) \equiv b$. Show that if $\gamma$ is a closed rectifiable curve in $G$ and $\gamma \sim \sigma_1$ then $\gamma \sim \sigma_2$. (Hint: connect $a$ and $b$ by a curve.)

2. Show that if we remove the requirement "$\Gamma(0, t) = \Gamma(1, t)$ for all $t$" from Definition 6.1 then the curve $\gamma_0(t) = e^{2\pi it}$, $0 \leq t \leq 1$, is homotopic to the constant curve $\gamma_1(t) \equiv 1$ in the region $G = \mathbb{C} - \{0\}$.

3. Let $\mathcal{C}$ = all rectifiable curves in $G$ joining $a$ to $b$ and show that Definition 6.11 gives an equivalence relation on $\mathcal{C}$.

4. Let $G = \mathbb{C} - \{0\}$ and show that every closed curve in $G$ is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$. 
6. Let \( \gamma(\theta) = \theta e^{i\theta} \) for \( 0 \leq \theta \leq 2\pi \) and \( \gamma(\theta) = 4\pi - \theta \) for \( 2\pi \leq \theta \leq 4\pi \). Evaluate \( \int_{\gamma} \frac{dz}{z^2 + \pi^2} \).

7. Let \( f(z) = [(z-\frac{1}{2} - i) \cdot (z-1-\frac{3}{2}i) \cdot (z-1+\frac{i}{2}) \cdot (z-\frac{3}{2}-i)]^{-1} \) and let \( \gamma \) be the polygon \([0, 2, 2+2i, 2i, 0]\). Find \( \int_{\gamma} f \).

8. Let \( G = \mathbb{C} - \{a, b\}, a \neq b \), and let \( \gamma \) be the curve in the figure below.

(a) Show that \( n(\gamma; a) = n(\gamma; b) = 0 \).

(b) Convince yourself that \( \gamma \) is not homotopic to zero. (Notice that the word is “convinced” and not “prove”. Can you prove it?) Notice that this example shows that it is possible to have a closed curve \( \gamma \) in a region such that \( n(\gamma; z) = 0 \) for all \( z \) not in \( G \) without \( \gamma \) being homotopic to zero. That is, the converse to Corollary 6.10 is false.

9. Let \( G \) be a region and let \( \gamma_0 \) and \( \gamma_1 \) be two closed smooth curves in \( G \). Suppose \( \gamma_0 \sim \gamma_1 \) and \( \Gamma \) satisfies (6.2). Also suppose that \( \gamma_t(s) = \Gamma(s, t) \) is smooth for each \( t \). If \( w \in \mathbb{C} - G \) define \( h(t) = n(\gamma_t; w) \) and show that \( h: [0, 1] \rightarrow \mathbb{Z} \) is continuous.

10. Find all possible values of \( \int_{\gamma} \frac{dz}{1 + z^2} \) where \( \gamma \) is any closed rectifiable curve in \( \mathbb{C} \) not passing through \( \pm i \).
Let $f$ be analytic in $B(a; R)$ and suppose that $f(a) = 0$. Show that $a$ is a zero of multiplicity $m$ iff $f^{(m-1)}(a) = \ldots = f(a) = 0$ and $f^{(m)}(a) \neq 0$. 
4. Suppose that $f: G \rightarrow \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any $z$ in $G$. 